INTRINSIC GEOMETRY OF IDEAL SPACE

MACMILLAN AND CO., LIMITED
LONDON · BOMBAY · CALCUTTA · MADRAS
MELBOURNE

THE MACMILLAN COMPANY
NEW YORK · BOSTON · CHICAGO
DALLAS · ATLANTA · SAN FRANCISCO

THE MACMILLAN COMPANY
OF CANADA, LIMITED
TORONTO

INTRINSIC GEOMETRY OF IDEAL SPACE

BY

A. R. FORSYTH
Sc.D., LL.D., Math.D., F.R.S.
EMERITUS PROFESSOR,
IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY, LONDON
FORMERLY SADLERIAN PROFESSOR
IN THE UNIVERSITY OF CAMBRIDGE

VOLUME I

MACMILLAN AND CO., LIMITED
ST. MARTIN'S STREET, LONDON
1935

PRINTED IN GREAT BRITAIN

PREFACE

THE present work is occupied with investigations of those intrinsic properties and differential measures of geometrical amplitudes which are connected with the corporate characteristics and the organic constituents of the amplitudes.

I.

Both for convenience of descriptive reference in the preliminary discussions, and for the determination of curvatures of initial rank (especially for the determination of the circular curvature of its geodesics), an amplitude is conceived as existing in a plenary uncurved (homaloidal) space which necessarily is of more extensive range than the amplitude itself. No attention is paid to position or orientation in that plenary space; the quest is for the magnitudes and the relations, which are essentially independent of mere location. Uncurved space is chosen as the standard medium of plenary extension for all curved amplitudes, mainly for two reasons: every measure of any curvature of such a space, regarded as an unrelated irresoluble entity, is zero: and all deviations from uniform evenness, that are exhibited by contained configurations, can be expressed by canonical measures which represent the magnitude and the import of the deviations independently of any controlling influence of properties of the plenary space. The homaloidal space is free from all native measures that can modify the geometrical character of included configurations: its standardising influence is impartially neutral.

The dimensionality of an amplitude, estimated by the integer that denotes the number of its own dimensions, is a central and fundamental characteristic determining the content of one group of properties—the gremial curvatures of its geodesics. But that integer provides no clue, other than an obvious lower limit, to the dimensionality of the plenary space: and therefore it provides no clue to the range of the remaining

(non-gremial) curvatures of the geodesics, to mention only one set of important magnitudes. Thus a curve is one-dimensional: it may exist in a plane: or its march may wind through some ampler range of space. We are accustomed to think of surfaces existing freely in what conventional experience deems to be uncurved triple space: yet they can exist equally freely in an ideal plenary space of more than three dimensions, even in a plenary space of illimitable extension. Still, when a curved amplitude is postulated in one or other of the customary modes, of which parametric representation is a frequent instance, its plenary space can be compounded from the aggregate of all the homaloids of successive orders of contact with the geodesics of the amplitude.

Moreover, a homaloidal space of any number of dimensions can always be merged into homaloidal spaces of any greater number of dimensions; and the inclusion can be effected in an unrestricted variety of ways. So far as the main investigations are concerned, it usually is convenient to assign to each amplitude its own individual plenary space, prescribed to be the least extensive homaloidal range that can contain the amplitude. Thus, to recur to the cited example of a plane curve, all essential properties can be derived without passing outside the plane of its existence: when that curve is postulated in a triple space, necessarily containing the plane, an equally complete discussion of the properties demands more elaborate analysis without ultimately furnishing additional knowledge of the geometry. But the requirements may be different if the same plane curve is postulated as existing on a surface in the triple space: its relations to the surface require other calculations. For instance, the flexure of a small circle on a spherical surface is the sole curvature of the circle relative to the sphere: yet that flexure cannot be inferred solely from the properties of the circle in its own plane.

Sometimes the dimensionality of the plenary space has no overt significance in an investigation. Thus the tangent flat of a region contains the tangent, the binormal, and the trinormal of every regional geodesic through the point of contact, irrespective of the range of the plenary space of the region. On the other hand, the nature of the

curve, which is the locus of the centres of circular curvature of all geodesics of a surface concurrent in the point of contact with a tangent plane, may be dominated by the range of the plenary space : when that space is triple, the locus is restricted to a line or parts of a line : when the space is quadruple, the locus is a plane lemniscate : when the dimension-integer of the space is a number greater than four, the locus is a skew quartic in a flat whatever the number.

Further, when a configuration is propounded metrically by means of a parametric differential expression for an element of its arc, the plenary space is not in evidence in any such incomplete definition : the expression itself contains no indication of the integer that measures the dimensionality of the space. Even in the simple case of a surface with an arbitrarily assigned arc-element, it has not yet proved practicable to devise the tests which settle the range of the plenary space. Some writers have definitely assumed an upper limit to the dimensionality of the plenary space of a manifold, having obtained it by counting the number of coefficients in the parametric expression connected with the arc-element of the manifold. The assumptions, almost deemed obvious, rest on no established proof and have remained untested and unchecked. Against their validity must be set the result that, if a propounded arc-element in two parameters is to characterise a surface within a triple homaloidal space, partial differential equations of the fifth order subsist between the coefficients in the expression of the arc-element.

In a converse aspect, any homaloidal range of ideal space more extensive than a straight line contains curved configurations, each necessarily of lower dimension-integer than the range itself. One definite restriction, directly imposed on such included configurations, is the limitation on the number of successive ranks of curvature which can appertain to any curve, organic or otherwise, in the configuration : even in the most general type, the number of such curvatures is one less than the dimensionality-integer of the plenary space. Consequently the geometry of such configurations is free from some of the difficulty that besets the intrinsic geometry of configurations in a plenary space of unknown dimensionality. (The analytical manipula-

tions, necessary in the various stages, are not reckoned as part of the difficulty, however laborious they may be; direct manipulation is always presumed to be practicable.) There is a similar consequence of specific knowledge of the range of the plenary space: it provides a corresponding limitation to the number of grades of differential relations which have to be constructed before the tale of independent geometrical magnitudes can be regarded as fully constituted.

In this respect there are two, and only two, types of configuration of which the intrinsic geometry can claim to have a semblance of completeness. One of the two types is provided by curves which exist in a plenary space of specifically assigned dimensionality: the generalised Frenet equations provide the means of obtaining the requisite number of successive curvatures together with associated properties, and they exact only direct calculations. The other of the two types is provided by the species of amplitudes styled primary: that is, amplitudes existing in a plenary space which is of only one dimension more extensive than themselves, such as Gauss surfaces in a triple space and regions in a quadruple space: in all such amplitudes, all the curvatures of a geodesic except the circular curvature are gremial. It is, of course, possible to proceed tentatively towards a complete geometrical system for amplitudes not included in either of these types. Progress, step by step, can always be achieved onwards from the beginning: but at present, the analysis appears to defy the kind of schematic completion attained in the differential geometry of curves. Thus the geometry of a primary domain is more easily developed than is the geometry of a surface existing freely in the quintuple space of the domain.

My book *Geometry of Four Dimensions* (it also included the geometry of curves in a general plenary space) was an adventure into the more amenable of these two realms of investigation. The present book, dealing with the geometry of configurations without a preliminary specification of the range of their plenary spaces, is an adventure into the less amenable of the realms. During the period of three-quarters of a century and more which has elapsed since the appearance of Riemann's thesis on the foundations of geometry and the publication of Christoffel's memoir on the transformation of homogeneous

quadratic differential expressions, much noteworthy progress in abstract geometry of multiple space has been achieved. Also for a period in more recent days, there was a vogue for one aspect of the subject, mainly due to the analytical requirements of theories of relativity: the mathematical physicists, in the analytical expression of their theories, had utilised some of the algebraic results of the abstract geometry in order to serve their immediate needs. To the earlier generations is due the achievement of having brought abstract geometry within the regular discipline of mathematics: to their successors is due the wealth of result that has already been attained. I have not attempted to produce a consecutive account of such researches. My aim, rather, has been to contribute to a systematic development of the subject, in which it is treated independently of its beginnings and of incidental applications, and can be duly recognised as a province in the domain of reasoned thought.

II.

The arrangement of the contents has been influenced, even directed in the main, by the actual course of its growth under the experience of earlier investigations in four-dimensional geometry. Dealing initially with the properties of a general amplitude and imposing no restriction on the dimensionality of its plenary space, I soon was convinced that some advantage, in simplicity and in clearness at that stage, would accrue from the detailed discussion of amplitudes of least extension such as surfaces, regions, domains, of two, three, four, dimensions respectively. The initial discussion of the general amplitude served a useful purpose in providing foundations; the discovery of characteristics, and of their discriminating effects, could be attained best from the simplest species of amplitudes as regards their dimensionality. The conviction was confirmed by the progress of the analysis which promptly compelled a separation of the curvatures of geodesics into a gremial group and a non-gremial remainder. Yet, even so, the results, which have been obtained for regions and for domains, are of reciprocal assistance towards the results for a general amplitude.

There may be a quite reasonable request for some explanation why, in dealing with amplitudes of specific dimensionality of successive ranges, a halt has been called after domains. Some reasons are too obvious to require formulation. One dominating reason has its root in the theory of invariantive concomitants of all classes constituting a complete system for a geometrical amplitude. The calculations connected with a (four-fold) domain provide adequate illustration and equally adequate indication of the categorical demands of that theory in its application to a general amplitude. For they introduce variables which are not subject to relations: distinct representatives are found in the parametric direction-variables of a line and the parametric orientation-variables of a region, and these two classes are contragredient to one another for domainal transformations. They also introduce variables which are of the type subject to identical relations: a representative class is provided by the orientation-variables of a domainal surface, and these variables are self-reciprocal for domainal transformations under which covariantive magnitudes persist. All the classes of variables which, after the conception first stated by Clebsch, can belong to the covariantive algebra of general amplitudes, are represented—though, of course, incompletely enumerated—by the types which occur in the discussion.

At one stage in the course of the work, there had been a hope that the complete system of concomitants of a representative general amplitude would have been constructed, partly for its own sake, partly for the light it can throw upon the relative independence of the geometrical measures. Towards the realisation of the hope, my intention had been to use Lie's theory of continuous groups. That theory is an analytical weapon of compelling power in such a field of research, though latterly it would seem to have fallen upon the evil days of neglect and even oblivion. Given reasonable manipulative skill, it provides systematic information concerning a normal tale of independent concomitants of each grade and thereby indicates the dependence, upon this aggregate, of other concomitants which may be obtained by synthesis, or geometrical inference, or organic derivation. It was with such an aim in mind (and a memoir of my own, published thirteen years ago, contained a partial treatment so far as concerns four-fold amplitudes) that,

from place to place throughout this book, there appears an insistence upon the invariantive character of many analytical measures which were to have served as special examples illustrative of the general theory. But time and circumstance have intervened to foil the ambition. And, moreover, such a range of the subject is so far-flung in its comprehensiveness that practically it would require a separate volume for adequate treatment.

III.

The analytical processes which have been adopted throughout are the amplification (though with one important modification) of those that had already been used in my *Geometry of Four Dimensions*. In origin, they are due to Gauss ; and I have found them simpler and more direct than is the vector analysis employed in many expositions of Riemannian geometry. The indicated modification of the processes is one that makes for a desirable brevity, subject to clearness. In all amplitudes, particularly in a general amplitude, the exclusion of considerations, of site and orientation of the amplitude in its plenary space, leaves all the space-coordinates on an equal standing ; and thus it has been found both feasible and convenient to proceed from a single abstract space-coordinate as typical of all such coordinates, and to deal with a single equation associated with this coordinate as typical of all the equations of the same kind. Also, I have abstained from the summation-convention by which the usual summation-symbol is omitted (leaving the process to be inferred from observation) when, in the representative term, there is a double occurrence of an indicial number. The reason for my practice is that, in the analysis, there are varied kinds of summation : with regard to space-coordinates : with regard to sets of parametric variables of their several kinds : with regard to terms in an expression, whether finite or not finite in number : with regard to groups of magnitudes of the same organic character. These are so frequent that, to me, there is a questionable economy as regards clearness, in shortening one type of expression by a single symbol (which, omitted, is supplied by a checking verification) while, at the same time, retaining the conventional symbol for all the other summations where it cannot be made conspicuous by its absence.

IV.

In every section, a substantial part of the analysis is devoted to the properties of geodesics, whether these belong to a central amplitude or to an included sub-amplitude. These properties all depend upon the curvatures of successive rank, as they occur in the (generalised) Frenet equations of a curve in the first instance; and these equations are simply modified so as to be more immediately applicable to a geodesic of the amplitude. As already remarked, these curvatures can be divided into a gremial group and the non-gremial remainder. For each curvature in the gremial group, the associated principal line of the geodesic lies within the tangent homaloid of the whole configuration to which the geodesic belongs; and the appropriate analysis for the gremial group is thus different in essential detail from the analysis required for the non-gremial group. When a geodesic belongs to a sub-amplitude (and, therefore, usually not to the central amplitude), its orthogonal frame is affected in relation to that central amplitude: the initial measure of its deviation from the amplitudinal geodesic is termed a flexure, and there are corresponding amplitudinal measures of later rank.

The general results constitute geodesics as convenient curves of reference within the amplitude of their existence, as significant as are straight lines in the Euclidean geometry of an earlier day. Subsequent investigations lead to an extension whereby, even as a preliminary consideration, geodesic surfaces and geodesic regions (to mention only a couple of such ancillary configurations) become convenient surfaces and regions of reference for the respective equi-dimensional configurations that do not possess the geodesic quality.

Two themes, in particular, have received much detailed attention. One of them is the Riemann measure of curvature: the other is the parallelism of geodesics.

A brief record of the origins of the Riemann measure will be found in the text. Its earliest occurrence is a statement in some manuscript fragments, left by Riemann and posthumously edited by Dedekind and

Weber. It consists of an analytical expression which is declared to be a measure of the curvature of a surface in a general amplitude: the surface in question now usually is termed a geodesic surface. The statement was not accompanied by any indication of the significance of the measure, or of the kind of curvature thus estimated, or even of the construction of the expression itself. The measure, alike in form and by its geometrical dimensions, is a measure of superficial area; the coefficients of the parametric combinations of variables are those combinations of derivatives of coefficients in the arc-element which are customarily denoted by the four-index symbols; and they have been simplified by using coefficients which occur in the central equation for the circular curvature of the amplitudinal geodesic. To obtain the interpretation, I have considered a small range of the amplitude in the vicinity of a point, instead of dealing with its descriptive measures at the point alone. A small triangle is framed, by drawing any two small geodesic arcs through the point and joining their extremities by another geodesic arc, necessarily small in a range free from singularities. Direct calculations, in approximations up to the third order of small quantities, lead to a determination of the length of the third side of the triangle; and the two angles of the geodesic triangle, other than the angle at the initial vertex, are calculated up to the second order of small quantities. In the limit, as the size of the triangle continues to diminish, the ratio of the angular excess of the triangle to its area is found to be the Riemann measure of curvature in the surface orientation of the triangle at its vertex. Because of the similarity of this result to the corresponding result for a spherical surface, the measure is called the sphericity of the amplitude in the specific orientation. (It may be added that, unlike the property for a spherical surface, the third side of the amplitudinal triangle does not lie in the geodesic surface at the vertex.) This interpretation is not a new result: as indicated in the proper place in the text, it appears to have been derived first by Levi-Civita from a result due to Pérès. The method of investigation is novel, as also are subsequent applications which are effected by the method.

To the parallelism of geodesics attention has been paid, partly because any formulation is a natural extension of the ancient theory,

partly because it bears promise of leading to developments in the quantitative geometry of amplitudes. Much of the earlier differential analysis leads to results which, though mainly only of a descriptive or qualitative type, are not numerical: they belong to any arbitrarily selected place in the amplitude, and they therefore possess a current significance, not merely a site-value. But their significance remains qualitative at any place, changing from place to place, yet not extending quantitatively over any range in the vicinity of a place.

Demands emerge for the discussion of ranges in an amplitude. Such discussion had been carried out for homaloids, also for amplitudes of constant sphericity of various types: in all cases for ranges of finite extent. For amplitudes of variable sphericity, consideration at present can be effected only for small ranges in the vicinity of any arbitrarily selected position: even so, the investigations belong to the beginnings of quantitative analysis. These have already appeared in the discussion of the Riemann measure, through the introduction of a small geodesic triangle in a manifold. The analytical processes, which there led to the inferred results, are effective also in the discussion of the properties and the measures connected with geodesic parallels.

The theory of amplitudinal parallelism was initiated by Levi-Civita. His original definition of a set of parallel geodesics in an amplitude associated them with directions that could be regarded as a set of unvarying lines for the amplitude, the association consisting of a conservation of inclinations between the successive geodesics and the selected unvarying lines. On a surface, this definition implies the simple property that all the geodesics, drawn through successive points on a basic geodesic, have the same inclination to that basic geodesic. There followed, immediately, a modification to Severi (the necessary references are given in the text). Under his definition, the amplitudinal geodesics through successive points on a basic geodesic, which are to be parallel to an assigned geodesic, are selected by means of the geodesic surface determined by the basic geodesic and the assigned geodesic; their directions at the basic points are tangential to this surface and make with the base the same angle as is made by the assigned geodesic. The Severi definition is not concerned with the

plenary space of the amplitude. The Levi-Civita parallel and the Severi parallel are distinct from one another in any amplitude which, being of more than two dimensions, is not of constant sphericity; and the angular deviation between them is of the second order of small quantities, the geodesic distance between the assigned geodesic and either parallel being considered a small quantity of the first order.

When it becomes necessary to construct a small geodesic parallelogram in any amplitude more extensive than a surface—it will suffice, for this statement, to specify a region—there is an immediate difficulty. When, through the second and third angular points of a geodesic triangle, regional geodesics are drawn parallel to the opposite sides, either by the Levi-Civita process or by the Severi process, they do not meet. The same difficulty occurs with some other types of parallels, drawn after alternative definitions of parallelism. Accordingly, a different method has been devised which, in the first place, leads to the construction of geodesic quadrilaterals by means of the foregoing triangle: the primary conditions of parallelism, common to the Levi-Civita parallel and the Severi parallel, are maintained; and a quadrilateral ensues from a requirement that the new geodesics intersect. These limitations still admit other requirements: and there is imposed the condition—a condition which can be satisfied—that the opposite sides of the geodesic quadrilateral shall be equal in length, one of the usual properties of a plane parallelogram.

Moreover, with this geodesic parallelogram as a boundary constituent, it is possible similarly to construct a (three-dimensional) cell in the region as a geodesic parallelepiped; and, without further conventions or added conditions, the same process is applied to the construction of a four-dimensional paralleloid in a domain, as indicative of its feasibility for a general amplitude. A path to quantitative mensuration in curved manifolds is thus opened; its further utility can be determined by the necessary approximations of an order higher than those which are required in the indicated investigations.

In bidding farewell to the subject, I would express my conviction that what has been achieved is little more than a beginning: the

results obtained are sufficient to indicate that ampler results remain to be discovered. Undoubtedly the manipulative labour occasionally is exacting, either by the methods used here or by the methods used elsewhere. But, as may be declared from my experience, such labour becomes comparatively simple and even increasingly attractive by growing acquaintance; and it can lead to the domains of new knowledge which will be the reward of fresh pioneers.

To the publishers, Macmillan & Co., my thanks are due for having undertaken the publication of a work so technical and so abstract. To them my thanks are tendered in full measure.

From the staff of the printers, Robert MacLehose & Co., I have received steady and unfailing assistance, as well as most courteous consideration, at every stage in their share of the book. To them also I tender my cordial thanks.

Finally, throughout the revision of the proof-sheets, I have had the invaluable help of my friend Professor E. H. Neville, of the University of Reading. To his devotion and watchful care, to his suggestions and criticisms and corrections, I am deeply indebted; and I would beg him to believe that his co-operation is treasured in my remembrance.

A. R. F.

LONDON, *April*, 1935.

CONTENTS

SECTION I

CHAPTER I

INTRODUCTORY

CHAPTER II

GENERAL AMPLITUDES

CHAPTER III

CURVATURES OF GEODESICS IN FREE GENERAL AMPLITUDES

CHAPTER IV

GEODESIC PROPERTIES RELATIVE TO THE TANGENT HOMALOID

CHAPTER V

RIEMANN'S MEASURE OF CURVATURE OF AN AMPLITUDE

CHAPTER VI

SUB-AMPLITUDES IN A GENERAL AMPLITUDE

CHAPTER VII

PRIMARY GENERAL AMPLITUDES

SECTION II

CHAPTER VIII

FREE SURFACES: INTRODUCTORY

CHAPTER IX

RANGES ON SURFACES: RIEMANN MEASURE OF CURVATURE

CHAPTER X

PARALLEL GEODESICS ON FREE SURFACES

CHAPTER XI

CURVATURES OF GEODESICS ON SURFACES

CHAPTER XII

CURVES OF CURVATURE ON SURFACES

CHAPTER XIII

PARAMETRIC CURVES ON A SURFACE

SECTION III

CHAPTER XIV

FREE REGIONS: PRELIMINARY

CHAPTER XV

TORSION, TILT, COIL, OF REGIONAL GEODESICS

SECTION I: AMPLITUDES

CHAPTER I

INTRODUCTORY

1. Results obtained in geometrical science during development from its initial creation, and progress achieved in many branches of pure and applied analysis, have compelled amplifications of the whole system of geometry. Considerations have entered which relate to an abstract space and are no longer limited to the older notions of space suggested by experience. This abstract space is purely ideal: its dimensional extent is unrestricted by extrinsic conditions. To some minds, the conceptions of multiple space are almost intuitive: to some, they constitute the obvious generalisation of the geometry that is familiar to experience: to some, they present no more than a fancied interpretation of algebraic relations. For all, they are cast into the mould of analytic forms, whatever be the vocabulary adopted in descriptive expression.

In this amplified geometry we have to deal with configurations of diverse types and of different ranges, all existing in some abstract space. One special aggregate of configurations is characterised by the property that, everywhere and everywhither in their range, straight lines can be drawn which are completely included in that range. This aggregate is the simplest of all, alike in geometrical quality and in analytical representation: in geometry, because all curvatures of geodesics are zero: in analysis, because the fundamental equations concerned with intrinsic relations are purely algebraical and mainly linear. The configurations, contained in this aggregate, are called homaloidal *: all others are said to be curved or have no specific title.

Types of configurations are frequently classed according to the rank of their dimensional extent. In particular, configurations of one dimension are called *lines* when they are homaloidal: if not homaloidal, they are called *curves*. Those of two dimensions are called *planes* or *surfaces*, according as they are homaloidal or not homaloidal; those of three dimensions are called *flats* or *regions*, in the like respective circumstances; and those of four dimensions are called *blocks* or *domains* in like manner. For configurations of more than four dimensions, the detailed discussion of properties has not been so specific for any one of them as to render the use of an individual title desirable; usually, the generic title of *amplitude* or

* Sometimes, the terms *flat* and *even* are used. In this treatise, the word *flat* is reserved for a three-dimensional homaloid.

manifold, with an added epithet of homaloidal when needed, proves sufficient. Sometimes a discrimination is effected by direct statement of the number of dimensions, such as is represented by an amplitude of m dimensions or a k-fold homaloid, with the implication that m and k are positive integers.

Any m-fold amplitude, m being greater than unity, may contain sub-amplitudes, the number of dimensions in each sub-amplitude being less than m. It is possible to have homaloidal sub-amplitudes within a non-homaloidal amplitude: thus there are ruled surfaces and planar regions. Usually, however, a sub-amplitude, contained in a curved amplitude, is also curved; and we have to examine the kinds of curvature, and to construct the estimate in each kind of curvature, whether the amplitude be plenary or be subsidiary. In a sub-amplitude, there are flexures relative to the plenary amplitude which do not appertain to the plenary amplitude itself; and there are curvatures of a plenary amplitude which also affect the sub-amplitude. For all of these, it is desirable to have appropriate spaces of reference, by comparison with which we can describe and can measure all such curvatures. The most convenient, among such types of spaces of reference, appear to be those which are homaloidal. As any homaloidal space can be completely merged in other homaloidal spaces of higher dimensionality, we select, as the space of reference for any configuration in an amplitude, the homaloidal space of lowest dimensionality containing the configuration and the amplitude; and the homaloidal space thus selected is termed the *plenary* space of the amplitude.

Yet it will be found that, for many of the intrinsic properties of a configuration as well as for many of the intrinsic measures of an amplitude, the actual dimensionality of the plenary space will have entirely disappeared without leaving any trace or sign. In this connection, it is sufficient to cite the branch of modern analysis commonly known as the Absolute Differential Calculus. But such a result does not hold for all properties and magnitudes, intrinsic to an amplitude: thus, in order to have a complete specification of so simple an intrinsic quantity as the circular curvature of a geodesic of an amplitude, we must have recourse to the plenary space of the amplitude, though its characteristic elements need not survive in evidence. The dimensions of a plenary space manifestly must be greater than those of every amplitude which it contains; but, in the absence of discriminating tests, no upper limit has yet been assigned *à priori* to the number of those dimensions.* Even so, it frequently occurs (especially with invariantive concomitants appertaining to the amplitude) that the dimensionality of the plenary space has disappeared from the constructed measure as completely as it has disappeared from recognition or detection in the simple expression for the arc-element of the amplitude itself.

Before initiating the discussion of amplitudes, whether of general or of specific

* A simple instance (§ 88) is given where a surface is devised, in so far as it can be represented by its arc-element, existing freely in a plenary space of an unlimited number of dimensions: no test is known for reducing that number.

order, some preliminary propositions of regular use will be stated. These propositions are comprised in two distinct sets. One of the sets relates to variables of different kinds and particularly to the inclinations of homaloids to one another: they will be stated briefly, as the appropriate developments will be effected in the application of the propositions. The other set contains a summary of some characteristic properties, connected with the curvature of skew curves in multiple plenary space of any number of dimensions; such properties must, of course, subsequently be adapted to curves (and especially to geodesics) which lie in configurations existing in that plenary space. For these preliminary explanations, it is tacitly assumed that all configurations are referred to sets of completely orthogonal axes in the space. An origin of general reference is of no significance. The axes themselves are only of incidental occurrence; and ultimately they cease to have any surviving importance, mainly because the measures obtained for intrinsic magnitudes are of the invariant type, independent of origin and independent also of axes of reference in the plenary space.

Types of variables.

2. For the expression of the concomitants, which appertain to the diverse configurations that are possible in multiple space, different types of variables are required. Initially, these are associated with the magnitudes and the relations of closed areas bounded by parallel edges in a plane, of closed parallelepipeds bounded by parallel faces in a flat, and so on; the number of edges, faces, and the like, for the successive figures, is the smallest possible consistent with the closing of the figure in each instance.

We begin with the expressions for the content of the figures. For the conterminous edges, we use $l_1, m_1, n_1, \ldots$ to denote the direction-cosines of a side of length r_1; then $l_2, m_2, n_2, \ldots$ to denote the direction-cosines of a second side of length r_2, the direction being distinct from the first; then $l_3, m_3, n_3, \ldots$ to denote the direction-cosines of a third side of length r_3, the direction not lying in the plane determined by the first two sides; and so on. We denote by c_{ij} the cosine of the angle between the conterminous sides r_i and r_j, so that

$$c_{ij} = l_i l_j + m_i m_j + n_i n_j + \ldots = \sum_l l_i l_j,$$

where the summation-sign $\sum_l$ extends over the products of homologous direction-cosines. Then the content of a parallelogram, bounded by r_1 and r_2 as conterminous edges, is

$$r_1 r_2 \begin{vmatrix} 1, & c_{12} \\ c_{21}, & 1 \end{vmatrix}^{\frac{1}{2}}.$$

To obtain the content of a parallelepiped, bounded by r_1, r_2, r_3 as conterminous

edges, we drop a perpendicular P, from a point distant r_3 from the corner along the third direction, upon the r_1r_2 parallelogram in the plane

$$\| \bar{y},\ l_1,\ l_2 \| = 0.$$

We therefore must make P^2, $=\sum(r_3l_3 - al_1 - \beta l_2)^2$, a minimum for all possible values of a and β; hence

$$\sum l_1(r_3l_3 - al_1 - \beta l_2) = 0, \quad \sum l_2(r_3l_3 - al_1 - \beta l_2) = 0,$$

that is,

$$r_3c_{13} - a - \beta c_{12} = 0, \qquad r_3c_{23} - ac_{12} - \beta = 0.$$

Let l_0, m_0, n_0, ... denote the direction of this perpendicular.

We have

$$l_0P = r_3l_3 - al_1 - \beta l_2;$$

consequently

$$l_0P \begin{vmatrix} 1, & c_{12} \\ c_{21}, & 1 \end{vmatrix} = \begin{vmatrix} l_3, & l_1, & l_2 \\ c_{13}, & 1, & c_{12} \\ c_{23}, & c_{12}, & 1 \end{vmatrix} r_3,$$

and so for the other direction-cosines m_0, n_0, ... , which now can be regarded as known. Further, the initial critical equations give

$$\sum l_1l_0 = 0, \quad \sum l_2l_0 = 0;$$

and therefore

$$P = r_3 \sum l_0l_3,$$

and

$$\left(\sum l_0l_3\right) \begin{vmatrix} 1, & c_{12} \\ c_{21}, & 1 \end{vmatrix} P = \begin{vmatrix} 1, & c_{13}, & c_{23} \\ c_{13}, & 1, & c_{12} \\ c_{23}, & c_{12}, & 1 \end{vmatrix} r_3.$$

But also

$$P \begin{vmatrix} 1, & c_{12} \\ c_{21}, & 1 \end{vmatrix} = \begin{vmatrix} \sum l_0l_3, & \sum l_0l_1, & \sum l_0l_2 \\ c_{13}, & 1, & c_{12} \\ c_{23}, & c_{12}, & 1 \end{vmatrix} r_3 = \left(\sum l_0l_3\right) \begin{vmatrix} 1, & c_{12} \\ c_{12}, & 1 \end{vmatrix} r_3;$$

and therefore

$$P^2 \begin{vmatrix} 1, & c_{12} \\ c_{21}, & 1 \end{vmatrix} = \begin{vmatrix} 1, & c_{13}, & c_{23} \\ c_{13}, & 1, & c_{12} \\ c_{23}, & c_{12}, & 1 \end{vmatrix} r_3^2 = \begin{vmatrix} 1, & c_{12}, & c_{13} \\ c_{21}, & 1, & c_{23} \\ c_{31}, & c_{32}, & 1 \end{vmatrix} r_3^2.$$

The content of the parallelepiped is

$$= P \,.\, r_1r_2 \begin{vmatrix} 1, & c_{12} \\ c_{21}, & 1 \end{vmatrix}^{\frac{1}{2}} = \begin{vmatrix} 1, & c_{12}, & c_{13} \\ c_{21}, & 1, & c_{23} \\ c_{31}, & c_{32}, & 1 \end{vmatrix}^{\frac{1}{2}} r_1r_2r_3.$$

Similarly for corresponding figures with parallel edges in spaces of four, and of more than four, dimensions.

The parametric variables, of various dimensions, in an amplitude characterised by means of parametric representation (§ 10), are compounded of sets of direction-variables, corresponding respectively to the directions of the different sets of sides in the preceding figures. For any one direction represented by $dx_1, \ldots, dx_n$, we have

$$ds^2 = \sum_i \sum_j A_{ij}\, dx_i\, dx_j.$$

We shall have minors of various orders of the determinant

$$\begin{vmatrix} A_{11}, & A_{12}, & A_{13}, & \ldots, & A_{1n} \\ A_{21}, & A_{22}, & A_{23}, & \ldots, & A_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n1}, & A_{n2}, & A_{n3}, & \ldots, & A_{nn} \end{vmatrix};$$

we therefore write

$$\begin{vmatrix} A_{ik}, & A_{jk} \\ A_{il}, & A_{jl} \end{vmatrix} = \begin{Bmatrix} i & j \\ k & l \end{Bmatrix},$$

$$\begin{vmatrix} A_{a\alpha}, & A_{b\alpha}, & A_{c\alpha} \\ A_{a\beta}, & A_{b\beta}, & A_{c\beta} \\ A_{a\gamma}, & A_{b\gamma}, & A_{c\gamma} \end{vmatrix} = \begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma \end{Bmatrix},$$

and so on, corresponding to the columns and the rows from which the constituents are selected.

We shall have sets of direction-variables, corresponding to the successive edges of the figures. At any point, we take infinitesimal parts of such edges $ds_1, ds_2, ds_3, \ldots$ in magnitude : the corresponding direction-variables are

$$\begin{matrix} dx_1^{(1)}, & dx_2^{(1)}, & dx_3^{(1)}, & \ldots, \\ dx_1^{(2)}, & dx_2^{(2)}, & dx_3^{(2)}, & \ldots, \\ dx_1^{(3)}, & dx_2^{(3)}, & dx_3^{(3)}, & \ldots, \end{matrix}$$

and so on. The content of a two-dimensional figure, thus bounded, is

$$= ds_1\, ds_2 \begin{vmatrix} 1, & c_{12} \\ c_{12}, & 1 \end{vmatrix}^{\frac{1}{2}}$$

$$= \begin{vmatrix} \sum A_{ij}\, dx_i^{(1)}\, dx_j^{(1)}, & \sum A_{ij}\, dx_i^{(2)}\, dx_j^{(1)} \\ \sum A_{ij}\, dx_i^{(1)}\, dx_j^{(2)}, & \sum A_{ij}\, dx_i^{(2)}\, dx_j^{(2)} \end{vmatrix}^{\frac{1}{2}} = P_{12},$$

where

$$P_{12}{}^2 = \sum \begin{Bmatrix} i & j \\ k & l \end{Bmatrix} \begin{vmatrix} dx_i^{(1)}, & dx_j^{(1)} \\ dx_i^{(2)}, & dx_j^{(2)} \end{vmatrix} \begin{vmatrix} dx_k^{(1)}, & dx_l^{(1)} \\ dx_k^{(2)}, & dx_l^{(2)} \end{vmatrix}.$$

These bilinear combinations of the direction-variables dx are *surface-variables* (or *orientation-variables* of a surface) : we can write

$$t_{ij} = \begin{vmatrix} dx_i^{(1)}, & dx_j^{(1)} \\ dx_i^{(2)}, & dx_j^{(2)} \end{vmatrix},$$

for all the values $i, j=1, 2, \ldots, n$, independently of one another; and then, if dS denote the element of area thus defined,

$$dS^2=\sum\begin{Bmatrix} i & j \\ k & l\end{Bmatrix} t_{ij}t_{kl}.$$

Thus, for a region in which the elementary arc is given by

$$ds^2=(A,\ B,\ C,\ F,\ G,\ H\between dp,\ dq,\ dr)^2,$$

an element of area is

$$dS^2=(a,\ b,\ c,\ f,\ g,\ h\between \begin{vmatrix} dq^{(1)}, & dr^{(1)} \\ dq^{(2)}, & dr^{(2)}\end{vmatrix},\ \begin{vmatrix} dr^{(1)}, & dp^{(1)} \\ dr^{(2)}, & dp^{(2)}\end{vmatrix},\ \begin{vmatrix} dp^{(1)}, & dq^{(1)} \\ dp^{(2)}, & dq^{(2)}\end{vmatrix})^2:$$

or, if we take

$$\xi,\ \eta,\ \zeta=\frac{1}{dS}\begin{Vmatrix} dp^{(1)}, & dq^{(1)}, & dr^{(1)} \\ dp^{(2)}, & dq^{(2)}, & dr^{(2)}\end{Vmatrix},$$

as orientation-coordinates of an element of surface dS, there is a permanent surface-relation

$$(a,\ b,\ c,\ f,\ g,\ h\between \xi,\ \eta,\ \zeta)^2=1$$

for the region, corresponding to the permanent arc-relation

$$(A,\ B,\ C,\ F,\ G,\ H\between p',\ q',\ r')^2=1.$$

The content of a three-dimensional figure, similarly bounded, is

$$=ds_1\,ds_2\,ds_3\begin{vmatrix} 1, & c_{12}, & c_{13} \\ c_{21}, & 1, & c_{23} \\ c_{31}, & c_{32}, & 1\end{vmatrix}^{\frac{1}{2}}$$

$$=\begin{vmatrix} \sum A_{ij}\,dx_i^{(1)}\,dx_j^{(1)}, & \sum A_{ij}\,dx_i^{(2)}\,dx_j^{(1)}, & \sum A_{ij}\,dx_i^{(3)}\,dx_j^{(1)} \\ \sum A_{ij}\,dx_i^{(2)}\,dx_j^{(2)}, & \sum A_{ij}\,dx_i^{(2)}\,dx_j^{(2)}, & \sum A_{ij}\,dx_i^{(3)}\,dx_j^{(2)} \\ \sum A_{ij}\,dx_i^{(3)}\,dx_j^{(3)}, & \sum A_{ij}\,dx_i^{(3)}\,dx_j^{(3)}, & \sum A_{ij}\,dx_i^{(3)}\,dx_j^{(3)}\end{vmatrix}^{\frac{1}{2}}=P_{123},$$

where

$$P_{123}{}^2=\sum\begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma\end{Bmatrix}\begin{vmatrix} dx_a^{(1)}, & dx_b^{(1)}, & dx_c^{(1)} \\ dx_a^{(2)}, & dx_b^{(2)}, & dx_c^{(2)} \\ dx_a^{(3)}, & dx_b^{(3)}, & dx_c^{(3)}\end{vmatrix}\begin{vmatrix} dx_\alpha^{(1)}, & dx_\beta^{(1)}, & dx_\gamma^{(1)} \\ dx_\alpha^{(2)}, & dx_\beta^{(2)}, & dx_\gamma^{(2)} \\ dx_\alpha^{(3)}, & dx_\beta^{(3)}, & dx_\gamma^{(3)}\end{vmatrix}.$$

These trilinear combinations of the direction-variables dx are called *volume-variables* (or *orientation-variables* of orientation of a volume): we can write

$$v_{ijk}=\begin{vmatrix} dx_i^{(1)}, & dx_j^{(1)}, & dx_k^{(1)} \\ dx_i^{(2)}, & dx_j^{(2)}, & dx_k^{(2)} \\ dx_i^{(3)}, & dx_j^{(3)}, & dx_k^{(3)}\end{vmatrix},$$

for all the combinations $i, j, k, =1, 2, \ldots, n$, independently of one another; and then, if dV represent the element of volume thus defined,

$$dV^2 = \sum \begin{Bmatrix} i & j & k \\ \alpha & \beta & \gamma \end{Bmatrix} v_{ijk} v_{\alpha\beta\gamma}.$$

Thus, for a domain in which the element of arc is

$$ds^2 = (A, B, C, D, F, G, H, L, M, N \between dp, dq, dr, dt)^2,$$

an element of volume is

$$dV^2 = (a, b, c, d, f, g, h, l, m, n \between dP, dQ, dR, dT)^2,$$

where

$$dP, dQ, dR, dT = \begin{Vmatrix} dp^{(1)}, & dq^{(1)}, & dr^{(1)}, & dt^{(1)} \\ dp^{(2)}, & dq^{(2)}, & dr^{(2)}, & dt^{(2)} \\ dp^{(3)}, & dq^{(3)}, & dr^{(3)}, & dt^{(3)} \end{Vmatrix},$$

with a corresponding permanent volume-relation

$$(a, b, c, d, f, g, h, l, m, n \between P_0, Q_0, R_0, T_0)^2 = 1,$$

where

$$(P_0, Q_0, R_0, T_0) dV = dP, dQ, dR, dT.$$

Similarly for the contents of like figures of more than three dimensions. Each increase in the number of sides leads to a new type of variable of extension: and, in every instance, the square of the content is a homogeneous quadratic function of the new variables.

Inclinations of homaloids: projections.

3. There are different grades of perpendicularity between two amplitudes, each of which is of dimensions greater than unity; they range from a comparatively limited rectangular property to complete orthogonality.

Consider two planes in any plenary homaloidal space of n dimensions, represented by the respective sets of equations

$$\Vert\, \bar{y} - a, l_1, l_2 \,\Vert = 0, \quad \Vert\, \bar{y} - b, l_3, l_4 \,\Vert = 0.$$

Any direction in the first plane is typified by the quantities

$$\alpha l_1 + \beta l_2,$$

where α and β are parametric; it is perpendicular to each of the two leading lines, and therefore to every line, in the second plane if

$$\alpha c_{13} + \beta c_{23} = 0, \quad \alpha c_{14} + \beta c_{24} = 0,$$

equations which are incompatible unless

$$c_{13} c_{24} - c_{14} c_{23} = 0.$$

Provided this condition is satisfied, it is possible to select a direction in the second plane typified by the quantities

$$\gamma l_3 + \delta l_4$$

which is perpendicular to each of the two leading lines, and therefore to every line, in the first plane; because all that is necessary is to choose the ratio $\gamma : \delta$ so that the relation

$$\gamma c_{13} + \delta c_{14} = 0$$

is satisfied and then, under the condition, the relation

$$\gamma c_{23} + \delta e_{24} = 0$$

is satisfied also, these being the necessary and sufficient relations.

In these circumstances, the two planes are said to be *perpendicular* to one another.

But, if there is to be a limitation to a specific line $\alpha l_1 + \beta l_2$ in the first plane and to a specific line $\gamma l_3 + \delta l_4$ in the second plane, the preliminary equations determining ratios $\alpha : \beta$ and $\gamma : \delta$ must not be evanescent. Should the equations for the ratio $\alpha : \beta$ be evanescent, then

$$c_{13} = 0, \ c_{23} = 0; \quad c_{14} = 0, \ c_{24} = 0;$$

in that event, the later equations for the ratio $\gamma : \delta$ also are evanescent. When the four relations are satisfied, every line in each plane is at right angles to every line in the other plane; and the two planes are then said to be *orthogonal* to one another.

There is an intermediate grade of perpendicularity in the case of two planes. It may be possible that the relation $\alpha c_{13} + \beta c_{23} = 0$ is evanescent, while the relation $\alpha c_{14} + \beta c_{24} = 0$ is not evanescent: that is, the relations $c_{13} = 0$, $c_{23} = 0$, are satisfied, shewing that every line in the first plane is at right angles to one line in the second plane but is not at right angles to every line in that second plane. We still regard the two planes as perpendicular, and not as orthogonal, to one another. The relation, which has been styled orthogonal, exists only when every line in each plane is at right angles to every line in the other plane.

Manifestly, the planes are orthogonal only if the four conditions

$$c_{13} = 0, \quad c_{23} = 0, \quad c_{14} = 0, \quad c_{24} = 0,$$

are satisfied; they are perpendicular if, though the four conditions are not simultaneously satisfied, or even though no one of them is satisfied, the single condition

$$c_{13} c_{24} - c_{14} c_{23} = 0$$

holds.

4. Corresponding considerations apply to two homaloids of different dimensions.

Thus a plane and a flat are represented by the respective sets of equations

$$\| \ \bar{y} - a, \ l_1, \ l_2 \ \| = 0, \quad \| \ \bar{y} - b, \ l_3, \ l_4, \ l_5 \ \| = 0.$$

Usually it is not possible to determine a line in the plane which is perpendicular

to the flat, that is, which is at right angles to every line in the flat; for the equations

$$ac_{13}+\beta c_{23}=0, \quad ac_{14}+\beta c_{24}=0, \quad ac_{15}+\beta c_{25}=0,$$

being the conditions that the planar line typified by the direction-variable $\alpha l_1+\beta l_2$ should be at right angles to the flat, cannot be simultaneously satisfied unless at least two relations are satisfied.

But it is always possible to determine one line in the flat which is at right angles to every line in the plane. Such a line has direction-cosines typified by $\alpha l_3+\beta l_4+\gamma l_5$, if the two equations

$$ac_{13}+\beta c_{14}+\gamma c_{15}=0, \quad ac_{23}+\beta c_{24}+\gamma c_{25}=0,$$

are satisfied; the two equations determine the necessary ratios $\alpha : \beta : \gamma$. When the two equations are effectively the same, so that relations

$$\frac{c_{23}}{c_{13}}=\frac{c_{24}}{c_{14}}=\frac{c_{25}}{c_{15}}$$

hold, the flat contains a plane

$$\left\| \bar{y}-b, \quad l_4-\frac{c_{14}}{c_{13}}l_3, \quad l_5-\frac{c_{15}}{c_{13}}l_3 \right\|=0$$

which is orthogonal to the given plane, while the plane contains one line

$$\left\| \bar{y}-a, \quad l_2-l_1\frac{c_{23}}{c_{13}} \right\|=0$$

which is at right angles to the flat. Accordingly, when the two relations hold, we say that the plane and the flat are *perpendicular* to one another.

If it should happen that both the equations in the ratios $\alpha : \beta : \gamma$ are evanescent, so that then the six conditions

$$c_{13}=0, \quad c_{23}=0, \quad c_{14}=0, \quad c_{24}=0, \quad c_{15}=0, \quad c_{25}=0,$$

hold, every line in the flat is at right angles to every line in the plane. In these circumstances, we declare that the plane and the flat are *orthogonal* to one another; and orthogonality between the plane and the flat is held to exist, only when the six conditions are satisfied.

Similarly for any pair of homaloids. Thus two flats

$$\| \bar{y}-a, \ l_1, \ l_2, \ l_3 \|=0, \quad \| \bar{y}-c, \ l_4, \ l_5, \ l_6 \|=0,$$

are said to be perpendicular to one another when the relation

$$\begin{vmatrix} c_{14}, & c_{15}, & c_{16} \\ c_{24}, & c_{25}, & c_{26} \\ c_{34}, & c_{35}, & c_{36} \end{vmatrix}=0$$

is satisfied; and they are said to be orthogonal when, and only when, each constituent of this determinant vanishes. And generally when two homaloids are such that every line in either is at right angles to every line in the other, then (and only then) the two homaloids are orthogonal to one another.

There are various stages of perpendicularity which yet are not complete orthogonality. Thus, when one of the two preceding flats contains a plane orthogonal to the other flat, the second flat contains a plane orthogonal to the first; and this property holds though the two flats may be not orthogonal to one another.

Ex. Shew that all homaloids, orthogonal to

$$\| \bar{y} - a, \; l_1, \; l_2, \dots, \; l_\mu \| = 0,$$

can be represented by the equations

$$\sum\{(\bar{y} - a')l_1\} = 0, \quad \sum\{(\bar{y} - a')l_2\} = 0, \dots, \quad \sum\{(\bar{y} - a')l_\mu\} = 0.$$

5. The inclination of two homaloids to one another is estimated by means of projections. Of the two homaloids, let one which has not the greater number of dimensions be projected upon the other, the projection being made by means of the leading lines of the projected homaloid. Lengths are taken along these leading lines through any point: and a figure is completed by means of parallel edges, these new edges being also conterminous at respective points. If V denote the content of this figure, and if V_0 denote the content of the figure made by the projected edges in the other homaloid, the inclination θ of the two homaloids is defined by the relation

$$V \cos \theta = V_0.$$

The inclination of two curved configurations at a common point is related to the foregoing definition, in the manner customary for the estimate of inclinations of curves and surfaces in homaloidal triple space. We take that tangent homaloid to each configuration which is of its own dimensions; the inclination of the configurations is taken to be the inclination of these tangent homaloids.

Returning to the consideration of the inclination of two homaloids, we must first obtain the projection, of an assigned portion of a line in one homaloid, upon the other homaloid. Let the line to be projected be

$$\frac{Y_1 - y_1}{m_1} = \frac{Y_2 - y_2}{m_2} = \dots = \frac{Y_n - y_n}{m_n} = D,$$

D denoting the portion to be projected; and let the homaloid, of r dimensions, upon which this portion is to be projected, be represented by the equations

$$\| \bar{y}_t - x_t, \; l_{t1}, \; l_{t2}, \dots, \; l_{tr} \| = 0,$$

so that any point in this r-fold homaloid is given by the relations involving r parameters

$$\bar{y}_t - x_t = a_1 l_{t1} + a_2 l_{t2} + \dots + a_r l_{tr},$$

for $t=1, \ldots, n$; and we shall assume that the point $\bar{y}_1, \ldots, \bar{y}_n$, thus represented, is the foot of the perpendicular from $Y_1, \ldots, Y_n$, upon the homaloid. In order that this assumption may accord with fact, the magnitude

$$\sum_t \{Y_t - (x_t + a_1 l_{t1} + a_2 l_{t2} + \ldots + a_r l_{tr})\}^2$$

must be a minimum for all possible values of the parameters a; hence

$$\sum_t l_{t\lambda}\{Y_t - (x_t + a_1 l_{t1} + a_2 l_{t2} + \ldots + a_r l_{tr})\} = 0,$$

for $\lambda = 1, \ldots, r$, the t-summations being for $t=1, \ldots, n$. Thus

$$\left.\begin{array}{r|l} \sum l_{t1}(Y_t - x_t) = & a_1 + a_2 c_{12} + a_3 c_{13} + \ldots + a_r c_{1r} \\ \sum l_{t2}(Y_t - x_t) = & a_1 c_{21} + a_2 + a_3 c_{23} + \ldots + a_r c_{2r} \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum l_{tr}(Y_t - x_t) = & a_1 c_{r1} + a_2 c_{r2} + a_3 c_{r3} + \ldots + a_r \end{array}\right| ,$$

equations sufficient to determine the parameters a for the foot of the perpendicular in question; while the coordinates of that foot are the quantities $\bar{y}_1, \ldots, \bar{y}_n$, with these determinate values of the parameters. We thus have

$$\begin{vmatrix} \bar{y}_\mu - x_\mu, & l_{\mu1}, & l_{\mu2}, & \ldots, & l_{\mu r} \\ \sum l_{t1}(Y_t - x_t), & 1, & c_{12}, & \ldots, & c_{1r} \\ \sum l_{t2}(Y_t - x_t), & c_{21}, & 1, & \ldots, & c_{2r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum l_{tr}(Y_t - x_t), & c_{r1}, & c_{r2}, & \ldots, & 1 \end{vmatrix} = 0,$$

for $\mu=1, \ldots, n$, as the expressions for the coordinates of the foot of the perpendicular from $Y_1, \ldots, Y_n$, upon the homaloid.

Let $(\bar{y}_\mu)$, for $\mu=1, \ldots, n$, denote the coordinates of the foot of the perpendicular upon the same homaloid from the point $y_1, \ldots, y_n$, the other extremity of the assigned portion of the line that is projected: then the foregoing equation holds when $\bar{y}_\mu$ is changed into $(\bar{y}_\mu)$ and Y_t is changed into y_t, for all values of $\mu, t, =1, \ldots, n$. Let D_0 denote the magnitude of the projection in the homaloid; and let $M_1, \ldots, M_n$, be its direction-cosines, so that

$$\bar{y}_\mu - (\bar{y}_\mu) = M_\mu D_0,$$

for all the values of μ. Also $Y_t - y_t = m_t D$, for all the values of t; and, therefore, subtracting the equations, we have

$$\begin{vmatrix} D_0 M_\mu, & l_{\mu1}, & l_{\mu2}, & \ldots, & l_{\mu r} \\ D\sum m_t l_{t1}, & 1, & c_{12}, & \ldots, & c_{1r} \\ D\sum m_t l_{t2}, & c_{21}, & 1, & \ldots, & c_{2r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ D\sum m_t l_{tr}, & c_{r1}, & c_{r2}, & \ldots, & 1 \end{vmatrix} = 0,$$

for $\mu=1, \dots, n$. Multiply this equation by m_μ and add for all the values of μ; then, if θ be the inclination of the line to its projection in the homaloid, so that

$$\cos\theta = \sum m_\mu M_\mu, \quad D_0 = D\cos\theta,$$

we have

$$\begin{vmatrix} \cos^2\theta, & \sum m_t l_{t1}, & \sum m_t l_{t2}, & \dots, & \sum m_t l_{tr} \\ \sum m_t l_{t1}, & 1, & c_{12}, & \dots, & c_{1r} \\ \sum m_t l_{t2}, & c_{21}, & 1, & \dots, & c_{2r} \\ \dots & \dots & \dots & \dots & \dots \\ \sum m_t l_{tr}, & c_{r1}, & c_{r2}, & \dots, & 1 \end{vmatrix} = 0.$$

But, according to definition, this angle θ is taken to be the inclination of the line to the homaloid: we therefore have an expression for the inclination of a line to a homaloid.

Ex. Verify that, if the line is at right angles to the flat

$$\begin{Vmatrix} \bar{y}_1 - x_1, & \bar{y}_2 - x_2, & \bar{y}_3 - x_3, & \dots \\ l_1, & m_1, & n_1, & \dots \\ l_2, & m_2, & n_2, & \dots \\ l_3, & m_3, & n_3, & \dots \end{Vmatrix},$$

in the sense that its projection vanishes, then it is at right angles to every line in the flat.

6. From this expression for the inclination of a line to a homaloid, we can deduce an expression for the inclination of two homaloids to one another. The process will be indicated adequately by finding the inclination between a plane and a flat, represented by the respective sets of equations

$$\| \bar{y} - a,\ l_1,\ l_2 \| = 0, \qquad \| \bar{y} - b,\ l_3,\ l_4,\ l_5 \| = 0.$$

We take a unit length along the guiding line in the direction $l_1, m_1, \dots$ in the plane; and we project it upon the flat. Denoting the direction-cosines of the projection by $\lambda_1, \mu_1, \dots$, and the length of the projection by $\cos\theta_1$ (so that θ_1 is the inclination of that guiding line to the flat), we have the two results

$$\begin{vmatrix} \lambda_1\cos\theta_1, & l_3, & l_4, & l_5 \\ c_{31}, & 1, & c_{34}, & c_{35} \\ c_{41}, & c_{43}, & 1, & c_{45} \\ c_{51}, & c_{53}, & c_{54}, & 1 \end{vmatrix} = 0, \qquad \begin{vmatrix} \cos^2\theta_1, & c_{13}, & c_{14}, & c_{15} \\ c_{31}, & 1, & c_{34}, & c_{35} \\ c_{41}, & c_{43}, & 1, & c_{45} \\ c_{51}, & c_{53}, & c_{54}, & 1 \end{vmatrix} = 0,$$

the first being typical of all the direction-cosines of the projection in the flat. We similarly take a unit length along the guiding line in the direction $l_2, m_2, \dots$, in

the plane; and we project it also upon the flat. Denoting the direction-cosines of the projection of this length by λ_2, μ_2, ... and the length of the projection by $\cos\theta_2$, we similarly have

$$\begin{vmatrix} \lambda_2\cos\theta_2, & l_3, & l_4, & l_5 \\ c_{32}, & 1, & c_{34}, & c_{35} \\ c_{42}, & c_{43}, & 1, & c_{45} \\ c_{52}, & c_{53}, & c_{54}, & 1 \end{vmatrix}=0, \qquad \begin{vmatrix} \cos^2\theta_2, & c_{23}, & c_{24}, & c_{25} \\ c_{32}, & 1, & c_{34}, & c_{35} \\ c_{42}, & c_{43}, & 1, & c_{45} \\ c_{52}, & c_{53}, & c_{54}, & 1 \end{vmatrix}=0,$$

the first of these likewise being typical of all the direction-cosines of this second projection in the flat.

We now write

$$\sum l_1 l_2=\cos\alpha, \quad \sum \lambda_1\lambda_2=\cos\beta.$$

The area of the parallelogram in the plane is $\sin\alpha$. The area, when projected, manifestly produces another parallelogram; and the area of the latter is

$$\cos\theta_1\cos\theta_2\sin\beta.$$

Let ϕ denote the inclination between the two parallelograms, one of which lies in the plane, and the other of which is its projection in the flat; as this inclination is determined by the ratio of the areas, we have

$$\cos\phi=\frac{\cos\theta_1\cos\theta_2\sin\beta}{\sin\alpha}.$$

This inclination ϕ is defined to be the inclination between the plane and the flat.

The expression of ϕ in terms solely of the parametric constants of the plane and the flat is obtained simply. By the properties of determinants, we have

$$\begin{vmatrix} \cos\theta_1\cos\theta_2\cos\beta, & c_{13}, & c_{14}, & c_{15} \\ c_{32}, & 1, & c_{34}, & c_{35} \\ c_{42}, & c_{43}, & 1, & c_{45} \\ c_{52}, & c_{53}, & c_{54}, & 1 \end{vmatrix}=0.$$

We denote the determinant

$$\begin{vmatrix} 1, & c_{34}, & c_{35} \\ c_{43}, & 1, & c_{45} \\ c_{53}, & c_{54}, & 1 \end{vmatrix}$$

by Δ, temporarily, and the minor of c_{ij} in Δ by δ_{ij}, for $i, j, =3, 4, 5$, with the obvious convention $c_{33}=1=c_{44}=c_{55}$. Then

$$\begin{aligned}\Delta\cos^2\theta_1&=(\delta_{33}, \delta_{44}, \delta_{55}, \delta_{45}, \delta_{53}, \delta_{34} \between c_{13}, c_{14}, c_{15})^2\\ &=(\delta \between c_{13}, c_{14}, c_{15})^2,\end{aligned}$$

with the usual notation ; also

$$\Delta \cos^2\theta_2 = (\delta \between c_{23}, c_{24}, c_{25})^2,$$

$$\Delta \cos\theta_1 \cos\theta_2 \cos\beta = (\delta \between c_{13}, c_{14}, c_{15} \between c_{23}, c_{24}, c_{25}).$$

Consequently

$$\Delta \cos^2\theta_1 \cos^2\theta_2 \sin^2\beta = (1, 1, 1, c_{45}, c_{53}, c_{34} \between \xi, \eta, \zeta)^2,$$

where

$$\xi, \ \eta, \ \zeta, = \left\| \begin{matrix} c_{13}, & c_{14}, & c_{15} \\ c_{23}, & c_{24}, & c_{25} \end{matrix} \right\|.$$

Also

$$\sin^2 a = \begin{vmatrix} 1, & c_{12} \\ c_{12}, & 1 \end{vmatrix};$$

and therefore we have

$$\cos^2\phi = \frac{(1, 1, 1, c_{45}, c_{53}, c_{34} \between \xi, \eta, \zeta)^2}{\begin{vmatrix} 1, & c_{12} \\ c_{12}, & 1 \end{vmatrix} \begin{vmatrix} 1, & c_{34}, & c_{35} \\ c_{43}, & 1, & c_{45} \\ c_{53}, & c_{54}, & 1 \end{vmatrix}},$$

a relation giving an expression for ϕ in the form required.

Ex. 1. Obtain an expression for the inclination ψ of the two planes

$$\| \bar{y} - a, \ l_1, \ l_2 \| = 0, \quad \| \bar{y} - b, \ l_3, \ l_4 \| = 0,$$

in the form

$$\begin{vmatrix} 1, & c_{12} \\ c_{12}, & 1 \end{vmatrix} \begin{vmatrix} 1, & c_{34} \\ c_{34}, & 1 \end{vmatrix} \cos^2\psi = \begin{vmatrix} c_{13}, & c_{14} \\ c_{23}, & c_{24} \end{vmatrix}^2;$$

and obtain also an expression for the inclination χ of the two flats

$$\| \bar{y} - a, \ l_1, \ l_2, \ l_3 \| = 0, \quad \| \bar{y} - b, \ l_4, \ l_5, \ l_6 \| = 0,$$

in the form

$$\begin{vmatrix} 1, & c_{12}, & c_{13} \\ c_{21}, & 1, & c_{23} \\ c_{31}, & c_{32}, & 1 \end{vmatrix} \begin{vmatrix} 1, & c_{45}, & c_{46} \\ c_{54}, & 1, & c_{56} \\ c_{64}, & c_{65}, & 1 \end{vmatrix} \cos^2\chi = \begin{vmatrix} c_{14}, & c_{24}, & c_{34} \\ c_{15}, & c_{25}, & c_{35} \\ c_{16}, & c_{26}, & c_{36} \end{vmatrix}^2.$$

Ex. 2. From the foregoing expression for the inclination of the plane and the flat, deduce the conditions of § 4 that the flat and the plane may be perpendicular to one another.

Curvatures of a curve in multiple space : the Frenet equations.

7. In the geometry of all configurations, geodesics have a significance as important as that of straight lines in the customary geometry of plane and solid figures ; and a knowledge of their essential properties as curves is a prime necessity, their relations as elements in a configuration being a subject of later investiga-

tion. To this end, we shall assume the results already established concerning the properties of skew curves in a plenary homaloidal space of n dimensions.*

For the analytical survey of such a curve, it suffices to consider the successive curvatures, constituted by the arc-rates of change in the inclinations of the various homaloids of contact in passage along the curve. For the full geometrical organic frame of a curve at any point, it is necessary also to take account of the successive orbicular curvatures, being those of the successive amplitudes of constant curvature which, each in its own rank, have the closest contact with the curve.

The curvatures, in the first of these categories, are given by a set of equations which are the extension, to multiple homaloidal space, of the Frenet-Serret equations for a curve in triple homaloidal space. They are as follows. We denote by

$$(l_r)_1,\ (l_r)_2,\ \dots\ ,\ (l_r)_n,$$

the spatial direction-cosines of the r-th principal line in the organic orthogonal frame of the curve; and we take l_r as typical of all the set of direction-cosines of this line. Thus when $r=1$, the line is the tangent of the curve; when $r=2$, the line is its prime normal (also called the radius of circular curvature); when $r=3$, the line is its binormal; when $r=4$, the line is its trinormal; and so on. The plane, through the two lines typified by l_1 and l_2, (containing also two consecutive tangents of the curve), is the osculating plane of the curve; the flat, through the three lines typified by l_1, l_2, l_3, (containing also two consecutive osculating planes of the curve), is its osculating flat; the block, through the four lines typified by l_1, l_2, l_3, l_4, (containing also two consecutive osculating flats of the curve), is its osculating block; and so on. The first of the curvatures under citation is the arc-rate of change of relative inclination of successive tangents of the curve; usually it is called the *circular curvature* and, in the tableau of the Frenet equations, it is denoted by $1/\rho_1$. The second of these curvatures is the arc-rate of change of relative inclination of successive osculating planes in passage along the curve: usually it is called the *torsion* and, in the tableau of the Frenet equations, it is denoted by $1/\rho_2$. The third of these curvatures is the arc-rate of change of relative inclination of successive osculating flats in passage along the curve; it is called the *tilt* and, in the Frenet tableau, it is denoted by $1/\rho_3$. The fourth of these curvatures is the arc-rate of change of relative inclination of successive osculating blocks in passage along the curve; it is called the *coil* and, in the Frenet tableau, it is denoted by $1/\rho_4$. These specific names are used chiefly in investigations concerned with the earliest curvatures in their ordered succession; and then it is often convenient to take $\rho_1,\ \rho_2,\ \rho_3,\ \rho_4,\ =\rho,\ \sigma,\ \tau,\ \kappa$, respectively. But, after

* They are set out in chapter xi of my *Geometry of Four Dimensions*, hereafter cited as *G.F.D.* The cited chapter deals specifically with curves in n-fold homaloidal space.

these earliest curvatures, the remainder continue to be discriminated solely by integer suffixes in the actual symbols of denotation.

The tableau of the generalised Frenet equations, which connect the typical direction-cosines, of the successive principal lines in the orthogonal organic frame of the curve, with the successive curvatures indicated, is

$$\left.\begin{aligned}
\frac{dl_1}{ds} &= \qquad\qquad\quad \frac{1}{\rho_1} l_2 \\
\frac{dl_2}{ds} &= -\frac{1}{\rho_1} l_1 \quad + \frac{1}{\rho_2} l_3 \\
\frac{dl_3}{ds} &= -\frac{1}{\rho_2} l_2 \quad + \frac{1}{\rho_3} l_4 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{dl_{n-1}}{ds} &= -\frac{1}{\rho_{n-2}} l_{n-2} + \frac{1}{\rho_{n-1}} l_n \\
\frac{dl_n}{ds} &= -\frac{1}{\rho_{n-1}} l_{n-1}
\end{aligned}\right\}.$$

The constituents, in the complete orthogonal array of direction-cosines

$$| (l_i)_j |,$$

satisfy the relations

$$\sum_{m=1}^{n} \{(l_i)_m\}^2 = 1, \quad \sum_{m=1}^{n} \{(l_i)_m (l_j)_m\} = 0,$$

for all values of i and j in the integer set $1, \dots, n$, provided i and j are not the same: and they also satisfy the relations

$$\sum_{i=1}^{n} \{(l_m)_i\}^2 = 1, \quad \sum_{m=1}^{n} \{(l_m)_i (l_m)_j\} = 0,$$

for the same range of values. The full determinant $| (l_i)_j |$ is equal to unity; and each individual constituent is equal to its minor.

For the determination of the sets of direction-cosines, after the first set typified by l_1, we use the space-coordinates of any point on the curve, as represented by $y_1, \dots, y_n$, referred to rectangular axes, so that

$$ds^2 = dy_1^{\,2} + dy_2^{\,2} + \dots + dy_n^{\,2};$$

and then

$$(l_1)_1, (l_1)_2, \dots, (l_1)_n = \frac{dy_1}{ds}, \quad \frac{dy_2}{ds}, \dots, \quad \frac{dy_n}{ds},$$

so that we may take

$$l_1 = \frac{dy}{ds}.$$

Then

$$l_2 = \rho_1 \frac{d^2y}{ds^2}, \quad l_3 = \rho_2 \left(\rho_1 \frac{d^3y}{ds^3} + \frac{d\rho_1}{ds} \frac{d^2y}{ds^2} + \frac{1}{\rho_1} \frac{dy}{ds} \right),$$

and so on: while the values of $\rho_1, \rho_2, \ldots$ are given by

$$\frac{1}{\rho_1^2} = \sum_{m=1}^{n} \left(\frac{d^2 y_m}{ds^2}\right)^2,$$

$$\frac{1}{\rho_2^2 \rho_1^2} + \frac{1}{\rho_1^4} + \left\{\frac{d}{ds}\left(\frac{1}{\rho_1}\right)\right\}^2 = \sum_{m=1}^{n} \left(\frac{d^3 y_m}{ds^3}\right)^2,$$

and similar equations derived through the preceding relations satisfied by the quantities $(l_i)_j$.

8. The preceding equations, of themselves, make all subsequent magnitudes dependent upon the assignment of the quantities typified by l_1 and, though in a less obvious manner, upon the expression of the point-coordinates of the curve in terms of some parameter. But in most of the applications, as for instance, when geodesics are under discussion, the parametric representation of the point-coordinates is not known and, indeed, may be the actual subject of an investigation. There is, however, one universal property of all geodesics in any amplitude which is of central importance: it relates to the prime normal, and is as follows. At any point P of an amplitude, let a homaloid be drawn, of the same number of dimensions as the amplitude, and touching the amplitude; and from a point Q in the curved amplitude, contiguous to P, let a perpendicular be drawn to this tangent homaloid. The limiting position of the direction of this perpendicular, as Q tends to coincide with P, coincides with the direction of the prime normal of the amplitudinal geodesic originating in the direction PQ. The direction-cosines of this perpendicular are quantities determined by the amplitude without any reference to the geodesic, as will appear in § 20 where the property is established. Let these direction-cosines be denoted by

$$(Y)_1, (Y)_2, \ldots,$$

and let the set be typified by the symbol Y. Then, for the Frenet tableau, we can take

$$l_2 = Y = y''\rho,$$

owing to the property postulated; and these quantities, typified by Y, can be regarded as data for the further investigation of the properties of the geodesic through the Frenet equations. In particular, for the earliest curvatures, we have

$$Y' = \frac{l_3}{\sigma} - \frac{y'}{\rho};$$

$$\begin{aligned} Y'' &= \frac{1}{\sigma}\left(-\frac{Y}{\sigma} + \frac{l_4}{\tau}\right) + l_3 \frac{d}{ds}\left(\frac{1}{\sigma}\right) - \frac{Y}{\rho^2} - y' \frac{d}{ds}\left(\frac{1}{\rho}\right) \\ &= \frac{l_4}{\sigma\tau} + l_3 \frac{d}{ds}\left(\frac{1}{\sigma}\right) - Y\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) - y' \frac{d}{ds}\left(\frac{1}{\rho}\right); \end{aligned}$$

and

$$Y''' = \frac{l_5}{\sigma\tau\kappa} + l_4\left\{\frac{2}{\tau}\frac{d}{ds}\left(\frac{1}{\sigma}\right) + \frac{1}{\sigma}\frac{d}{ds}\left(\frac{1}{\tau}\right)\right\} + l_3\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} + \frac{1}{\tau^2}\right)\right\}$$
$$- \frac{3}{2} Y \frac{d}{ds}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) - y'\left\{\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) - \frac{1}{\rho}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)\right\},$$

in each of which formulæ it will be noted that the typical direction-cosines, of the successive principal lines of the curve, occur linearly. Owing to the relations among these direction-cosines, which may be exhibited in the form

$$\begin{aligned} &\sum y'^2 = 1\,; \\ &\sum y'Y = 0, \quad \sum Y^2 = 1\,; \\ &\sum y'l_3 = 0, \quad \sum Yl_3 = 0, \quad \sum l_3{}^2 = 1\,; \end{aligned}$$

and so on, we obtain the simple results

$$\sum Y'^2 = \frac{1}{\rho^2} + \frac{1}{\sigma^2},$$

$$\sum Y''^2 = \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^2 + \frac{\rho'^2}{\rho^4} + \frac{\sigma'^2}{\sigma^4} + \frac{1}{\sigma^2\tau^2}.$$

These results, as well as other inferences to be made from the equations just given, will be used in the development of properties of amplitudes of specific orders of dimension; they belong to geodesics in the amplitudes and, in formal expression, are unaffected by the rank of the amplitude.

9. The orbicular curvatures in the second of the two categories in § 7 relate to the geometrical frame of the curve, mainly through enclosed amplitudes geometrically descriptive of the curve. The first of these curvatures is that of the circle of closest contact, a circle which lies in the osculating plane; its radius is called the radius of circular curvature, being equal to ρ_1 or ρ, and its centre is the centre of circular curvature. The second of these curvatures is that of the sphere of closest contact, a sphere which lies in the osculating flat; its radius R is called the radius of spherical curvature, being given by the relation

$$R^2 = \rho^2 + \sigma^2\left(\frac{d\rho}{ds}\right)^2,$$

and its centre is the centre of spherical curvature. The third of the curvatures is that of the globe of closest contact, a globe which lies in the osculating block; its radius Γ is called the radius of globular curvature, being given by the relation

$$\Gamma^2 = R^2 + \tau^2\left(\frac{R}{\sigma}\frac{dR}{d\rho}\right)^2,$$

and its centre is the centre of globular curvature. And so for the various orbiculate curvatures in succession, the law of formation of which is definitely established.*

* *G.F.D.*, vol. i, § 200.

CHAPTER II

General Amplitudes

Parametric representation; primary magnitudes.

10. We consider an amplitude of n dimensions, existing in a plenary homaloidal space of $n+n'$ dimensions, where n' may be any positive integer $\geqslant 1$. For the most part, no special account is taken of the value of n'; though when $n'=1$, there is simplification of properties, the amplitude then being called *primary*, in relation to the plenary space. That space is tacitly referred to orthogonal axes: the coordinates, which distinguish a position (being a point) in the space, are denoted by $y_1, y_2, y_3, \ldots$, being $n+n'$ in number. When the point belongs to the n-fold amplitude, the space-coordinates are conceived to be expressed in terms of n parametric magnitudes $x_1, x_2, \ldots, x_n$: these parameters, which help to characterise the amplitude, are assumed to be independent of one another for its complete range.

It is also assumed that the n-fold amplitude is not completely contained in any more extensive amplitude of $n+\nu$ dimensions (where $1 \leqslant \nu < n'$) which itself exists in the plenary homaloidal space. The amplitude then is said to exist freely in the plenary space of $n+n'$ dimensions; thus a surface can exist freely in a homaloidal space of four dimensions without being merged into a region in that space. The analytical equivalent of the general assumption is that the determinants in the array

$$\left\| \begin{array}{cccc} \dfrac{\partial y_1}{\partial x_1}, & \dfrac{\partial y_2}{\partial x_1}, & \ldots\ldots\ldots, & \dfrac{\partial y_{n+n'}}{\partial x_1} \\ \dfrac{\partial y_1}{\partial x_2}, & \dfrac{\partial y_2}{\partial x_2}, & \ldots\ldots\ldots, & \dfrac{\partial y_{n+n'}}{\partial x_2} \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{\partial y_1}{\partial x_n}, & \dfrac{\partial y_2}{\partial x_n}, & \ldots\ldots\ldots, & \dfrac{\partial y_{n+n'}}{\partial x_n} \end{array} \right\|$$

do not simultaneously vanish.

Summations will often extend over space-coordinates and their derivatives with respect to parameters; for their expression, the selected space-coordinate is taken to be y_m. For the expression of a set of equations, of which there is one and only one in the set corresponding to each of the space-coordinates, a representative equation alone will be stated as affecting a typical variable y, the implication being that y can assume all the values $y_1, y_2, y_3, \ldots$ in succession, so as to constitute the whole set of equations.

There are magnitudes, intrinsic to the amplitude, and involving symmetric combinations of derivatives of the space-coordinates with respect to the parameters of the amplitude. Its *first-order* (or *primary*) *magnitudes* are denoted by A_{ij}; and they are defined in significance by the relation

$$A_{ij}=\sum_m \frac{\partial y_m}{\partial x_i}\frac{\partial y_m}{\partial x_j},$$

for all the values $i, j, =1, \ldots, n$, taken independently of one another, the indicated m-summation extending over all the space-coordinates. Manifestly $A_{ij}=A_{ji}$: thus, in all, there are $\frac{1}{2}n(n+1)$ primary magnitudes.

We denote by Ω the determinant of the primary magnitudes, so that

$$\Omega=\begin{vmatrix} A_{11}, & A_{12}, & \ldots, & A_{1n} \\ A_{21}, & A_{22}, & \ldots, & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1}, & A_{n2}, & \ldots, & A_{nn} \end{vmatrix}.$$

In accordance with customary usage, we denote the co-factor of A_{ij} in Ω by a_{ij}, so that, for all combinations of i and j,

$$a_{ii}=\frac{\partial\Omega}{\partial A_{ii}},\quad a_{ij}=\frac{1}{2}\frac{\partial\Omega}{\partial A_{ij}}.$$

The determinant, similar to Ω and constructed with the quantities a_{ij} as constituents, is equal to Ω^{n-1}; and the minors of this new a_{ij}-determinant are connected with complementary minors of Ω by known relations. Moreover, established results in the theory of determinants allow Ω to be expressed as the sum of the squares of all the determinants of n rows and columns that can be composed from the preceding matrix-array of the first parametric derivatives of the space-coordinates. Further, principal minors of Ω of all orders (that is, minors of all orders taken along the principal diagonal of Ω) are expressible as the sums of squares of corresponding determinants of the same respective orders formed from the same matrix-array; for example,

$$A_{11}A_{22}-A_{12}^2=\sum_i\sum_j\left(\frac{\partial y_i}{\partial x_1}\frac{\partial y_j}{\partial x_2}-\frac{\partial y_i}{\partial x_2}\frac{\partial y_j}{\partial x_1}\right)^2.$$

Consequently Ω, and each of its principal minors of all orders (including the constituents of the principal diagonal), are essentially positive magnitudes. Also the minors of Ω of all orders are expressible as the sums of products of corresponding determinants of the same respective orders formed from the same matrix-array; for example, the combination

$$A_{ij}A_{kl}-A_{ik}A_{jl}=\sum_a\sum_b\left(\frac{\partial y_a}{\partial x_i}\frac{\partial y_b}{\partial x_l}-\frac{\partial y_a}{\partial x_l}\frac{\partial y_b}{\partial x_i}\right)\left(\frac{\partial y_a}{\partial x_j}\frac{\partial y_b}{\partial x_k}-\frac{\partial y_a}{\partial x_k}\frac{\partial y_b}{\partial x_j}\right),$$

for all values of i, j, k, l, occurs in connection with the Riemann measure of curvature of an amplitude, estimated in a superficial orientation.

11. The earliest immediate occurrence of the primary magnitudes A_{ij} is to be found in the analytical expression for the element of arc in the n-fold amplitude. This arc-element is taken to be the element of linear distance, in the plenary homaloidal space, between two contiguous points each of which belongs to the amplitude; and therefore it is given, initially, by the relation

$$ds^2 = dy_1{}^2 + dy_2{}^2 + dy_3{}^2 + \ldots + dy_{n+n'}^2,$$

when the space-coordinates of the two points are $y_1, y_2, y_3, \ldots, y_{n+n'}$, for one point and $y_1+dy_1,\ y_2+dy_2,\ y_3+dy_3,\ \ldots,\ y_{n+n'}+dy_{n+n'}$, for the other. Because the points belong to the amplitude, each of the space-coordinates y is a function of n independent parametric variables $x_1, \ldots, x_n$, the aggregate of functions being subject to the exclusive conditions stated in § 10; and therefore, for each element dy_m, there exists an expression of the form

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 + \ldots + \frac{\partial y}{\partial x_n} dx_n,$$

where small quantities of order higher than the first are neglected, and the complete variations $dx_1, \ldots, dx_n$, of the n parameters give all possible variations in the amplitude, measured from the first point. Hence

$$ds^2 = \sum_m \left(\frac{\partial y_m}{\partial x_1} dx_1 + \frac{\partial y_m}{\partial x_2} dx_2 + \ldots + \frac{\partial y_m}{\partial x_n} dx_n \right)^2,$$

the summation being taken over the aggregate of space-coordinates. When the right-hand side is developed, the coefficient of $dx_r{}^2$ is

$$= \sum_m \left(\frac{\partial y_m}{\partial x_r} \right)^2 = A_{rr},$$

and the coefficient of $dx_i\, dx_j$

$$= 2 \sum_m \frac{\partial y_m}{\partial x_i} \frac{\partial y_m}{\partial x_j} = 2A_{ij}.$$

The term $A_{rr}\, dx_r{}^2$ can be written $A_{rr}\, dx_r\, dx_r$; and, because $A_{ij} = A_{ji}$, the term $2A_{ij}\, dx_i\, dx_j$ can be written $A_{ij}\, dx_i\, dx_j + A_{ji}\, dx_j\, dx_i$; hence the relation is often expressed in the form

$$ds^2 = \sum_i \sum_j A_{ij}\, dx_i\, dx_j,$$

where the double summation extends over all the values $i, j, = 1, \ldots, n$, taken independently of one another.

From this expression for ds^2, all specific occurrence of the dimensional extent of the plenary homaloidal space has disappeared; the expression contains only the differential elements of the n parameters and, as coefficients of the binary products of these elements, only functions of these parameters. Accordingly when such an expression is postulated for the arc-element of an amplitude, there are no manifest indications of the dimensional range of a plenary homaloidal space containing the

amplitude. It seems desirable to insist on this fact at the outset, because of a course of argument sometimes adduced to obtain an estimate of this dimensional range. The summary of this argument, briefly, is as follows. Let there be N functions y_m such that, for all values of i and j in the range $1, \ldots, n$, the equations

$$\sum_{m=1}^{N} \frac{\partial y_m}{\partial x_i} \frac{\partial y_m}{\partial x_j} = A_{ij},$$

where A_{ij} are the coefficients in a postulated expression for the arc-element, are satisfied. The total number of such equations is the total number of coefficients A_{ij}, that is, their number is $\frac{1}{2}n(n+1)$; and it is argued that, if

$$N \geqslant \tfrac{1}{2}n(n+1),$$

there are sufficient quantities y to satisfy the aggregate of equations. In particular, because $\frac{1}{2}n(n+1)$ is an integer, it will be sufficient * to take $N = \frac{1}{2}n(n+1)$.

Now this argument implies that, because there are as many unknown quantities as there are equations, therefore the equations necessarily provide values of these quantities. But each of the $\frac{1}{2}n(n+1)$ equations is a partial differential equation: and, in constituting the system, no account is taken of the fact that all relations of the type

$$\frac{\partial}{\partial x_j}\left(\frac{\partial y_m}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial y_m}{\partial x_j}\right)$$

must be satisfied for all values of m, i, j. Further, each of the equations is of a very special form, because every coefficient of a combination of partial derivatives is actually equal to unity, a particularity tending to exclude the kind of negative condition usually incumbent on the forms of partial differential equations in relation to their characters. And, certainly, existence-theorems, applicable to a completely general system, are never adduced, to cover or to justify the assumption that the $\frac{1}{2}n(n+1)$ equations do possess a common set of solutions.

An extreme instance may be adduced: it is the earliest instance that can arise. Let there be two parameters x_1, x_2, taken as p and q for the moment; there are three coefficients A_{11}, A_{12}, A_{22}, taken as E, F, G, so that $N = 3$. The argument would imply that the three equations

$$\left.\begin{aligned}\left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2 + \left(\frac{\partial z}{\partial p}\right)^2 &= E\\ \frac{\partial x}{\partial p}\frac{\partial x}{\partial q} + \frac{\partial y}{\partial p}\frac{\partial y}{\partial q} + \frac{\partial z}{\partial p}\frac{\partial z}{\partial q} &= F\\ \left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2 + \left(\frac{\partial z}{\partial q}\right)^2 &= G\end{aligned}\right\},$$

* Thus $N = 10$ when $n = 4$; and, in discussions of the relativity equation, it is declared that the configuration can exist in a ten-fold plenary homaloidal space.

involving three dependent variables x, y, z, always possess at least one set of solutions giving x, y, z, as functions of p and q, without any external limitations of any kind. As the only magnitudes that can be subject to limitation are E, F, G, and as no limitations are imposed, it would appear that E, F, G, can be assumed arbitrarily. Such a result seems not to accord with an inference that will be drawn from the Gauss theory of surfaces in triple homaloidal space. The postulated existence of x, y, z, as functions of two parameters p and q provides such a surface. At a later stage (§ 89), in this connection, it will be proved that certainly one relation must be satisfied by E, F, G, if the converse inference as to the number of point-variables is to be valid: in other words, the assumption is not unconditionally true even for the particular instance when $n=2$.

Nor is it worth while taking, for the general case, a value of N greater than $\frac{1}{2}n(n+1)$. All the stated hesitations, in accepting the assumption, still remain whatever be the value of N. Moreover, a special example is given (§ 88) where, for particular values of E, F, G, there are equations

$$\sum_{m=1}^{\infty}\left(\frac{\partial y_m}{\partial p}\right)^2=E,\quad \sum_{m=1}^{\infty}\frac{\partial y_m}{\partial p}\frac{\partial y_m}{\partial q}=F,\quad \sum_{m=1}^{\infty}\left(\frac{\partial y_m}{\partial q}\right)^2=G,$$

which formally are initiated from a plenary space of an unlimited number of dimensions.

Accordingly, when an expression for the arc-element of an amplitude of n dimensions is postulated in the form

$$ds^2=\sum_i \sum_j A_{ij}\,dx_i\,dx_j,$$

for $i, j, =1, \ldots, n$, in all combinations, the determination of the smallest number of dimensions of the homaloidal plenary space, which can contain the amplitude, remains a matter for investigation.

Christoffel symbols.

12. Of the various combinations, which involve second derivatives of space-coordinates with respect to amplitudinal parameters, there are two persistently recurring sets, specially connected with the first derivatives of the primary magnitudes A_{ij}. To represent the magnitudes in the respective sets, the symbols *

$$[ij, k],\quad \{ij, k\},$$

where $i, j, k, =1, \ldots, n$, in all combinations, are used; and their significance is as follows.

* The symbols in the text are modifications, in form but not in significance, of the symbols used by Christoffel in his now classical memoir, *Crelle*, vol. lxx (1869), pp. 46-70.

The first of the two symbols, usually called Christoffel symbols, is defined by the relation

$$[ij, k] = \sum_m \frac{\partial y_m}{\partial x_k} \frac{\partial^2 y_m}{\partial x_i \partial x_j},$$

for all the numerical values of i, j, k, as stated. Manifestly, the integers i and j can be interchanged in $[ij, k]$, without affecting its value. Now, from the defined value of A_{ij}, we have

$$\begin{aligned} \frac{\partial A_{ij}}{\partial x_k} &= \sum_m \frac{\partial^2 y_m}{\partial x_i \partial x_k} \frac{\partial y_m}{\partial x_j} + \sum_m \frac{\partial^2 y_m}{\partial x_j \partial x_k} \frac{\partial y_m}{\partial x_i} \\ &= [ik, j] + [jk, i]; \end{aligned}$$

and, similarly,

$$\frac{\partial A_{jk}}{\partial x_i} = [ji, k] + [ki, j] = [ij, k] + [ik, j],$$

$$\frac{\partial A_{ki}}{\partial x_j} = [kj, i] + [ij, k] = [jk, i] + [ij, k].$$

Consequently, we have

$$[pq, r] = \tfrac{1}{2} \left(\frac{\partial A_{qr}}{\partial x_p} + \frac{\partial A_{pr}}{\partial x_q} - \frac{\partial A_{pq}}{\partial x_r} \right),$$

for all arrangements of the indices $p, q, r, = 1, \ldots, n$, taken independently of one another. The preceding value is formally complete when p, q, r, are three distinct integers from the set; but it is valid, if any two be equal, and if the three be equal, because then

$$[pp, r] = \frac{\partial A_{pr}}{\partial x_p} - \tfrac{1}{2} \frac{\partial A_{pp}}{\partial x_r},$$

$$[pq, p] = \tfrac{1}{2} \frac{\partial A_{pp}}{\partial x_q},$$

$$[pp, p] = \tfrac{1}{2} \frac{\partial A_{pp}}{\partial x_p}.$$

The second of the specified Christoffel symbols is defined, in relation to the quantities $[ij, k]$, by the equation

$$\{ij, k\} = \frac{1}{\Omega} \sum_{\mu=1}^{n} a_{k\mu} [ij, \mu].$$

Because i and j are interchangeable in every term on the right-hand side without affecting the value of the term, it follows that the value of $\{ij, k\}$ is unaffected by the interchange of the integers i and j. As the quantities $\{ij, k\}$ are linearly expressible in terms of the quantities $[ij, \mu]$, the integer-pair ij being the same throughout, so the quantities $[ij, \mu]$ are linearly expressible in terms of the quantities $\{ij, k\}$, again with the conservation of the integer-pair ij. To obtain the latter expressions, we take the n relations for $\{ij, k\}$ in the successive values

1, ..., n for the integer k; when these are resolved for the n magnitudes $[ij, \mu]$ which occur linearly, they yield the expressions

$$[ij, \mu] = \sum_{k=1}^{n} A_{k\mu} \{ij, k\},$$

where the integer-pair ij is conserved throughout. This form is, of course, analytically equivalent to the relations defining $\{ij, k\}$ and can therefore be regarded also as implicitly defining the second Christoffel symbol. The first parametric derivatives of the primary magnitudes A_{ij} have been expressed in terms of the quantities $[ij, \mu]$, and therefore they can be expressed in terms of the quantities $\{ij, k\}$; in fact, we have

$$\begin{aligned}\frac{\partial A_{ij}}{\partial x_k} &= [ki, j] + [kj, i] \\ &= \sum_{p=1}^{n} A_{jp} \{ki, p\} + \sum_{q=1}^{n} A_{iq} \{kj, q\} \\ &= \sum_{p=1}^{n} [A_{jp} \{ki, p\} + A_{ip} \{kj, p\}],\end{aligned}$$

where, as now will be observed, the integer-pair ij from the left-hand side is distributed over the terms on the right-hand side, the distributed parts being connected with the index of the deriving variable on the left-hand side.

Moreover, we have

$$\begin{aligned}&\frac{\partial A_{ik}}{\partial x_j} + \frac{\partial A_{jk}}{\partial x_i} - \frac{\partial A_{ij}}{\partial x_k} \\ &\quad = \sum_{p=1}^{n} [A_{kp} \{ji, p\} + A_{ip} \{jk, p\}] + \sum_{p=1}^{n} [A_{kp}\{ij, p\} + A_{jp}\{ik, p\}] \\ &\qquad\qquad - \sum_{p=1}^{n} [A_{jp} \{ik, p\} + A_{ip} \{jk, p\}] \\ &\quad = 2\sum_{p=1}^{n} A_{kp} \{ji, p\}.\end{aligned}$$

The left-hand side $= 2\ [ij, k]$; and the result is in accordance with the definition (p. 24) of the second Christoffel symbol.

Note. These Christoffel symbols belong to the amplitude of general dimensionality represented by the unspecific integer n. When we have to discuss the properties of amplitudes of specific dimensionality, in particular, surfaces, regions, and domains, it is convenient to employ special symbols without, of course, modifying the defined significance.

Thus for surfaces, which are two-fold, $n = 2$; and, for them, we take

$$\Gamma_{ij} = \{ij, 1\}, \quad \Delta_{ij} = \{ij, 2\},$$

for the three combinations $i, j, =1, 2$. For regions, which are three-fold, $n=3$; and, for them, we take

$$\Gamma_{ij}=\{ij, 1\}, \quad \Delta_{ij}=\{ij, 2\}, \quad \Theta_{ij}=\{ij, 3\},$$

for the six combinations $i, j, =1, 2, 3$. For domains, which are four-fold, $n=4$; and, for them, we take

$$\Gamma_{ij}=\{ij, 1\}, \quad \Delta_{ij}=\{ij, 2\}, \quad \Theta_{ij}=\{ij, 3\}, \quad \Phi_{ij}=\{ij, 4\},$$

for all the ten combinations $i, j, =1, 2, 3, 4$. For amplitudes of more than four dimensions, we revert to the general Christoffel symbols as defined.

13. The derivatives of the determinant Ω and its first minors are required; they are expressible in terms of the Christoffel symbols. We have

$$\begin{aligned}
\frac{\partial \Omega}{\partial x_k} &= \sum_i \sum_j a_{ij} \frac{\partial A_{ij}}{\partial x_k} \\
&= \sum_i \sum_j a_{ij}\{[ki, j]+[kj, i]\} \\
&= \sum_i \sum_j a_{ij}[ki, j] + \sum_j \sum_i a_{ji}[ki, j] \\
&= 2\sum_i \sum_j a_{ij}[ki, j].
\end{aligned}$$

But, by the definition of the symbol $\{pq, r\}$,

$$\frac{1}{\Omega} \sum_j a_{ij}[ki, j] = \{ki, i\};$$

and therefore

$$\begin{aligned}
\frac{1}{2\Omega}\frac{\partial \Omega}{\partial x_k} &= \sum_i \{ki, i\} \\
&= \{k1, 1\} + \{k2, 2\} + \ldots\ldots + \{kn, n\},
\end{aligned}$$

for all values of k.

To obtain derivatives of the minors of Ω, we proceed from the identical relations

$$A_{\mu 1}a_{1s} + A_{\mu 2}a_{2s} + \ldots + A_{\mu r}a_{rs} + \ldots + A_{\mu n}a_{ns} = 0, \text{ or } \Omega,$$

the right-hand side being Ω when $\mu=s$, and being 0 when μ and s are different. Using a typical relation when μ and s are different, we have

$$\begin{aligned}
&A_{\mu 1}\frac{\partial a_{1s}}{\partial x_k} + A_{\mu 2}\frac{\partial a_{2s}}{\partial x_k} + \ldots + A_{\mu r}\frac{\partial a_{rs}}{\partial x_k} + \ldots + A_{\mu n}\frac{\partial a_{ns}}{\partial x_k} \\
&+ a_{1s}\sum_p [A_{p1}\{k\mu, p\} + A_{\mu p}\{k1, p\}] \\
&+ a_{2s}\sum_p [A_{p2}\{k\mu, p\} + A_{\mu p}\{k2, p\}] \\
&+ \ldots\ldots + a_{ns}\sum_p [A_{pn}\{k\mu, p\} + A_{\mu p}\{kn, p\}] = 0;
\end{aligned}$$

and when we take the relation for $\mu=s$, there is a term

$$\frac{\partial \Omega}{\partial x_k}$$

on the right-hand side. On the left-hand side, the aggregate coefficient of the typical term $\{k\mu, p\}$ is

$$=a_{1s}A_{p1}+a_{2s}A_{p2}+\ldots+a_{xs}A_{pn}=0, \text{ or } \Omega,$$

being equal to 0 when p is different from s, and to Ω when $p=s$. Thus the differentiated relation is

$$A_{\mu 1}\frac{\partial a_{1s}}{\partial x_k}+A_{\mu 2}\frac{\partial a_{2s}}{\partial x_k}+\ldots+A_{\mu r}\frac{\partial a_{rs}}{\partial x_k}+\ldots+A_{\mu n}\frac{\partial a_{xs}}{\partial x_k}$$
$$=-\Omega\{k\mu, s\}-\sum_p A_{\mu p}[\sum_l a_{sl}\{kl, p\}],$$

when μ and s are different; and when $\mu=s$, there is an additional term $\frac{\partial \Omega}{\partial x_k}$ on the right-hand side. Resolving the whole set of n relations for the n quantities $\frac{\partial a_{1s}}{\partial x_k}, \frac{\partial a_{2s}}{\partial x_k}, \ldots, \frac{\partial a_{xs}}{\partial x_k}$, we have

$$\Omega\frac{\partial a_{rs}}{\partial x_k}-\frac{\partial \Omega}{\partial x_k}a_{rs}+\Omega\sum_\mu a_{\mu r}\{k_\mu, s\}=-\sum_p\sum_\mu a_{\mu r}A_{\mu p}[\sum_l a_{sl}\{kl, p\}].$$

Once more using the identical relations concerning minors, we have

$$\sum_\mu a_{\mu r}A_{\mu p}=0 \text{ when } p \text{ and } r \text{ are different, } =\Omega \text{ when } p=r,$$

so that the right-hand side

$$=-\Omega\sum_l a_{sl}\{kl, r\};$$

consequently

$$\frac{\partial a_{rs}}{\partial x_k}-\frac{1}{\Omega}\frac{\partial \Omega}{\partial x_k}a_{rs}=-\sum_\mu a_{\mu r}\{k\mu, s\}-\sum_l a_{sl}\{kl, r\},$$

that is,

$$\Omega\frac{\partial}{\partial x_k}\left(\frac{a_{rs}}{\Omega}\right)=-\sum_\mu [a_{\mu r}\{k\mu, s\}+a_{\mu s}\{k\mu, r\}],$$

for all combinations of values of r, s, k, from the set $1, 2, \ldots, n$, independently of one another, while the summation for μ extends over all the values $1, 2, \ldots, n$.

At a later stage, we use a symbol g_{kl} according to the definition

$$g_{kl}=\sum_i x_i'\{il, k\},$$

for all the values $k, l, =1, \ldots, n$. With this symbol so defined, it follows that

$$\Omega\frac{d}{ds}\left(\frac{a_{ij}}{\Omega}\right)=-\sum_\mu (g_{i\mu}a_{\mu j}+g_{j\mu}a_{\mu i}).$$

Riemann four-index symbols.

14. The Riemann four-index symbols * emerge from various sources. In the simplest form, which occurs in the Gauss characteristic equation of a surface lying in homaloidal triple space, the single symbol is connected with the second parametric derivatives of the primary magnitudes. For the general n-fold amplitude, we can proceed as follows.

When the primary magnitude A_{il}, defined by the equation

$$A_{il}=\sum_m \frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_l},$$

the summation extended over all the space-coordinates typified by y, is differentiated doubly, with respect to any two parameters x_j and x_k, we have

$$\frac{\partial^2 A_{il}}{\partial x_j\,\partial x_k}=\sum_m \frac{\partial^3 y}{\partial x_i\,\partial x_j\,\partial x_k}\frac{\partial y}{\partial x_l}+\sum_m \frac{\partial^3 y}{\partial x_l\,\partial x_j\,\partial x_k}\frac{\partial y}{\partial x_i}$$
$$+\sum_m \frac{\partial^2 y}{\partial x_i\,\partial x_j}\frac{\partial^2 y}{\partial x_l\,\partial x_k}+\sum_m \frac{\partial^2 y}{\partial x_i\,\partial x_k}\frac{\partial^2 y}{\partial x_l\,\partial x_j},$$

in which summations occur involving third derivatives of the typical space-variable. One of these summations recurs when the suffixes i and j are interchanged; the other recurs when the suffixes k and l are interchanged; and the new summations, which are introduced by these interchanges, themselves recur when i and l are changed into j and k. Accordingly, we construct the corresponding expressions for

$$\frac{\partial^2 A_{ik}}{\partial x_j\,\partial x_l},\quad \frac{\partial^2 A_{jl}}{\partial x_i\,\partial x_k},\quad \frac{\partial^2 A_{jk}}{\partial x_i\,\partial x_l},$$

with the aim (by linear combinations) of eliminating summations involving the third derivatives which occur; and we find

$$\frac{1}{2}\left(\frac{\partial^2 A_{il}}{\partial x_j\,\partial x_k}-\frac{\partial^2 A_{ik}}{\partial x_j\,\partial x_l}-\frac{\partial^2 A_{jl}}{\partial x_i\,\partial x_k}+\frac{\partial^2 A_{jk}}{\partial x_i\,\partial x_l}\right)=\sum_m\left(\frac{\partial^2 y}{\partial x_i\,\partial x_k}\frac{\partial^2 y}{\partial x_j\,\partial x_l}-\frac{\partial^2 y}{\partial x_j\,\partial x_k}\frac{\partial^2 y}{\partial x_i\,\partial x_l}\right),$$

a quantity which involves only second derivatives of the typical space-variable.

According to one definition, the four-index symbol (ij, kl) is propounded as representing the magnitude

$$\frac{1}{2}\left(\frac{\partial^2 A_{il}}{\partial x_j\,\partial x_k}-\frac{\partial^2 A_{ik}}{\partial x_j\,\partial x_l}-\frac{\partial^2 A_{jl}}{\partial x_i\,\partial x_k}+\frac{\partial^2 A_{jk}}{\partial x_i\,\partial x_l}\right)+\sum_\lambda\sum_\mu A_{\lambda\mu}\left[\{il,\mu\}\{jk,\lambda\}-\{ik,\mu\}\{jl,\lambda\}\right],$$

where, in the last summation, λ and μ take all the values $1, \ldots, n$, independently

* The magnitudes, denoted by the symbols, first appeared in the classical memoir by Riemann on the foundations of geometry; they occur in the measure of curvature which he postulates for a general amplitude, *Ges. Werke* (1876), p. 261.

of one another: obviously the value of the total sum is unaltered by a formal interchange of λ and μ. We therefore have

$$(ij,\, kl)=\sum_m \left(\frac{\partial^2 y}{\partial x_i\,\partial x_k}\,\frac{\partial^2 y}{\partial x_j\,\partial x_l}-\frac{\partial^2 y}{\partial x_j\,\partial x_k}\,\frac{\partial^2 y}{\partial x_i\,\partial x_l}\right)$$
$$+\sum_\lambda \sum_\mu A_{\lambda\mu}\left[\{il,\,\mu\}\{jk,\,\lambda\}-\{ik,\,\mu\}\{jl,\,\lambda\}\right].$$

A simplified modification of this form can be obtained by introducing certain quantities which occur in the fundamental equations for the circular curvature of a geodesic of the amplitude. It will appear (§ 17) that there are combinations η_{ij} of the second derivatives of the typical space-variable y and of the Christoffel symbol $\{ab,\, c\}$, in the form defined by the equation

$$\eta_{ij}=\frac{\partial^2 y}{\partial x_i\,\partial x_j}-\sum_t \frac{\partial y}{\partial x_t}\{ij,\, t\},$$

there being such a combination $\eta_{ij}^{(m)}$ for each of the variables y_m. Meanwhile, merely accepting the symbols η_{ij} as thus defined, we have

$$\frac{\partial^2 y}{\partial x_i\,\partial x_j}=\eta_{ij}+\sum_t \frac{\partial y}{\partial x_t}\{ij,\, t\}.$$

For all values of i, j, r, we have

$$\sum_m \frac{\partial y_m}{\partial x_r}\eta_{ij}^{(m)}=\sum_m \frac{\partial y_m}{\partial x_r}\frac{\partial^2 y_m}{\partial x_i\,\partial x_j}-\sum_m \frac{\partial y_m}{\partial x_r}\left[\sum_t \frac{\partial y}{\partial x_t}\{ij,\, t\}\right]=[ij,\, r]-\sum_t A_{rt}\{ij,\, t\}=0,$$

by the formula in § 12. Consequently

$$\sum_m \frac{\partial^2 y_m}{\partial x_i\,\partial x_k}\,\frac{\partial^2 y_m}{\partial x_j\,\partial x_l}=\sum_m\left[\eta_{ik}^{(m)}+\sum_\lambda \frac{\partial y_m}{\partial x_\lambda}\{ik,\,\lambda\}\right]\left[\eta_{jl}^{(m)}+\sum_\mu \frac{\partial y_m}{\partial x_\mu}\{jl,\,\mu\}\right]$$
$$=\sum_m \eta_{ik}\eta_{jl}+\sum_\lambda\sum_\mu A_{\lambda\mu}\{ik,\,\lambda\}\{jl,\,\mu\}.$$

In the same way

$$\sum_m \frac{\partial^2 y_m}{\partial x_j\,\partial x_k}\,\frac{\partial^2 y_m}{\partial x_i\,\partial x_l}=\sum_m \eta_{jk}\eta_{il}+\sum_\lambda\sum_\mu A_{\lambda\mu}\{jk,\,\lambda\}\{il,\,\mu\}.$$

Consequently

$$(ij,\, kl)=\sum_m(\eta_{ik}\eta_{jl}-\eta_{il}\eta_{jk}),$$

which proves to be a useful alternative equivalent for the general Riemann four-index symbol, expressed solely in terms of the magnitudes $\eta_{\alpha\beta}$. These magnitudes $\eta_{\alpha\beta}$ occur in the equations (§ 17), which determine the direction-cosines Y_m of the prime normal and the magnitude ρ of the radius of circular curvature of the amplitudinal geodesic, and which are

$$\frac{Y_m}{\rho}=\sum_i\sum_j \eta_{ij}^{(m)}x_i'x_j'$$

for all the values $m=1, 2, \ldots$, of the dimension-range of the plenary space, the quantities x' being the direction-variables of the geodesic.

15. Modified equivalent forms can be obtained for the Riemann four-index symbols. Proceeding from $[\alpha\beta, \gamma]$, as defined by

$$[\alpha\beta, \gamma] = \sum \frac{\partial y}{\partial x_\gamma} \frac{\partial^2 y}{\partial x_\alpha \partial x_\beta},$$

and differentiating two such Christoffel magnitudes, we have

$$\frac{\partial}{\partial x_l}[ik, j] = \sum \frac{\partial^2 y}{\partial x_j \partial x_l} \frac{\partial^2 y}{\partial x_i \partial x_k} + \sum \frac{\partial y}{\partial x_j} \frac{\partial^3 y}{\partial x_i \partial x_k \partial x_l},$$

$$\frac{\partial}{\partial x_k}[il, j] = \sum \frac{\partial^2 y}{\partial x_j \partial x_k} \frac{\partial^2 y}{\partial x_i \partial x_l} + \sum \frac{\partial y}{\partial x_j} \frac{\partial^3 y}{\partial x_i \partial x_k \partial x_l};$$

hence

$$\frac{\partial}{\partial x_l}[ik, j] - \frac{\partial}{\partial x_k}[il, j] = \sum \left(\frac{\partial^2 y}{\partial x_j \partial x_l} \frac{\partial^2 y}{\partial x_i \partial x_k} - \frac{\partial^2 y}{\partial x_j \partial x_k} \frac{\partial^2 y}{\partial x_i \partial x_l} \right)$$
$$= (ij, kl) - \sum_\lambda \sum_\mu A_{\lambda\mu} [\{il, \mu\}\{jk, \lambda\} - \{ik, \mu\}\{jl, \lambda\}],$$

by an earlier result. Now

$$\sum_\lambda A_{\lambda\mu}\{jk, \lambda\} = [jk, \mu], \quad \sum_\lambda A_{\lambda\mu}\{jl, \lambda\} = [jl, \mu],$$

so that

$$\sum_\lambda \sum_\mu A_{\lambda\mu}[\{il, \mu\}\{jk, \lambda\} - \{ik, \mu\}\{jl, \lambda\}] = \sum_\mu \Big(\{il, \mu\}[jk, \mu] - \{ik, \mu\}[jl, \mu] \Big);$$

consequently we have an expression for the Riemann symbol in the form

$$(ij, kl) = \frac{\partial}{\partial x_l}[ik, j] - \frac{\partial}{\partial x_k}[il, j] + \sum_\mu \{il, \mu\}[jk, \mu] - \sum_\mu \{ik, \mu\}[jl, \mu].$$

Also, by substituting for the symbols $\{ab, c\}$ in terms of the symbols $[ab, \gamma]$ on the right-hand side, we obtain the form

$$(ij, kl) = \frac{\partial}{\partial x_l}[ik, j] - \frac{\partial}{\partial x_k}[il, j] + \frac{1}{\Omega} \sum_\lambda \sum_\mu a_{\lambda\mu} \{[il, \lambda][jk, \mu] - [ik, \lambda][jl, \mu]\}.$$

Further varied forms can be derived from these expressions by interchanges of i and j, by interchanges of k and l, and by the interchange of ij with kl, always without affecting the actual value of the magnitude which all the forms represent.

16. Each of the integers i, j, k, l, has a range from 1 to n; but the various possible combinations do not produce algebraically independent symbols. It is easy to verify the relations of identity, satisfied by each Riemann symbol, contained in the set

$$(ij, kl) = (ji, lk) = (kl, ij) = (lk, ji)$$
$$= -(ij, lk) = -(ji, kl) = -(lk, ij) = -(kl, ji),$$

as well as the algebraical relation satisfied by different symbols

$$(ij, kl) + (ik, lj) + (il, jk) = 0.$$

The first set of relations shews that the magnitude denoted by the symbol vanishes, if $i=j$, or if $k=l$; while it is possible to have $i=k$ or l, and $j=k$ or l, separately or together (provided i is not equal to j), without evanescence of the represented magnitude. Hence there are $\frac{1}{2}n(n-1)$, $=N$, possible combinations ij, and then (as kl can be the same as ij) there are $\frac{1}{2}N(N+1)$ possible combinations kl to be associated with each of the N combinations ij. The relations in the identical set shew that all distinct values will be obtained by fixing any one of the combinations ij; hence, so far as the set of identical relations will allow, there are $\frac{1}{2}N(N+1)$ possible symbols representing non-vanishing magnitudes. On the other hand, among these symbols thus retained, there exists one algebraical relation for every possible combination of i, j, k, l; that is, there are $\frac{1}{24}n(n-1)(n-2)(n-3)$ such algebraical relations. Consequently, the number of linearly independent magnitudes represented by this Riemann four-index symbol (ij, kl)

$$\begin{aligned}&=\tfrac{1}{2}N(N+1)-\tfrac{1}{24}n(n-1)(n-2)(n-3)\\&=\tfrac{1}{12}n^2(n^2-1).\end{aligned}$$

Thus there is one such symbol for a surface: there are six such symbols for a region; and there are twenty such symbols for a domain.

(i) When the amplitude is a surface, and the arc-element is taken in the form

$$ds^2=A\,dp^2+2H\,dp\,dq+B\,dq^2,$$

the superficial parameters being p, q, there is only a single Riemann four-index symbol. In conformity with the general notation, it is represented by $(12, 12)$; and, for the value, we have

$$\begin{aligned}(12, 12)&=-(21, 12)=-(12, 21)=(21, 21)\\&=-\tfrac{1}{2}(A_{22}-2H_{12}+B_{11})\\&\quad+A(\Gamma_{12}{}^2-\Gamma_{11}\Gamma_{22})+H(2\Gamma_{12}\Delta_{12}-\Gamma_{11}\Delta_{22}-\Gamma_{22}\Delta_{11})+B(\Delta_{12}{}^2-\Delta_{11}\Delta_{22})\end{aligned}$$

on using the alternative notation for the Christoffel symbols $\{ij, k\}$, and now adopting the double-suffix notation

$$A_{11}=\frac{\partial^2 A}{\partial p^2},\quad A_{12}=\frac{\partial^2 A}{\partial p\,\partial q},\quad A_{22}=\frac{\partial^2 A}{\partial q^2},$$

and similarly for H and B, to denote parametric derivatives. The terms in the last line can be represented symbolically in the abbreviated notation

$$-(*\between 11\between 22)+(*\between 12\between 12),$$

and also in a less frequently used form

$$-\sum A\Gamma_{11}\Gamma_{22}+\sum A\Gamma_{12}{}^2,$$

the earlier of which is preferable though the coefficients of the arc-element are latent. Thus the single superficial Riemann symbol is given by

$$(12, 12)=-\tfrac{1}{2}(A_{22}-2H_{12}+B_{11})-(*\between 11\between 22)+(*\between 12\between 12).$$

(ii) When the amplitude is a region and its arc-element is taken in the form

$$ds^2 = A\,dp^2 + 2H\,dp\,dq + B\,dq^2 + 2G\,dp\,dr + 2F\,dq\,dr + C\,dr^2,$$

the regional parameters being p, q, r, we still adopt the corresponding abbreviated notation for the modified Christoffel symbols and the double-suffix notation

$$A_{ij} = \frac{\partial^2 A}{\partial x_i\,\partial x_j},$$

where $x_1 = p$, $x_2 = q$, $x_3 = r$, for second parametric derivatives of the primary magnitudes: thus

$$\begin{aligned}(*\between ij \between kl) = A\Gamma_{ij}\Gamma_{kl} &+ H(\Gamma_{ij}\Delta_{kl} + \Gamma_{kl}\Delta_{ij}) + G(\Gamma_{ij}\Theta_{kl} + \Gamma_{kl}\Theta_{ij}) \\ &+ B\Delta_{ij}\Delta_{kl} \qquad\qquad\quad + F(\Delta_{ij}\Theta_{kl} + \Delta_{kl}\Theta_{ij}) \\ &\qquad\qquad\qquad\qquad\qquad + C\Theta_{ij}\Theta_{kl},\end{aligned}$$

for all the values of $i, j, k, l, = 1, 2, 3$, in all combinations.

There are six Riemann four-index symbols appertaining to a region: their values are

$$\left.\begin{aligned}(23, 23) &= -(32, 23) = -(23, 32) = (32, 32) \\ &= -\tfrac{1}{2}(B_{33} - 2F_{23} + C_{22}) - (*\between 22 \between 33) + (*\between 23 \between 23)\end{aligned}\right\},$$

$$\left.\begin{aligned}(31, 31) &= -(13, 31) = -(31, 13) = (13, 13) \\ &= -\tfrac{1}{2}(C_{11} - 2G_{31} + A_{33}) - (*\between 33 \between 11) + (*\between 31 \between 31)\end{aligned}\right\},$$

$$\left.\begin{aligned}(12, 12) &= -(21, 12) = -(12, 21) = (21, 21) \\ &= -\tfrac{1}{2}(A_{22} - 2H_{12} + B_{11}) - (*\between 11 \between 22) + (*\between 12 \between 12)\end{aligned}\right\},$$

$$\left.\begin{aligned}(12, 31) &= (21, 13) = (31, 12) = (13, 21) \\ &\qquad = -(21, 31) = -(12, 13) = -(13, 12) = -(31, 21) \\ &= \tfrac{1}{2}(A_{23} - G_{12} - H_{13} + F_{11}) - (*\between 12 \between 13) + (*\between 11 \between 23)\end{aligned}\right\},$$

$$\left.\begin{aligned}(23, 12) &= (32, 21) = (12, 23) = (21, 32) \\ &\qquad = -(32, 12) = -(23, 21) = -(21, 23) = -(12, 32) \\ &= \tfrac{1}{2}(B_{31} - H_{23} - F_{21} + G_{22}) - (*\between 23 \between 21) + (*\between 22 \between 31)\end{aligned}\right\},$$

$$\left.\begin{aligned}(31, 23) &= (13, 32) = (23, 31) = (32, 13) \\ &\qquad = -(13, 23) = -(31, 32) = -(32, 31) = -(23, 13) \\ &= \tfrac{1}{2}(C_{12} - F_{31} - G_{32} + H_{33}) - (*\between 31 \between 32) + (*\between 33 \between 12)\end{aligned}\right\}.$$

(iii) When the amplitude is a domain and its arc-element is taken in the form

$$ds^2 = (A, B, C, D, F, G, H, L, M, N \between dp, dq, dr, dt)^2,$$

the domainal parameters being p, q, r, t, there are twenty linearly independent Riemann four-index symbols. (There are, in fact, twenty-one non-vanishing symbols; but three of them are subject to the linear relation

$$(12, 34) + (13, 42) + (14, 23) = 0,$$

which reduces the independent aggregate by a single unit.) Their expressions will be stated at a later stage (§ 275).

It will be noted that, neither for the surface nor for the region, has the simple expression

$$(ij, kl) = \sum (\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}),$$

for the symbol in terms of the magnitudes η_{ab}, been used. The significance of these magnitudes is to be established, in connection with geodesics in an amplitude and their radius of circular curvature, alike in magnitude and direction. To that investigation we now proceed.

Equations of geodesics.

17. The intrinsic equations of geodesics in the n-fold amplitude are obtained by making the length of the arc, joining two points and measured in the amplitude, a minimum. For the immediate purpose, our concern is solely with the equations current along the geodesic, not with the additional qualitative tests of Legendre and Weierstrass, nor with the additional quantitative test of Jacobi, as devised in the calculus of variations. These intrinsic equations arise as critical (Euler) equations which must be satisfied, if the integral

$$\int \left(\sum_i \sum_j A_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} \right)^{\frac{1}{2}} dt$$

is to possess a minimum value ; and they are

$$\frac{d}{dt} \left\{ \frac{\sum_j A_{kj} \dfrac{dx_j}{dt}}{\left(\sum_i \sum_j A_{ij} \dfrac{dx_i}{dt} \dfrac{dx_j}{dt} \right)^{\frac{1}{2}}} \right\} - \frac{1}{2} \frac{\sum_i \sum_j \dfrac{\partial A_{ij}}{\partial x_k} \dfrac{dx_i}{dt} \dfrac{dx_j}{dt}}{\left(\sum_i \sum_j A_{ij} \dfrac{dx_i}{dt} \dfrac{dx_j}{dt} \right)^{\frac{1}{2}}} = 0,$$

for each of the values $k = 1, 2, \ldots, n$. After the critical equations have been obtained, the arc can be made the independent variable ; and, when this change is effected, the typical equation becomes

$$\frac{d}{ds} \left(\sum_j A_{kj} x_j' \right) = \tfrac{1}{2} \sum_i \sum_j \frac{\partial A_{ij}}{\partial x_k} x_i' x_j',$$

still for all the n values of k. The left-hand side of this typical equation is

$$\sum_j (A_{kj} x_j'') + \sum_j \left(\sum_l \frac{\partial A_{kj}}{\partial x_l} x_l' \right) x_j',$$

where, in the second term, the full coefficient of $x_i' x_j'$ is

$$\frac{\partial A_{kj}}{\partial x_i} + \frac{\partial A_{ki}}{\partial x_j}.$$

On the right-hand side of this typical equation, the full coefficient of $x_i'x_j'$ is

$$\frac{\partial A_{ij}}{\partial x_k}.$$

Thus the typical equation becomes

$$\sum_j A_{kj}x_j'' + \tfrac{1}{2}\sum_i \sum_j x_i'x_j' \left(\frac{\partial A_{kj}}{\partial x_i} + \frac{\partial A_{ki}}{\partial x_j} - \frac{\partial A_{ij}}{\partial x_k}\right) = 0,$$

that is, it can be written

$$\sum_j (A_{kj}x_j'') + \sum_i \sum_j [ij, k]\, x_i'x_j' = 0,$$

where now, as earlier, the double summation is for all the values $1, 2, \ldots, n$, of i, j, taken independently of one another.

When these equations are resolved for the n magnitudes x'', they lead to the equivalent equations

$$x_k'' + \sum_i \sum_j \{ij, k\}\, x_i'x_j' = 0,$$

where the summations on the right-hand sides extend over the values $i, j = 1, 2, \ldots, n$, taken independently of one another.

These equations, the intrinsic equations of a geodesic in the amplitude, are n in number. But they are not independent of one another, when the permanent equation

$$U = \sum_i \sum_j A_{ij}x_i'x_j' = 1$$

is taken into account; for we have

$$\frac{dU}{ds} = 0$$

for all variations in the region, and it is not difficult to verify that this general result can be expressed in the form

$$\sum_{k=1}^{n} \frac{\partial U}{\partial x_k'} [x_k'' + \sum_i \sum_j \{ij, k\}\, x_i'x_j'] = 0,$$

shewing that, in virtue of the permanent equation, only $n-1$ of the n intrinsic equations can be considered independent. We also may regard $U=1$ as an integral of the n intrinsic equations; but it provides no information generally characteristic of geodesics, for it only expresses the fact that they lie in the amplitude. On the other hand, should (for instance) n independent first integrals of the set of the n equations of the second order have been obtained, either the integrals must actually be in accord with $U=1$, or their constants must be so limited as to secure this accord.

Before proceeding to discuss the significance of these characteristic equations of geodesics, it is convenient to extend the foregoing analysis, in order to obtain the corresponding intrinsic equations of a geodesic in an amplitude of fewer than

n dimensions, which itself is contained within the n-fold amplitude under consideration. Such a sub-amplitude is represented, in that origination, by means of a set of μ relations

$$\theta_a(x_1, x_2, \ldots, x_n)=0, \quad (a=1, 2, \ldots, \mu;\ \mu < n),$$

among the parameters of the containing manifold.

For this purpose, it is necessary to make the integral

$$\left(\sum_i \sum_j A_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt}\right)^{\frac{1}{2}} dt$$

a minimum, subject to the foregoing μ conditional relations $\theta=0$. By the known theorems established in the calculus of variations, the critical conditions are similar to the earlier conditions, the difference consisting of additive terms which occur because of the existence of the relations $\theta=0$; and these critical conditions, after the earlier analysis, are seen to take the form

$$\sum_j A_{kj}x_j'' + \sum_i \sum_j [ij, k]\, x_i'x_j' = \lambda_1 \frac{\partial\theta_1}{\partial x_k} + \lambda_2 \frac{\partial\theta_2}{\partial x_k} + \ldots + \lambda_\mu \frac{\partial\theta_\mu}{\partial x_k},$$

for the values $k=1, 2, \ldots, n$, now using x'' to denote variation along these geodesics in the sub-amplitude.

As was the fact with the plenary amplitude, these n equations are equivalent to only $n-1$ independent equations when account is taken of the permanent relation

$$\sum A_{ij}x_i'x_j'=1.$$

Moreover, the quantities $\lambda_1, \ldots, \lambda_\mu$, are multipliers, the values of which are left undetermined in the formation of the critical equations; these values are obtained in association with the set of μ conditional relations $\theta=0$, but the investigation is deferred at this stage.

Ex. 1. The geodesics in the amplitude

$$ds^2 = \frac{1}{D^2}(dx_1^2 + dx_2^2 + \ldots + dx_n^2),$$

where

$$D = 1 + \frac{1}{4\kappa}(x_1^2 + x_2^2 + \ldots + x_n^2),$$

can be found by direct substitution in the equations of § 17. The amplitude itself is primary, being an amplitude in a plenary homaloidal space of $n+1$ dimensions; the explicit form is due to Riemann; and it is one of the very special forms of amplitudes which have a constant Riemann measure of curvature (sphericity) in superficial orientations*. But the procedure from the fundamental equations is simple.

* *G.F.D.*, vol. ii, § 442.

We take a new independent variable u, so that

$$D^2\left(\frac{ds}{du}\right)^2=\sum\{x_u^{(k)}\}^2=U^2,$$

where $x_u^{(k)}$ denotes $\dfrac{dx_k}{du}$; and the integral to minimise is

$$\int F\,du,$$

where F denotes U/D. The typical critical equation is

$$\frac{\partial F}{\partial x_k}-\frac{d}{du}\left\{\frac{\partial F}{\partial x_u^{(k)}}\right\}=0,$$

for all values of k. Now

$$\frac{\partial F}{\partial x_u^{(k)}}=\frac{1}{D}\cdot\frac{x_u^{(k)}}{U}=\frac{1}{D^2}\frac{x_u^{(k)}}{\dfrac{ds}{du}}=\frac{x_k'}{D^2},$$

and

$$\frac{\partial F}{\partial x_u}=-\frac{1}{D^2}\frac{\partial D}{\partial x_k}U=-\frac{x_k}{2\kappa D}\frac{ds}{du};$$

and therefore the critical equation becomes

$$\frac{x_k}{2\kappa D}+\frac{d}{ds}\left(\frac{x_k'}{D^2}\right)=0,$$

the quantities x' denoting derivatives with regard to the arc of the geodesic. The equations occur for the values $k=1, \ldots, n$; and they have the amplitudinal integral

$$x_1'^2+x_2'^2+\ldots+x_n'^2=D^2.$$

The last is a universal integral, devoid of arbitrary constants; and effectively, if retained, it reduces the number of independent critical equations to $n-1$.

Let i, j, k, denote any three different indices. We have the first integrals

$$\begin{aligned}x_jx_k'-x_kx_j'&=AD^2,\\ x_kx_i'-x_ix_k'&=BD^2,\\ x_ix_j'-x_jx_i'&=CD^2,\end{aligned}$$

where A, B, C, are arbitrary constants. Thus

$$Ax_i+Bx_j+Cx_k=0,$$

for any three of the parameters. But only a limited number of such relations can be independent; we therefore take

$$x_m=a_mx_1+c_mx_2,$$

for $m=3, \ldots, n$, the constants a and c being independent. These can be regarded as integrals of the system, leaving x_1 and x_2 for complete determination. We retain the integral

$$x_1x_2'-x_2x_1'=cD^2,$$

where c is an arbitrary constant, independent of the constants a_m, c_m; and it is necessary also to determine D.

We have

$$\begin{aligned} 4\kappa(D-1) &= x_1^2 + x_2^2 + \ldots + x_n^2 \\ &= Ax_1^2 + 2Hx_1x_2 + Bx_2^2, \end{aligned}$$

on using the adopted integrals, where

$$\begin{aligned} A-1 &= a_3^2 + \ldots + a_n^2, \\ H &= a_3c_3 + \ldots + a_nc_n, \\ B-1 &= c_3^2 + \ldots + c_n^2. \end{aligned}$$

We shall require $AB - H^2$. Its value V^2 is given by

$$V^2 = 1 + \sum_m a_m^2 + \sum_m c_m^2 + \sum_\lambda \sum_\mu (a_\lambda c_\mu - a_\mu c_\lambda)^2,$$

manifestly a positive quantity: thus V is real, and we shall take it to be the positive square root of this value of V^2.

Similarly, we have

$$\begin{aligned} D^2 &= x_1'^2 + x_2'^2 + \ldots + x_n'^2 \\ &= Ax_1'^2 + 2Hx_1'x_2' + Bx_2'^2, \end{aligned}$$

and

$$2\kappa D' = Ax_1x_1' + H(x_1x_2' + x_2x_1') + Bx_2x_2'.$$

Consequently

$$4\kappa(D-1)D^2 - 4\kappa^2 D'^2 = V^2(x_1x_2' - x_2x_1')^2 = V^2c^2D^4,$$

a differential equation of the first order. It is

$$4\kappa^2 D'^2 = -\kappa(2D - D^2)^2 + (\kappa - V^2c^2)D^4.$$

When κ is positive, reality of the variables requires $\kappa > V^2c^2$, and we take

$$V^2c^2 = \kappa \sin^2 a;$$

when κ is negative, there is no such limitation.

We shall discuss the case, when κ is positive; and it is found to require circular functions of the current parameter ultimately chosen. (The case, when κ is negative, involves hyperbolic functions.) With the assumed value of V, the equation is

$$\kappa\left(\frac{dD}{ds}\right)^2 = -D^2 + D^3 - \tfrac{1}{4}D^4 \sin^2 a.$$

Let a new variable t be taken, so that

$$t\kappa^{\frac{1}{2}} = s;$$

and as s has no specific origin of measurement, the arbitrary constant of integration can be absorbed into s. The integral is thus

$$\frac{2}{D} = 1 + \cos a \cos t,$$

so that the value of D is known in terms of the arc s; and it satisfies the equations

$$\frac{D^2}{4(D-1)} = \frac{1}{1 - \cos^2 a \cos^2 t},$$

$$D - 1 = \frac{1 - \cos a \cos t}{1 + \cos a \cos t}.$$

To determine x_1 and x_2, we have

$$\frac{d}{ds}\left(\frac{x_2}{x_1}\right)=c\frac{D^2}{x_1^{\,2}},$$

$$4\kappa(D-1)=Ax_1^2+2Hx_1x_2+Bx_2^2,$$

so that, if

$$x_2=x_1\tan q,\quad c\kappa^{\frac{1}{2}}=\gamma,$$

we have

$$\frac{\sec^2 q}{A+2H\tan q+B\tan^2 q}\frac{dq}{dt}=\frac{\gamma}{\kappa}\frac{D^2}{4(D-1)}=\frac{\gamma}{\kappa}\frac{1}{1-\cos^2\alpha\cos^2 t}.$$

Hence

$$\frac{1}{V}\tan^{-1}\frac{H+B\tan q}{V}=\frac{\gamma}{\kappa\sin\alpha}\tan^{-1}\left(\frac{\tan t}{\sin\alpha}\right)+\text{constant}.$$

But

$$V\gamma=Vc\kappa^{\frac{1}{2}}=\kappa\sin\alpha,$$

and it is convenient to take a new current parameter u for the arc such that

$$\tan u=\frac{\tan t}{\sin\alpha}=\frac{1}{\sin\alpha}\tan\left(\frac{s}{\kappa^{\frac{1}{2}}}\right);$$

and now we have

$$H+B\tan q=V\tan(u+\beta),$$

where β is an arbitrary constant of integration.

Thus the primitive of the intrinsic equations of the amplitudinal geodesics is composed of the set

$$x_m=a_mx_1+c_mx_2,\quad (m=3,\ldots,n),$$

$$x_1=\frac{2}{V}(B\kappa)^{\frac{1}{2}}(D-1)^{\frac{1}{2}}\cos(u+\beta),$$

$$x_2=\frac{2}{V}\left(\frac{\kappa}{B}\right)^{\frac{1}{2}}(D-1)^{\frac{1}{2}}\{V\sin(u+\beta)-H\cos(u+\beta)\},$$

$$D-1=\frac{1-\cos\alpha\cos t}{1+\cos\alpha\cos t},\quad \tan t=\sin\alpha\tan u,$$

and t is equal to $s\kappa^{-\frac{1}{2}}$.

Ex. 2. Obtain the corresponding equations for the amplitude of constant sphericity in any superficial orientation, when the sphericity is negative.

18. One property may be noted in passing. The element of arc, represented by

$$ds=\left(\sum_i\sum_j A_{ij}\,dx_i\,dx_j\right)^{\frac{1}{2}},$$

is taken with the positive sign of the radical; and therefore a zero value is a least magnitude if not a characteristic minimum. Let

$$\int f\,dt=\int\left(\sum_i\sum_j A_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt}\right)^{\frac{1}{2}}dt,$$

so that nul-lines * are given by $f=0$: the integral itself denoting the length of the amplitudinal arc between the limits of the curve resulting from the expression of the parameters in terms of t. The initial form of the critical equations characteristic of a geodesic is

$$\frac{\partial f}{\partial x_k}-\frac{d}{dt}\left(\frac{\partial f}{\partial x_k'}\right)=0,$$

where x_k' denotes $\dfrac{dx_k}{dt}$, and $k=1, \ldots, n$. Now

$$\frac{\partial f}{\partial x_k'}=\frac{1}{f}\left(\sum_a A_{ka}\frac{dx_a}{dt}\right),$$

and therefore

$$-\frac{d}{dt}\left(\frac{\partial f}{\partial x_k'}\right)=\frac{1}{f^2}\left(\sum_a A_{ka}\frac{dx_a}{dt}\right)\frac{df}{dt}-\frac{1}{f}\frac{d}{dt}\left(\sum_a A_{ka}\frac{dx_a}{dt}\right).$$

But

$$\frac{d}{dt}\left(\sum_a A_{ka}\frac{dx_a}{dt}\right)=\sum_a\left(A_{ka}\frac{d^2x_a}{dt^2}\right)+\sum_a\sum_\beta\frac{\partial A_{ka}}{\partial x_\beta}\frac{dx_a}{dt}\frac{dx_\beta}{dt};$$

and

$$\frac{\partial f}{\partial x_k}=\frac{1}{2f}\sum_i\sum_j\left(\frac{\partial A_{ij}}{\partial x_k}\frac{dx_i}{dt}\frac{dx_j}{dt}\right).$$

When these values are substituted, the typical critical equation becomes

$$\frac{1}{f}\left(\sum_a A_{ka}\frac{dx_a}{dt}\right)\frac{df}{dt}=\sum_a\left(A_{ka}\frac{d^2x_a}{dt^2}\right)$$
$$+\sum_a\sum_\beta\left(\frac{\partial A_{ka}}{\partial x_\beta}\frac{dx_a}{dt}\frac{dx_\beta}{dt}\right)-\tfrac{1}{2}\sum_i\sum_j\left(\frac{\partial A_{ij}}{\partial x_k}\frac{dx_i}{dt}\frac{dx_j}{dt}\right).$$

In the second line, the complete coefficient of the combination $\dfrac{dx_\lambda}{dt}\dfrac{dx_\mu}{dt}$ is

$$\begin{aligned}&=\frac{\partial A_{k\lambda}}{\partial x_\mu}+\frac{\partial A_{k\mu}}{\partial x_\lambda}-\frac{\partial A_{\lambda\mu}}{\partial x_k}\\&=2\sum_{a=1}A_{ka}\{\lambda\mu, a\},\end{aligned}$$

by the formula on p. 25; and therefore the right-hand side can be expressed in the form

$$\sum_a A_{ka}\left[\frac{d^2x_a}{dt^2}+\sum_\lambda\sum_\mu\{\lambda\mu, a\}\frac{dx_\lambda}{dt}\frac{dx_\mu}{dt}\right].$$

Thus the typical critical equation becomes

$$\left(\sum_a A_{ka}\frac{dx_a}{dt}\right)\frac{df}{dt}=f\sum_a A_{ka}\left[\frac{d^2x_a}{dt^2}+\sum_\lambda\sum_\mu\{\lambda\mu, a\}\frac{dx_\lambda}{dt}\frac{dx_\mu}{dt}\right],$$

holding for $k=1, \ldots, n$. Two inferences can be drawn, among others, by specialising the arbitrarily assumed variable t along the curve.

* There are two such lines in every superficial orientation in the amplitude.

In the first place, we deal with real curves, and take df to denote the arc of the curve, so that now

$$f=(\sum\sum A_{ij}x_i'x_j')^{\frac{1}{2}}=1,$$

by the definition of the arc; and therefore $\frac{df}{dt}=0$, on this assumption. The equation becomes

$$\sum_a A_{ka}[x_a''+\sum_\lambda\sum_\mu\{\lambda\mu, a\}\,x_\lambda'x_\mu']=0,$$

for $k=1, \ldots, n$; on resolution of the n equations, we obtain the customary characteristic equations (§ 17) of amplitudinal geodesics.

In the second place, consider the nul-lines of the amplitude which are given by $f=0$, the equation holding along a curve so that, also,

$$\frac{df}{dt}=0.$$

The foregoing critical equations are satisfied by these values; and we infer that the equations, which are characteristic of the geodesics in any amplitude, are satisfied by the nul-lines of the amplitude.

19. Because geodesics are of significant importance throughout the geometry of configurations of all types, one general proposition may be stated at this stage in its comprehensive form. It is an inference from the existence-theorem (often called Cauchy's theorem) concerning the integrals of a system of ordinary differential equations in any number of dependent variables.*

The intrinsic differential equations of a geodesic in an amplitude of n dimensions are the set

$$\frac{d^2x_m}{ds^2}=-\sum_i\sum_j\{ij, m\}\frac{dx_i}{ds}\frac{dx_j}{ds}, \quad (m=1, \ldots, n),$$

where the summation on the right-hand side is for the values $i, j, =1, \ldots, n$, taken independently of one another. (When the configuration, to which a geodesic belongs, is of $n-r$ dimensions and is contained within the postulated amplitude of x dimensions, it can be represented analytically by means of n appropriate equations

$$x_m=f_m(z_1, z_2, \ldots, z_{n-r}),$$

where $z_1, \ldots, z_{n-r}$, are $n-r$ independent parameters; and then the intrinsic equations of its geodesics are the $n-r$ equations in the variables z_μ of the same type as the preceding set in the n variables x_m. The cited set can therefore be regarded as universally representative, for the present purpose.)

* For the existence-theorem, see my *Theory of Differential Equations*, vol. ii, chap. ii.

We know (§ 17) that these n equations are equivalent to only $n-1$ independent equations when the permanent arc-relation

$$\sum_i \sum_j A_{ij} \frac{dx_i}{ds} \frac{dx_j}{ds} = 1$$

is retained. We therefore may regard this arc-relation as an integral of the system of equations; the "initial values" of the magnitudes, to which they are assigned, will then be subject to a corresponding relation.

The set of n equations may be modified to a canonical set of $2n$ equations, in the form

$$\frac{dx_m}{ds} = x_m', \quad \frac{dx_m'}{ds} = -\sum_i \sum_j \{ij, m\} x_i' x_j', \quad (m=1, \ldots, n),$$

involving $2n$ dependent variables $x_1, \ldots, x_n, x_1', \ldots, x_n'$, with s as the independent variable. By Cauchy's theorem, such a set of equations possesses a set of integrals for a range of the amplitude in which the magnitudes $\{ij, m\}$ are regular, regarded as analytic functions of their arguments (effectively a range of the amplitude that is free from geometrical singularities of any type); and the integrals are characterised by the properties:

(i), the values of $x_1, \ldots, x_m, x_1', \ldots, x_m'$, are regular functions of s within the range:

(ii), for an assigned value of s, the variables x_m assume assigned values c_m (for $m=1, \ldots, n$) such that, at those values and in their vicinity in the range, the magnitudes $\{ij, m\}$ are regular functions of their arguments; effectively, this is a mode of determining an origin of measurement of s along a geodesic:

(iii), for that assigned value of s, the variables x_m' assume assigned values $t_1, \ldots, t_n$, respectively, such that

$$\sum_i \sum_j A_{ij}(c_1, \ldots, c_n) t_i t_j = 1,$$

a relation in accordance with the necessary arc-relation of the amplitude:

(iv), the integrals thus obtained are unique, i.e., they are the only integrals determined by the conditions; and they are regular functions of the independent variable s.

When this analytical theorem is interpreted geometrically, by reference to the geometric significance of the system of differential equations, we infer that, in any configuration, a geodesic is uniquely determinate by the assignment of an initial position $(c_1, \ldots, c_n)$ and the assignment of its direction at that initial position by direction-variables in the configuration. Further, any number of geodesics can be drawn through any point O in the configuration; each geodesic

so drawn is unique through the assignment, at O, of its direction by means of a set of direction-variables satisfying the arc-relation at the point.

The result, in the combined circumstances as stated, is of validity for all configurations, whether a general n-fold amplitude, or specific amplitudes such as surfaces, regions, domains; and, hereafter, it will be assumed systematically, without any explicit re-statement of the distinctive uniqueness of a geodesic as determined by assigned initial conditions.

Tangent homaloid of the amplitude and prime normals of geodesics.

20. We now shall prove that, at a point O on an amplitudinal geodesic, the direction of its prime normal coincides with the limiting position of a perpendicular drawn, from a contiguous point Q of the geodesic upon the tangential n-fold homaloid of the amplitude at O, the limit being attained as Q coincides with O. In establishing the result, the magnitude as well as the spatial direction-cosines of the radius of circular curvature of the geodesic will be obtained.

A line, drawn through the point O and touching the amplitude at O, is represented by the set of equations

$$\bar{y}_m - y_m = \lambda y_m' = \frac{\partial y_m}{dx_1}\lambda x_1' + \frac{\partial y_m}{\partial x_2}\lambda x_2' + \dots + \frac{\partial y_m}{\partial x_n}\lambda x_n',$$

for all the values $m=1, \dots, n+n'$, corresponding to the dimensions of the homaloidal plenary space, the typical variable $\bar{y}_m$ being current along the line while y_m is the typical variable of the amplitudinal point O. Hence all points on all such lines, and therefore all such lines themselves, are contained in the homaloid represented by the n' independent equations in the set

$$\left\| \bar{y}_m - y_m, \quad \frac{\partial y_m}{\partial x_1}, \;\dots\dots, \; \frac{\partial y_m}{\partial x_n} \right\| = 0;$$

and this homaloid manifestly is of n dimensions. Any point in the homaloid can be represented by the equations

$$\bar{y}_m - y_m = a_1 \frac{\partial y_m}{\partial x_1} + \dots\dots + a_n \frac{\partial y_m}{\partial x_n},$$

where $a_1, \dots, a_n$ can be regarded as the (linear) parameters of this n-fold homaloid. Accordingly, these equations (in either of the two equivalent sets) are the equations of the n-fold homaloid touching the amplitude at O.

Moreover, every direction, which passes through O and lies in the homaloid, is given by the equations

$$l_m = \lambda_1 \frac{\partial y_m}{\partial x_1} + \dots\dots + \lambda_n \frac{\partial y_m}{\partial x_n},$$

for its spatial direction-cosines, where $m=1, \dots, n+n'$, and $\lambda_1, \dots, \lambda_n$, are para-

meters. In particular, the tangent line to the curve OQ is given by the value $\lambda_i = \frac{dx_i}{ds} = x_i'$, thus providing the spatial direction-cosines of the tangent in the form

$$y_m' = \frac{\partial y_m}{\partial x_1} x_1' + \ldots\ldots + \frac{\partial y_m}{\partial x_n} x_n'$$

for all the values of m.

In the amplitude we take a point Q with space-coordinates $\eta_1, \eta_2, \eta_3, \ldots$, contiguous to the point O with space-coordinates $y_1, y_2, y_3, \ldots$, along the geodesic through O drawn in the direction specified by $x_1', \ldots, x_n'$; and we denote by t the small arc-distance OQ measured along that geodesic. From Q let a perpendicular be dropped upon the tangent n-fold homaloid; let the intercept length of the perpendicular be denoted by P, its spatial direction-cosines by $Y_1, Y_2, Y_3, \ldots$, to be represented typically by Y; and let the coordinates of the foot of the perpendicular, in the n-fold homaloid, be $\bar{y}_1, \bar{y}_2, \bar{y}_3, \ldots$, where, as in general,

$$\bar{y}_m = y_m + a_1 \frac{\partial y_m}{\partial x_1} + a_2 \frac{\partial y_m}{\partial x_2} + \ldots + a_n \frac{\partial y_m}{\partial x_m},$$

the appropriate values of the parameters $a_1, \ldots, a_n$, in which have to be determined.

In order that the line joining $\eta_1, \eta_2, \eta_3, \ldots$ to $\bar{y}_1, \bar{y}_2, \bar{y}_3, \ldots$ may be perpendicular to the tangent homaloid, the magnitude

$$\sum_m (\eta_m - \bar{y}_m)^2,$$

$$= \sum_m \left(\eta_m - y_m - \sum_r a_r \frac{\partial y_m}{\partial x_r} \right)^2,$$

must be a minimum for all possible values of the parameters $a_1, \ldots, a_n$; the critical conditions, necessary and sufficient to satisfy this demand, are

$$\sum_m \frac{\partial y_m}{\partial x_k} \left(\eta_m - y_m - \sum_r a_r \frac{\partial y_m}{\partial x_r} \right) = 0,$$

for the values $k = 1, \ldots, n$.

These n equations of condition can be taken in the form

$$\sum_m \frac{\partial y_m}{\partial x_k} (\eta_m - \bar{y}_m) = 0.$$

But we have

$$\eta_m - \bar{y}_m = Y_m P,$$

for each of the values of m; and therefore

$$\sum_m Y_m \frac{\partial y_m}{\partial x_k} = 0,$$

for the n values of k. Accordingly, the direction of the specified perpendicular is

at right angles to every direction in the tangent homaloid : it is therefore orthogonal to that homaloid.

Returning to the first form of the critical equations, we can take the typical equation in the form

$$\sum_m \left\{\frac{\partial y_m}{\partial x_k}\left(\sum_r a_r \frac{\partial y_m}{\partial x_r}\right)\right\} = \sum_m \frac{\partial y_m}{\partial x_k}(\eta_m - y_m),$$

that is,

$$a_1 A_{1k} + a_2 A_{2k} + \dots + a_n A_{nk} = \sum_m \frac{\partial y_m}{\partial x_k}(\eta_m - y_m),$$

holding for $k=1, \dots, n$. Now for any curve through O, drawn in the amplitude through the direction $x_1', \dots, x_n'$, a point $\eta_1, \eta_2, \eta_3, \dots$ contiguous to O at a small arc-distance t along the curve, we have

$$\begin{aligned}\eta_m &= y_m + y_m' t + \tfrac{1}{2} y_m'' t^2 + \dots \\ &= y_m + y_m' t + \tfrac{1}{2} y_m'' t^2,\end{aligned}$$

(up to the second power of the small quantity t inclusive), for all the space-coordinates. Hence, after substitution in the expression

$$\sum_m \frac{\partial y_m}{\partial x_k}(\eta_m - y_m)$$

on the right-hand side, the coefficient of the first power of t

$$\begin{aligned}&= \sum_m \frac{\partial y_m}{\partial x_k} y_m' \\ &= \sum_m \frac{\partial y_m}{\partial x_k}\left(\frac{\partial y_m}{\partial x_1} x_1' + \frac{\partial y_m}{\partial x_2} x_2' + \dots + \frac{\partial y_m}{\partial x_n} x_n'\right) \\ &= A_{1k} x_1' + A_{2k} x_2' + \dots + A_{nk} x_n' ;\end{aligned}$$

and therefore the critical equations, with the second power of t retained, become

$$(a_1 - x_1' t) A_{1k} + (a_2 - x_2' t) A_{2k} + \dots + (a_n - x_n' t) A_{nk} = \tfrac{1}{2} t^2 \sum_m \frac{\partial y_m}{\partial x_k} y_m'',$$

still for the values $k=1, \dots, n$. Again, for the variation of space-coordinates along any curve in the amplitude, we have

$$y_m'' = \sum_i \left(\frac{\partial y_m}{\partial x_i} x_i''\right) + \sum_i \sum_j \frac{\partial^2 y_m}{\partial x_i \partial x_j} x_i' x_j',$$

with the now customary double-summation range for i and j, the values of the quantities x_i'' being settled by the law of the curve ; and therefore we have

$$\sum_m \frac{\partial y_m}{\partial x_k} x_m'' = \sum_i \sum_m \frac{\partial y_m}{\partial x_k}\frac{\partial y_m}{\partial x_i} x_i'' + \sum_i \sum_j \sum_m \frac{\partial y_m}{\partial x_k}\frac{\partial^2 y_m}{\partial x_i \partial x_j} x_i' x_j'.$$

Now

$$\sum_m \frac{\partial y_m}{\partial x_k}\frac{\partial y_m}{\partial x_i} = A_{ik}, \quad \sum_m \frac{\partial y_m}{\partial x_k}\frac{\partial^2 y_m}{\partial x_i \partial x_j} = [ij, k],$$

on the introduction of the Christoffel symbol $[ab, c]$; and therefore the critical equations, still up to the second power of t inclusive, can be written

$$(a_1 - x_1't - \tfrac{1}{2}x_1''t^2)A_{1k} + \ldots\ldots + (a_n - x_n't - \tfrac{1}{2}x_n''t^2)A_{nk}$$
$$= \tfrac{1}{2}t^2 \sum_i \sum_j [ij, k]\, x_i' x_j',$$

for $k=1, \ldots, n$. When these n equations are resolved to give the individual quantities $a_r - x_r't - \frac{1}{2}x_r''t^2$, which occur linearly, we find

$$a_r - x_r't - \tfrac{1}{2}x_r''t^2 = \tfrac{1}{2}t^2 \sum_i \sum_j \{ij, r\}\, x_i' x_j',$$

on the introduction of the Christoffel symbol $\{ab, c\}$, the result holding for the values $r=1, \ldots, n$.

Accordingly, we take

$$\begin{aligned} Y_m P &= \eta_m - \bar{y}_m \\ &= \eta_m - y_m - \sum_r \left(a_r \frac{\partial y_m}{\partial x_r}\right) \\ &= t y_m' + \tfrac{1}{2}t^2 y_m'' - \sum_r \left(a_r \frac{\partial y_m}{\partial x_r}\right), \end{aligned}$$

accurately up to the second power of the small quantity t inclusive. In this expression let the values of the quantities a_r, which have just been obtained, be substituted. Then the total coefficient of the first power of t on the right-hand side

$$= y_m' - \sum_r \left(x_r' \frac{\partial y_m}{\partial x_r}\right),$$

which is equal to zero: that is, the term involving the first power of t disappears. Next, the total coefficient of $\frac{1}{2}t^2$ on the right-hand side

$$= y_m'' - \sum_r \left(\frac{\partial y_m}{\partial x_r} x_r''\right) - \sum_r \sum_i \sum_j \frac{\partial y_m}{\partial x_r} \{ij, r\}\, x_i' x_j';$$

and, whatever be the curve, we have

$$y_m'' - \sum_r \left(\frac{\partial y_m}{\partial x_r} x_r''\right) = \sum_i \sum_j \frac{\partial^2 y_m}{\partial x_i\, \partial x_j} x_i' x_j',$$

so that the total coefficient of $\frac{1}{2}t^2$, on the right-hand side of the expression for $Y_m P$,

$$= \sum_i \sum_j \frac{\partial^2 y_m}{\partial x_i\, \partial x_j} x_i' x_j' - \sum_r \sum_i \sum_j \frac{\partial y_m}{\partial x_r} \{ij, r\}\, x_i' x_j'.$$

We write

$$\eta_{ij}^{(m)} = \frac{\partial^2 y_m}{\partial x_i\, \partial x_j} - \sum_r \{ij, r\} \frac{\partial y_m}{\partial x_r}$$

for all admissible values of m and of i, j; we use η_{ij} to denote the typical member

of the set $\eta_{ij}{}^{(m)}$ to be associated with the typical variable denoted by y; and we now have

$$Y_m P = \tfrac{1}{2}t^2 \sum_i \sum_j \eta_{ij}{}^{(m)} x_i' x_j'.$$

These quantities η_{ij} are the quantities used in § 14 for the expression there given for the Riemann four-index symbol.

Now let a plane be drawn through the point O, the point Q, and the foot of the perpendicular from Q on the homaloid, so that the plane contains a tangent line to the homaloid and a perpendicular upon that tangent line. It is (necessarily) the osculating plane of the element of curved section at O of the homaloid by the plane; and if ρ denote the radius of circular curvature of that plane section at O, we have

$$2P\rho = t^2,$$

in the limit as Q draws to coincidence with O along the specified arc. Consequently, in that limit, we have

$$\frac{Y_m}{\rho} = \sum_i \sum_j \eta_{ij}{}^{(m)} x_i' x_j',$$

for all the $n+n'$ values of m.

21. Next, consider the radius of circular curvature ρ_1 of the amplitudinal geodesic and its direction-cosines in the plenary space. The latter are

$$\rho_1 y_m''$$

for all the values of m. For any curve, we have

$$y_m'' = \sum_r \frac{\partial y_m}{\partial x_r} x_r'' + \sum_i \sum_j \frac{\partial^2 y_m}{\partial x_i\, \partial x_j} x_i' x_j';$$

also, along a geodesic of the amplitude,

$$x_r'' = -\sum_i \sum_j \{ij, r\} x_i' x_j'$$

for all the n values of r; hence, along a geodesic in the amplitude in the direction $x_1', \ldots, x_n'$, we have

$$\begin{aligned} y_m'' &= \sum_i \sum_j \left[\frac{\partial^2 y_m}{\partial x_i\, \partial x_j} - \sum_r \frac{\partial y_m}{\partial x_r} \{ij, r\} \right] x_i' x_j' \\ &= \sum_i \sum_j \eta_{ij}{}^{(m)} x_i' x_j'. \end{aligned}$$

If then $l_1, l_2, l_3, \ldots$ denote the direction-cosines of the geodesic in the direction $x_1', \ldots, x_n'$, so that $l_m = \rho_1 y_m''$, the relations

$$\frac{Y_m}{\rho} = y_m'' = \frac{l_m}{\rho_1},$$

for all the $n+n'$ values of m, subsist between direction-cosines of the prime normal of the geodesic and the limiting position of the perpendicular from Q on

the tangent homaloid as Q tends to coincidence with O along the geodesic. Hence we have

$$\rho_1 = \rho, \quad Y_m = l_m,$$

the latter relation holding for all the values of m. These last equations establish the proposition, enunciated in § 8, concerning the coincidence of the directions, of the foregoing perpendicular and of the prime normal of the geodesic—a result that dominates the consideration of the further curvatures of the geodesic.

Moreover, we have $\rho_1 = \rho$; or the circular curvature, at O, of the section of the amplitude by the plane, through the tangent and the prime normal of the geodesic, is equal to the circular curvature of the geodesic. This normal section does not recur in investigations that follow: the only characteristic of importance, which it possesses, is its circular curvature, being the same as that of the geodesic; all its other curvatures are zero, because it is a plane curve: and this circular curvature, as a property of the geodesic, remains. Accordingly, we pass from this section of the amplitude; and there remain the results, relating to the spatial direction-cosines of the prime normal of the geodesic and to the circular curvature of the geodesic, as contained in the set of equations typified by

$$\frac{Y_m}{\rho} = \sum_i \sum_j \eta_{ij}^{(m)} x_i' x_j',$$

together with the property that the prime normal is orthogonal to the tangent manifold of the amplitude, expressed by the relations

$$\sum Y_m \frac{\partial y_m}{\partial x_r} = 0,$$

holding for the values $r = 1, \ldots, n$.

NOTE 1. As already remarked (§§ 14, 15), these quantities η_{ij}, occurring as coefficients of the parametric combinations $x_i' x_j'$ in the preceding equations, are subsidiary also to the expression of the various Riemann symbols according to the general law

$$(ij, kl) = \sum (\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}),$$

the summation on the right-hand side extending over the $n + n'$ dimensions of the homaloidal plenary space.

In particular, a surface has a single Riemann four-index symbol; its value can be expressed in the form

$$(12, 12) = \sum (\eta_{11}\eta_{22} - \eta_{12}^2).$$

A region has six (non-vanishing) and linearly independent Riemann four-index symbols; their values can be expressed in the forms

$$\left.\begin{array}{ll} (23, 23) = \sum (\eta_{22}\eta_{33} - \eta_{23}^2), & (12, 31) = \sum (\eta_{12}\eta_{13} - \eta_{11}\eta_{23}) \\ (31, 31) = \sum (\eta_{33}\eta_{11} - \eta_{31}^2), & (23, 12) = \sum (\eta_{23}\eta_{21} - \eta_{22}\eta_{31}) \\ (12, 12) = \sum (\eta_{11}\eta_{22} - \eta_{12}^2), & (31, 23) = \sum (\eta_{31}\eta_{32} - \eta_{33}\eta_{12}) \end{array}\right\}.$$

NOTE 2. When the n-fold amplitude is primary, that is, such as to have its homaloidal plenary space of only one dimension greater than itself, then $n'=1$; and the range of values of m is $1, \dots, n+1$. The quantities $Y_1, \dots, Y_{n+1}$, are then proportional to the determinants in the array

$$\left\| \begin{matrix} \frac{\partial y_1}{\partial x_1}, & \frac{\partial y_2}{\partial x_1}, & \cdots\cdots, & \frac{\partial y_{n+1}}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2}, & \frac{\partial y_2}{\partial x_2}, & \cdots\cdots, & \frac{\partial y_{n+1}}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_1}{\partial x_n}, & \frac{\partial y_2}{\partial x_n}, & \cdots\cdots, & \frac{\partial y_{n+1}}{\partial x_n} \end{matrix} \right\| ;$$

in fact,

$$\Omega^{\frac{1}{2}} Y_1 = \frac{\partial(y_2, \dots, y_{n+1})}{\partial(x_1, \dots, x_n)},$$

and so for the others. Consequently, the direction-cosines $Y_1, \dots, Y_{n+1}$, of the prime normal of a geodesic in a primary n-fold amplitude, do not involve the parametric quantities $x_1', \dots, x_n'$, specifying the direction of the geodesic; and so all the geodesics of a primary amplitude have a common prime normal. The simplest examples are provided by a surface in triple homaloidal space and by a region in quadruple homaloidal space.

But the property is not possessed by geodesics of an amplitude when the homaloidal plenary space is of dimensions which exceed, by more than one unit, those of the amplitude. Thus the prime normals of a surface, in quadruple homaloidal space, merely lie in a plane orthogonal to the surface; and as will appear for a surface, in a plenary homaloidal space of more than four dimensions, the prime normals of superficial geodesics lie in a flat orthogonal to the surface (§ 105).

It may be added that, for a primary n-fold amplitude, the aggregate, composed of any n guiding lines of the tangential n-fold homaloid of the amplitude (such as the directions of the parametric curves), together with the common direction of the prime normals of all amplitudinal geodesics through a point, constitute a set of axes of reference for the plenary homaloidal space; these axes, however, are usually not orthogonal to one another, at least in the form initially suggested in relation to the tangent homaloid of the amplitude. But an orthogonal system of spatial axes is provided by the orthogonal organic frame of a geodesic.

Derivatives of the quantities $\{ij, k\}$: quantities $\{ijk, \mu\}$.

22. The Christoffel symbols involve second-order derivatives of the space-coordinates with respect to the parameters of the amplitude; and in § 14, during a construction of the Riemann four-index symbol, combinations were made which

removed all the third-order derivatives of those coordinates that occurred incidentally. Yet third-order derivatives must be taken into account. Thus they are involved implicitly in some of the expressions for the successive curvatures in the Frenet equations: they likewise are necessary for the discussion of ranges in the amplitude, not merely along characteristic curves such as geodesics, but in the immediate general vicinity of any point. For the present, it will suffice to consider them as arising, in certain combinations analogous in form to the Christoffel symbols $\{ij, l\}$, in relations deduced by differentiation from the intrinsic equations of geodesics, and in expressions for the grades of curvature of amplitudinal geodesics in the succession of curvatures established in the Frenet equations.

We begin with the definition of certain symbols $\begin{bmatrix} ijk \\ l \end{bmatrix}$ according to the general relations

$$\sum_m \frac{\partial y_m}{\partial x_\mu} \frac{\partial^3 y_m}{\partial x_i \partial x_j \partial x_k} = \sum_l A_{\mu l} \begin{bmatrix} ijk \\ l \end{bmatrix},$$

for all values $i, j, k, \mu, = 1, \ldots, n$, taken independently of l, the l-summation being for the same range, and the m-summation for $m = 1, \ldots, n+n'$. Obviously the value of the symbol must be unaltered by the interchange of the indices i, j, k, in any expression obtained for that value. When we differentiate the analogous relation (§ 12) for the Christoffel symbols, in the form

$$\sum_m \frac{\partial y_m}{\partial x_\mu} \frac{\partial^2 y_m}{\partial x_i \partial x_j} = \sum_l A_{\mu l} \{ij, l\},$$

with respect to a parameter x_k, we have

$$\begin{aligned} \sum \frac{\partial^2 y}{\partial x_\mu \partial x_k} \frac{\partial^2 y}{\partial x_i \partial x_j} + \sum \frac{\partial y}{\partial x_\mu} \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} &= \sum_l A_{\mu l} \frac{\partial}{\partial x_k} \{ij, l\} + \sum_l \{ij, l\} \frac{\partial A_{\mu l}}{\partial x_k} \\ &= \sum_l A_{\mu l} \frac{\partial}{\partial x_k} \{ij, l\} + \sum_l \sum_p \{ij, l\} [A_{lp}\{k\mu, p\} + A_{\mu p}\{kl, p\}] \end{aligned}$$

by the formula in § 12. By a mere interchange of the equivalent indices l and p, we have

$$\sum_l \sum_p \{ij, l\} A_{\mu p}\{kl, p\} = \sum_p \sum_l \{ij, p\} A_{\mu l}\{kp, l\};$$

when this is used, the right-hand side can be changed to

$$\sum_l A_{\mu l} \left[\frac{\partial}{\partial x_k} \{ij, l\} + \sum_p \{ij, p\} \{kp, l\} \right] + \sum_l \sum_p A_{lp} \{ij, l\} \{k\mu, p\}.$$

We have had (§ 14) the relation

$$\sum \frac{\partial^2 y}{\partial x_\mu \partial x_k} \frac{\partial^2 y}{\partial x_i \partial x_j} - \sum_l \sum_p A_{lp} \{ij, l\} \{k\mu, p\} = \sum \eta_{\mu k} \eta_{ij};$$

and thus the foregoing derived equation becomes

$$\sum \eta_{\mu k}\eta_{ij} + \sum \frac{\partial y}{\partial x_\mu} \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = \sum_l A_{\mu l}\left[\frac{\partial}{\partial x_k}\{ij, l\} + \sum_p \{ij, p\}\{kp, l\}\right].$$

For the second summation on this left-hand side, let the preceding equivalent expression defining the quantities $\begin{bmatrix} ijk \\ l \end{bmatrix}$ be substituted; and then let the n such equations, for $\mu = 1, \ldots, n$, be resolved so as to express the new magnitudes for the n values of l. We thus find

$$\begin{bmatrix} ijk \\ l \end{bmatrix} = \frac{\partial}{\partial x_k}\{ij, l\} + \sum_p \{ij, p\}\{kp, l\} - \frac{1}{\Omega}\sum_\mu a_{\mu l}\left(\sum \eta_{\mu k}\eta_{ij}\right),$$

a relation holding for all the values i, j, k, l, chosen independently of one another from the set $1, \ldots, n$.

On the left-hand side, the three symbols i, j, k, can be interchanged without affecting the value of the symbol; hence we have also

$$\begin{bmatrix} ijk \\ l \end{bmatrix} = \frac{\partial}{\partial x_i}\{jk, l\} + \sum_p \{jk, p\}\{ip, l\} - \frac{1}{\Omega}\sum_\mu a_{\mu l}\left(\sum \eta_{\mu i}\eta_{jk}\right),$$

and also

$$\begin{bmatrix} ijk \\ l \end{bmatrix} = \frac{\partial}{\partial x_j}\{ki, l\} + \sum_p \{ki, p\}\{jp, l\} - \frac{1}{\Omega}\sum_\mu a_{\mu l}\left(\sum \eta_{\mu j}\eta_{ki}\right).$$

These distinct values of the third-order symbol provide relations between first derivatives of the Christoffel symbols $\{ab, c\}$. In particular, the equality of the second and the third expressions yields the relation

$$\begin{aligned} \frac{\partial}{\partial x_j}\{ki, l\} - \frac{\partial}{\partial x_i}\{kj, l\} + \sum_p [\{ki, p\}\{jp, l\} - \{kj, p\}\{ip, l\}] \\ = \frac{1}{\Omega}\sum_\mu a_{\mu l}(\eta_{\mu j}\eta_{ik} - \eta_{\mu i}\eta_{kj}) \\ = \frac{1}{\Omega}\sum_\mu a_{\mu l}(ij, k\mu), \end{aligned}$$

on introducing the Riemann four-index symbol. Sometimes in this connection a symbol, called the *Riemann four-index symbol of the second kind* and denoted by

$$\{kl, ij\},$$

is introduced to represent the quantity on the left-hand side (and then the earlier four-index symbol is said to be of the first kind). For the two sets of symbols, we have the relations

$$\{kl, ij\} = \frac{1}{\Omega}\sum_\mu a_{\mu l}(k\mu, ij),$$

$$(kl, ij) = \sum_\mu A_{l\mu}\{k\mu, ij\}.$$

Whichever form be adopted, the relation gives a value of the difference between the first parametric derivatives of two Christoffel symbols requiring only four indices i, j, k, l, in all: and the difference, thus expressed, will be found important in constructing a significance for the Riemann measure of curvature of an amplitude.

It is to be noted that the relation

$$\{kl, ij\} + \{il, jk\} + \{jl, ki\} = 0$$

follows immediately from the corresponding relation (§ 16) for the symbol (ab, cd).

For example, appertaining to a region with its arc-element represented in the form

$$ds^2 = (A, B, C, F, G, H \between dp, dq, dr)^2,$$

we have quantities Γ_{ij}, Δ_{ij}, Θ_{ij}, such that (§ 16)

$$\Gamma_{ij} = \{ij, 1\}, \quad \Delta_{ij} = \{ij, 2\}, \quad \Theta_{ij} = \{ij, 3\}.$$

To compare with the general form, we take

$$dp = dx_1, \quad dq = dx_2, \quad dr = dx_3,$$

and we have

$$\begin{aligned} a_{11}, a_{12}, a_{13}, &= a, h, g, \\ a_{21}, a_{22}, a_{23}, &= h, b, f, \\ a_{31}, a_{32}, a_{33}, &= g, f, c. \end{aligned}$$

Then the general formula in the text provides the relations

$$\begin{aligned} \frac{\partial \Gamma_{ki}}{\partial x_j} - \frac{\partial \Gamma_{kj}}{\partial x_i} &= \frac{1}{\Omega}[a(ij, k1) + h(ij, k2) + g(ij, k3)], \\ \frac{\partial \Delta_{ki}}{\partial x_j} - \frac{\partial \Delta_{kj}}{\partial x_i} &= \frac{1}{\Omega}[h(ij, k1) + b(ij, k2) + f(ij, k3)], \\ \frac{\partial \Theta_{ki}}{\partial x_j} - \frac{\partial \Theta_{kj}}{\partial x_i} &= \frac{1}{\Omega}[g(ij, k1) + f(ij, k2) + c(ij, k3)], \end{aligned}$$

where, on the right-hand sides, the symbols (ij, kl) which do not vanish are the six Riemann symbols

$$(23, 23), \quad (31, 31), \quad (12, 12), \quad (12, 31), \quad (23, 12), \quad (31, 23),$$

of the region as stated in § 16.

23. The preceding relations, however, only express the differences between derivatives of a couple of Christoffel symbols $\{ab, c\}$, while it is desirable to have the value of each such derivative by itself. These values can be established in connection with the expressions for the magnitudes x_μ''', for $\mu = 1, \ldots, n$, along a geodesic in the amplitude; and they can be deduced from the relations

$$x_\mu'' = -\sum_i \sum_j \{ij, \mu\} x_i' x_j',$$

being the intrinsic parametric equations of the geodesic.

When this typical relation is differentiated along the geodesic, we have

$$x_\mu''' = -\sum_i \sum_j x_i' x_j' \left[\sum_k x_k' \frac{\partial}{\partial x_k} \{ij, \mu\} \right]$$
$$+ \sum_l \sum_m \{lm, \mu\} x_m' \left[\sum_a \sum_b \{ab, l\} x_a' x_b' \right]$$
$$+ \sum_l \sum_m \{lm, \mu\} x_l' \left[\sum_\alpha \sum_\beta \{\alpha\beta, m\} x_\alpha' x_\beta' \right],$$

all the summations being for the range 1, ... , n, for each of the integers independently of the others. The right-hand side manifestly is a homogeneous expression, of the third combined order in the n variables x'; we therefore write

$$x_\mu''' = -\sum_i \sum_j \sum_k \{ijk, \mu\} x_i' x_j' x_k',$$

where the summation is for all the values 1, ... , n, of i, j, k, independently of one another and where the symbol $\{ijk, \mu\}$ is unaltered by interchanges of i, j, k, among one another. Thus the total coefficient of $x_1'^3$ is $-\{111, \mu\}$; the total coefficient of $x_1'^2 x_2'$ is $-3\{112, \mu\}$; the total coefficient of $x_1' x_2' x_3'$ is $-6\{123, \mu\}$; and so for all the terms.

Now consider the most general type of term, that which involves the combination $x_i' x_j' x_k'$. In the final expression, its coefficient is

$$-6\{ijk, \mu\}.$$

In the derived value for x_μ''', the total coefficient is

$$-2\frac{\partial}{\partial x_k}\{ij, \mu\} - 2\frac{\partial}{\partial x_i}\{jk, \mu\} - 2\frac{\partial}{\partial x_j}\{ki, \mu\}$$

from the first line,

$$2\sum_p \{ij, p\}\{kp, \mu\} + 2\sum_p \{jk, p\}\{ip, \mu\} + 2\sum_p \{ki, p\}\{jp, \mu\}$$

from the second line, and this same quantity

$$2\sum_p \{ij, p\}\{kp, \mu\} + 2\sum_p \{jk, p\}\{ip, \mu\} + 2\sum_p \{ki, p\}\{jp, \mu\}$$

also from the third line. Now (§ 22) we had a magnitude

$$\begin{bmatrix} lmn \\ \theta \end{bmatrix} = \frac{\partial}{\partial x_n}\{lm, \theta\} + \sum_p \{lm, p\}\{xp, \theta\} - \frac{1}{\Omega}\sum_t a_{\theta t}\left(\sum \eta_{tn}\eta_{lm}\right),$$

for all values of l, m, n, θ, as well as the usual summation 1, ... , n, for t; when this magnitude is used to remove the derivatives of the symbols $\{ab, c\}$ from the contribution of the first line, that contribution becomes

$$-6\begin{bmatrix} ijk \\ \mu \end{bmatrix} + 2\sum_p \{ij, p\}\{kp, \mu\} + 2\sum_p \{jk, p\}\{ip, \mu\} + 2\sum_p \{ki, p\}\{jp, \mu\}$$
$$-\frac{2}{\Omega}\sum_t \left[a_{\mu t}\left(\sum \eta_{tk}\eta_{ij}\right)\right] - \frac{2}{\Omega}\sum_t \left[a_{\mu t}\left(\sum \eta_{ti}\eta_{jk}\right)\right] - \frac{2}{\Omega}\sum_t \left[a_{\mu t}\left(\sum \eta_{tj}\eta_{ki}\right)\right].$$

When this modified contribution is combined with the contributions of the second and third line, so as to obtain the total coefficient, and when this aggregate is equated to $-6\{ijk, \mu\}$, we have

$$\begin{aligned}\begin{bmatrix} ijk \\ \mu \end{bmatrix} = \{ijk, \mu\} & \\ &+ \sum_p [\{ij, p\}\{kp, \mu\} + \{jk, p\}\{ip, \mu\} + \{ki, p\}\{jp, \mu\}] \\ &- \frac{1}{3\Omega} \sum_t a_{\mu t} \{\sum (\eta_{ti}\eta_{jk} + \eta_{tj}\eta_{ki} + \eta_{tk}\eta_{ij})\}.\end{aligned}$$

Let the earlier value (p. 50)

$$\frac{\partial}{\partial x_k}\{ij, \mu\} + \sum_p \{ij, p\}\{kp, \mu\} - \frac{1}{\Omega}\sum_t a_{\mu t}(\sum \eta_{tk}\eta_{ij})$$

for the same third-order symbol be equated to the value, which has just been constructed; then we find an expression for any first derivative of any Christoffel symbol $\{ab, c\}$ in the form

$$\begin{aligned}\frac{\partial}{\partial x_k}\{ij, \mu\} = \{ijk, \mu\} &+ \sum_p [\{ki, p\}\{jp, \mu\} + \{kj, p\}\{ip, \mu\}] \\ &- \frac{1}{3\Omega}\sum_t a_{\mu t}\{\sum (\eta_{ti}\eta_{jk} + \eta_{tj}\eta_{ik} - 2\eta_{tk}\eta_{ij})\}.\end{aligned}$$

The last summations on the right-hand side can be modified by the introduction of the Riemann symbol

$$(ij, kl) = \sum (\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}),$$

so that the whole term becomes

$$-\frac{1}{3\Omega}\sum_t a_{\mu t}[(ti, jk) + (tj, ik)].$$

If the Riemann symbol of the second kind $\{ab, cd\}$ be introduced (§ 22), this whole term becomes

$$-\tfrac{1}{3}\{i\mu, kj\} - \tfrac{1}{3}\{j\mu, ki\}.$$

Hence, as final alternative expressions for any first derivative of a Christoffel symbol, in terms of the Riemann four-index symbols and of the coefficients in x_μ''' and x_μ'' we have

$$\begin{aligned}\frac{\partial}{\partial x_k}\{ij, \mu\} &= \{ijk, \mu\} + \sum_p [\{ki, p\}\{jp, \mu\} + \{kj, p\}\{ip, \mu\}] \\ &\qquad - \frac{1}{3\Omega}\sum_t a_{\mu t}[(ti, jk) + (tj, ik)] \\ &= \{ijk, \mu\} + \sum_p [\{ki, p\}\{jp, \mu\} + \{kj, p\}\{ip, \mu\}] \\ &\qquad - \tfrac{1}{3}\{i\mu, kj\} - \tfrac{1}{3}\{j\mu, ki\},\end{aligned}$$

while the quantities $\{ijk, \mu\}$ are the coefficients in the values of the magnitudes x_μ''', given by

$$x_\mu''' = -\sum_i \sum_j \sum_k \{ijk, \mu\} x_i' x_j' x_k',$$

taken along the amplitudinal geodesic.

From the first form of expression for a derivative of $\{ij, \mu\}$ in terms of the symbols (ab, cd), and noting that $\{ijk, \mu\}$ is unaltered by interchange of the symbols i, j, k, we easily derive the relation which led (p. 50) to the introduction of the Riemann four-index symbol of the second kind.

24. Just as we have had the value of y''—it is Y/ρ—for the typical variable, it is convenient to have the value of y''', also for the typical variable.

In the first place, on differentiating the relation

$$\eta_{ij} = \frac{\partial^2 y}{\partial x_i \partial x_j} - \sum_r \frac{\partial y}{\partial x_r} \{ij, r\}$$

with respect to x_k, we have

$$\frac{\partial \eta_{ij}}{\partial x_k} = \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} - \sum_r \frac{\partial^2 y}{\partial x_r \partial x_k} \{ij, r\} - \sum_r \frac{\partial y}{\partial x_r} \frac{\partial}{\partial x_k} \{ij, r\}.$$

The second term

$$= \sum_r \eta_{rk} \{ij, r\} + \sum_r \sum_p \frac{\partial y}{\partial x_p} \{rk, p\} \{ij, r\}$$

$$= \sum_r \eta_{rk} \{ij, r\} + \sum_r \sum_p \frac{\partial y}{\partial x_r} \{pk, r\} \{ij, p\}.$$

When this is substituted for the second term; and when the values of the derivatives $\{ij, r\}$, as given in § 23, are substituted in the third term; a slight re-arrangement gives the result

$$\begin{aligned}\frac{\partial \eta_{ij}}{\partial x_k} = \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} &- \sum_r \frac{\partial y}{\partial x_r} \{ijk, r\} - \sum_r \eta_{rk} \{ij, r\} \\ &- \sum_r \sum_p \frac{\partial y}{\partial x_r} [\{jk, p\}\{ip, r\} + \{ki, p\}\{jp, r\} + \{ij, p\}\{kp, r\}] \\ &+ \frac{1}{3\Omega} \sum_r \sum_t \frac{\partial y}{\partial x_r} a_{rt} [(ti, jk) + (tj, ik)].\end{aligned}$$

Next, differentiating the typical equation

$$y'' = \sum_i \sum_j \eta_{ij} x_i' x_j'$$

along the geodesic, we have

$$\begin{aligned}y''' = \sum_i \sum_j \sum_k x_i' x_j' x_k' \frac{\partial \eta_{ij}}{\partial x_k} &- \sum_l \sum_m \eta_{lm} x_l' \left[\sum_a \sum_b \{ab, m\} x_a' x_b'\right] \\ &- \sum_l \sum_m \eta_{lm} x_m' \left[\sum_\alpha \sum_\beta \{\alpha\beta, l\} x_\alpha' x_\beta'\right].\end{aligned}$$

On the right-hand side, we select the total coefficient of the combination $x_i'x_j'x_k'$, as in § 23; it is

$$=2\frac{\partial\eta_{ij}}{\partial x_k}+2\frac{\partial\eta_{jk}}{\partial x_i}+2\frac{\partial\eta_{ki}}{\partial x_j}$$
$$-4\sum_p\left[\eta_{ip}\{jk,p\}+\eta_{jp}\{ki,p\}+\eta_{kp}\{ij,p\}\right].$$

Let this magnitude be denoted by $6\eta_{ijk}$; and, on the right-hand side, let substitution of the values of the derivatives of η_{jk}, η_{ki}, η_{ij}, be made. Then, after some arrangement, we find the right-hand side equal to

$$6\frac{\partial^3 y}{\partial x_i\partial x_j\partial x_k}-6\sum_r\frac{\partial y}{\partial x_r}\{ijk,r\}$$
$$-6\sum_p\left[\eta_{ip}\{jk,p\}+\eta_{jp}\{ki,p\}+\eta_{kp}\{ij,p\}\right]$$
$$-6\sum_r\sum_p\frac{\partial y}{\partial x_r}\left[\{ip,r\}\{jk,p\}+\{jp,r\}\{ki,p\}+\{kp,r\}\{ij,p\}\right]$$
$$+\frac{2}{3\Omega}\sum_r\sum_t\frac{\partial y}{\partial x_r}a_{rt}\left[(ti,jk)+(tj,ik)+(tj,ki)+(tk,ji)+(tk,ij)+(ti,kj)\right].$$

The group of terms in the last line vanishes identically, because of the relation

$$(ab,cd)=-(ab,dc);$$

and therefore we have

$$\eta_{ijk}=\frac{\partial^3 y}{\partial x_i\partial x_j\partial x_k}-\sum_r\frac{\partial y}{\partial x_r}\{ijk,r\}$$
$$-\sum_p\left[\eta_{ip}\{jk,p\}+\eta_{jp}\{ki,p\}+\eta_{kp}\{ij,p\}\right]$$
$$-\sum_r\sum_p\frac{\partial y}{\partial x_r}\left[\{ip,r\}\{jk,p\}+\{jp,r\}\{ki,p\}+\{kp,r\}\{ij,p\}\right].$$

With this value of η_{ijk}, the expression for y''' taken along the amplitudinal geodesic is

$$y'''=\sum_i\sum_j\sum_k\eta_{ijk}x_i'x_j'x_k',$$

where y is the typical space-variable.

We also note the relation

$$\frac{\partial\eta_{ij}}{\partial x_k}=\eta_{ijk}+\sum_p\left[\eta_{ip}\{jk,p\}+\eta_{jp}\{ik,p\}\right]+\frac{1}{3\Omega}\sum_r\sum_t\frac{\partial y}{\partial x_r}a_{rt}\left[(ti,jk)+(tj,ik)\right],$$

holding for all the values of i, j, k.

25. Parametric derivatives of the quantity, denoted by the Riemann four-index symbol (ij, kl), are expressible in terms of these magnitudes η_{ijk}. We have (§ 14)

$$(ij,kl)=\sum(\eta_{ik}\eta_{jl}-\eta_{il}\eta_{jk})$$

where the summation without any specified index implies the dimension-range of the plenary space. Hence

$$\frac{\partial}{\partial x_m}(ij, kl) = \sum \left(\eta_{jl} \frac{\partial \eta_{ik}}{\partial x_m} + \eta_{ik} \frac{\partial \eta_{jl}}{\partial x_m} - \eta_{il} \frac{\partial \eta_{jk}}{\partial x_m} - \eta_{jk} \frac{\partial \eta_{il}}{\partial x_m} \right).$$

When substitution is made for the derivatives of η, as just obtained, all the terms on the right-hand side involving derivatives $\frac{\partial y}{dx_r}$ vanish. The aggregate of terms involving the Christoffel symbols $\{ab, c\}$ is

$$\begin{aligned} &= \sum_p \sum \eta_{jl} [\eta_{ip} \{km, p\} + \eta_{kp} \{im, p\}] \\ &\quad + \sum_p \sum \eta_{ik} [\eta_{jp} \{lm, p\} + \eta_{lp} \{jm, p\}] \\ &\quad - \sum_p \eta_{il} [\eta_{jp} \{km, p\} + \eta_{kp} \{jm, p\}] \\ &\quad - \sum_p \eta_{jk} [\eta_{ip} \{lm, p\} + \eta_{lp} \{im, p\}] \\ &= - \sum_p \{im, p\} (kl, jp) + \sum_p \{jm, p\} (kl, ip) \\ &\qquad\qquad - \sum_p \{km, p\} (ij, lp) + \sum_p \{lm, p\} (ij, kp). \end{aligned}$$

The aggregate of terms involving quantities of the type η_{abc} is obvious; we therefore have

$$\begin{aligned} &\frac{\partial}{\partial x_m}(ij, kl) \\ &\qquad = \sum (\eta_{ik}\eta_{jlm} - \eta_{il}\eta_{jkm} - \eta_{jk}\eta_{ilm} + \eta_{jl}\eta_{ikm}) \\ &\qquad\qquad - \sum_p \{im, p\} (kl, jp) + \sum_p \{jm, p\} (kl, ip) \\ &\qquad\qquad - \sum_p \{km, p\} (ij, lp) + \sum_p \{lm, p\} (ij, kp). \end{aligned}$$

It is convenient to use a symbol to denote the expression in the first line; following Bianchi*, we define

$$(ij, klm) = \sum (\eta_{ik}\eta_{jlm} - \eta_{il}\eta_{jkm} - \eta_{jk}\eta_{ilm} + \eta_{jl}\eta_{ikm}).$$

We at once have the identities

$$\begin{aligned} (ij, klm) &= (kl, ijm) = -(ji, klm) = -(ij, lkm), \\ &= (ji, lkm) = (lk, jim) = -(kl, jim) = -(lk, ijm); \\ &(ij, mkl) + (ij, lmk) + (ij, klm) = 0, \\ &(ij, klm) + (ik, ljm) + (il, jkm) = 0, \\ &(ij, klm) + (jk, ilm) + (ki, jlm) = 0, \end{aligned}$$

* *Lezioni di geometria differenziale*, vol. i (1902), p. 351, with a correction noted in the list at the end of vol. ii.

representative of many others. When the symbol is introduced into the foregoing relation, we have

$$\begin{aligned}\frac{\partial}{\partial x_m}(ij, kl) = (ij, klm)& \\ &- \sum_p \{im, p\}(kl, jp) - \sum_p \{km, p\}(ij, lp) \\ &+ \sum_p \{jm, p\}(kl, ip) + \sum_p \{lm, p\}(ij, kp),\end{aligned}$$

as the expression for any parametric derivative of the quantity denoted by a Riemann four-index symbol.

Ex. Verify the relation, due to Bianchi (*l.c.*),

$$\begin{aligned}\frac{\partial}{\partial x_m}(ij,\ kl) + \frac{\partial}{\partial x_k}(ij,\ lm) + \frac{\partial}{\partial x_l}(ij,\ mk)& \\ = \sum_t [\{ik, t\}(jt, ml) + \{il, t\}(jt, km) + \{im, t\}(jt, lk)]& \\ - \sum_t [\{jk, t\}(it, lm) + \{jl, t\}(it, mk) + \{jm, t\}(it, kl)].&\end{aligned}$$

Derivatives of $\{ijk, \mu\}$ *: quantities* $\{ijkl, \mu\}$.

26. The magnitudes $\begin{bmatrix} ijk \\ a \end{bmatrix}$, defined (§ 22) by the relation

$$\sum \frac{\partial y}{\partial x_p} \frac{\partial^3 y}{\partial x_i\, \partial x_j\, \partial x_k} = \sum_a A_{pa} \begin{bmatrix} ijk \\ a \end{bmatrix},$$

are connected with the magnitudes $\{ijk, a\}$ by the equation (§ 23)

$$\begin{bmatrix} ijk \\ a \end{bmatrix} = \{ijk, a\} + \sum_\beta \Phi(a, \beta) - \frac{1}{3\Omega} \sum_\theta a_{a\theta} f(ijk\theta),$$

where

$$\Phi(a, \beta) = \{i\beta, a\}\{jk, \beta\} + \{j\beta, a\}\{ki, \beta\} + \{k\beta, a\}\{ij, \beta\},$$
$$f(ijk\theta) = \sum (\eta_{\theta i}\eta_{jk} + \eta_{\theta j}\eta_{ki} + \eta_{\theta k}\eta_{ij});$$

and therefore

$$\sum \frac{\partial y}{\partial x_p} \frac{\partial^3 y}{\partial x_i\, \partial x_j\, \partial x_k} = \sum_a A_{pa} [\{ijk, a\} + \sum_\beta \Phi(a, \beta)] - \tfrac{1}{3} f(ijkp).$$

Again, we had (§ 24)

$$\frac{\partial^3 y}{\partial x_i\, \partial x_j\, \partial x_k} = \eta_{ijk} + \sum_r \left[\frac{\partial y}{\partial x_r}\{ijk, r\}\right] + \sum_\lambda T_\lambda + \sum_r \sum_\lambda \left[\frac{\partial y}{\partial x_r}\Phi(r, \lambda)\right],$$

where

$$T_\lambda = \eta_{i\lambda}\{jk, \lambda\} + \eta_{j\lambda}\{ki, \lambda\} + \eta_{k\lambda}\{ij, \lambda\};$$

consequently

$$\sum \frac{\partial y}{\partial x_s} \frac{\partial^3 y}{\partial x_i\, \partial x_j\, \partial x_k} = \sum \frac{\partial y}{\partial x_s}\eta_{ijk} + \sum_r A_{rs} [\{ijk, r\} + \sum_\lambda \Phi(r, \lambda)],$$

so that, comparing the two values of the left-hand side,

$$\sum \frac{dy}{ds}\eta_{ijk} = -\tfrac{1}{3} f(ijks).$$

Also, as

$$\frac{\partial^2 y}{\partial x_m\, \partial x_t} = \eta_{mt} + \sum_s \frac{\partial y}{\partial x_s}\{mt, s\},$$

we have

$$\begin{aligned}
\sum \frac{\partial^2 y}{\partial x_m\, \partial x_t}\, \frac{\partial^3 y}{\partial x_i\, \partial x_j\, \partial x_k}
&= \sum \eta_{mt}\eta_{ijk} + \sum_\lambda \sum (\eta_{mt} T_\lambda) + \sum_s \left[\{mt, s\} \left(\sum \frac{\partial y}{\partial x_s}\eta_{ijk}\right)\right] \\
&\qquad + \sum_r \sum_s A_{rs}\{mt, s\}[\{ijk, r\} + \sum_\lambda \Phi(r, \lambda)] \\
&= \sum \eta_{mt}\eta_{ijk} + \sum_\lambda \sum (\eta_{mt} T_\lambda) - \tfrac{1}{3} \sum_s \{mt, s\} f(ijks) \\
&\qquad + \sum_r \sum_s A_{rs}\{mt, s\}[\{ijk, r\} + \sum_\lambda \Phi(r, \lambda)].
\end{aligned}$$

Next, we differentiate the foregoing equation

$$\sum \frac{\partial y}{\partial x_p}\, \frac{\partial^3 y}{\partial x_i\, \partial x_j\, \partial x_k} = \sum_a A_{pa}[\{ijk, a\} + \sum_\beta \Phi(a, \beta)] - \tfrac{1}{3} f(ijkp)$$

with respect to x_l; and we have

$$\begin{aligned}
\sum \frac{\partial y}{\partial x_p}\, \frac{\partial^4 y}{\partial x_i\, \partial x_j\, \partial x_k\, \partial x_l} &+ \sum \frac{\partial^2 y}{\partial x_p\, \partial x_l}\, \frac{\partial^3 y}{\partial x_i\, \partial x_j\, \partial x_k} \\
&= \sum_a A_{pa}\left[\frac{\partial}{\partial x_l}\{ijk, a\} + \sum_\beta \frac{\partial}{\partial x_l}\Phi(a, \beta)\right] - \tfrac{1}{3}\frac{\partial}{\partial x_l} f(ijkp) \\
&\qquad + \sum_a \sum_\gamma [\{ijk, a\} + \sum_\beta \Phi(a, \beta)][A_{p\gamma}\{la, \gamma\} + A_{a\gamma}\{lp, \gamma\}].
\end{aligned}$$

At this stage, we introduce a new symbol $\begin{bmatrix} ijkl \\ \omega \end{bmatrix}$, on the analogy of $[ij, \mu]$ for second point-derivatives and of $\begin{bmatrix} ijk \\ \mu \end{bmatrix}$ for third point-derivatives, and defined by the equation

$$\sum \frac{\partial y}{\partial x_p}\, \frac{\partial^4 y}{\partial x_i\, \partial x_j\, \partial x_k\, \partial x_l} = \sum_\omega A_{p\omega} \begin{bmatrix} ijkl \\ \omega \end{bmatrix}.$$

Inserting this value, and substituting for the sum of the products of second point-derivatives and third point-derivatives, we have

$$\begin{aligned}
\sum_a A_{pa}\begin{bmatrix} ijkl \\ a \end{bmatrix} &+ \sum \eta_{pl}\eta_{ijk} + \sum_\lambda (\sum \eta_{pl} T_\lambda) - \tfrac{1}{3}\sum_s \{pl, s\} f(ijks) \\
&\qquad + \sum_s \sum_a A_{sa}\{pl, s\}[\{ijk, a\} + \sum_\lambda \Phi(a, \lambda)] \\
&= \sum_a A_{pa}\left[\frac{\partial}{\partial x_l}\{ijk, a\} + \sum_\beta \frac{\partial}{\partial x_l}\Phi(a, \beta)\right] - \tfrac{1}{3}\frac{\partial}{\partial x_l} f(ijkp) \\
&\qquad + \sum_a \sum_\gamma [\{ijk, a\} + \sum_\beta \Phi(a, \beta)][A_{p\gamma}\{la, \gamma\} + A_{a\gamma}\{lp, \gamma\}].
\end{aligned}$$

Now the two aggregates

$$\sum_s \sum_a A_{sa}\{pl, s\}[\{ijk, a\} + \sum_\lambda \Phi(a, \lambda)], \quad \sum_a \sum_\gamma A_{a\gamma}\{lp, \gamma\}[\{ijk, a\} + \sum_\beta \Phi(a, \beta)],$$

are the same in effect, the sole difference being in the use of equivalent symbols; also, by a mere change of symbols,

$$\sum_a \sum_\gamma [\{ijk, a\} + \sum_\beta \Phi(a, \beta)] A_{p\gamma}\{la, \gamma\}$$
$$= \sum_\epsilon \sum_a A_{pa}\{l\epsilon, a\}[\{ijk, \epsilon\} + \sum_\beta \Phi(\epsilon, \beta)];$$

and therefore the equation can be written

$$\sum_a A_{pa}\Theta(a) = -(\sum \eta_{pi}\eta_{ijk}) - \sum_\lambda \sum \eta_{pl}T_\lambda + \tfrac{1}{3}\sum_s \{pl, s\} f(ijks) - \tfrac{1}{3}\frac{\partial}{\partial x_l} f(ijkp),$$

where

$$\Theta(a) = \begin{bmatrix} ijkl \\ a \end{bmatrix} - \frac{\partial}{\partial x_l}\{ijk, a\} - \sum_\beta \frac{\partial}{\partial x_l}\Phi(a, \beta)$$
$$- \sum_\epsilon \{l\epsilon, a\}[\{ijk, \epsilon\} + \sum_\beta \Phi(\epsilon, \beta)].$$

As regards the right-hand side of the equation, we require a simpler form of the terms

$$\tfrac{1}{3}\sum_s \{pl, s\} f(ijks) - \tfrac{1}{3}\frac{\partial}{\partial x_l} f(ijk, p).$$

In framing the derivative of f, we use the relations

$$\frac{\partial \eta_{ij}}{\partial x_k} = \eta_{ijk} + \sum_\gamma [\eta_{i\gamma}\{jk, \gamma\} + \eta_{ja}\{ik, \gamma\}]$$
$$+ \frac{1}{3\Omega}\sum_r \sum_t \frac{\partial y}{\partial x_r} a_{rt}[(ti, jk) + (tj, ik)],$$

and

$$\sum \eta_{\lambda\mu}\frac{\partial y}{\partial x_r} = 0,$$

the latter for all values of λ, μ, r, a relation which makes all the terms in $\dfrac{\partial f}{\partial x_l}$ involving the Riemann four-index symbols disappear. When the terms are gathered together, the expression can be taken in the form

$$-\tfrac{1}{3}\sum(\eta_{pi}\eta_{jkl} + \eta_{pj}\eta_{ikl} + \eta_{pk}\eta_{ijl}) - \tfrac{1}{3}\sum(\eta_{jk}\eta_{pil} + \eta_{ki}\eta_{pjl} + \eta_{ij}\eta_{pkl})$$
$$-\tfrac{1}{3}\sum_\gamma [\{il, \gamma\} f(jk\gamma p) + \{jl, a\} f(ki\gamma p) + \{kl, a\} f(ij\gamma p)],$$

which we shall denote by

$$-\tfrac{1}{3}g_p(ijk, l).$$

Thus the right-hand side of the equation is $-U_p(ijk, l)$, where

$$U_p(ijk, l) = \sum \eta_{pl}\{\eta_{ijk} + \sum_\lambda T_\lambda(ijk)\} - \tfrac{1}{3}g_p(ijk, l),$$

now using $T_\lambda(ijk)$ in place of the former T_λ.

Thus the equation is

$$\sum_a A_{pa}\Theta(a) = -U_p(ijk, l),$$

and therefore, resolving for the various quantities Θ for the different values of a, we have

$$\Theta(a) = -\frac{1}{\Omega}\sum_p a_{pa}U_p(ijk, l),$$

that is,

$$\begin{bmatrix} ijkl \\ a \end{bmatrix} = \frac{\partial}{\partial x_l}\{ijk, a\} + \sum_\epsilon \{l\epsilon, a\}\{ijk, \epsilon\} - \frac{1}{\Omega}\sum_p a_{pa}U_p(ijk, l)$$
$$+ \sum_\epsilon \sum_\beta \{l\epsilon, a\}\Phi(ijk, \epsilon, \beta) + \sum_\beta \frac{\partial}{\partial x_l}\Phi(ijk, a, \beta),$$

now inserting the characteristic numbers in the function Φ; and this result we take in the form

$$\begin{bmatrix} ijkl \\ a \end{bmatrix} = \frac{\partial}{\partial x_l}\{ijk, a\} + P_a(ijk, l),$$

where, in the function P_a, only known magnitudes occur.

The left-hand side is unaltered by the interchange of the symbols i, j, k, l; and the right-hand side is unaltered by the change of the symbols i, j, k, among themselves alone. We thus have

$$\frac{\partial}{\partial x_l}\{ijk, a\} + P_a(ijk, l)$$
$$= \frac{\partial}{\partial x_i}\{jkl, a\} + P_a(jkl, i)$$
$$= \frac{\partial}{\partial x_j}\{kli, a\} + P_a(kli, j)$$
$$= \frac{\partial}{\partial x_k}\{lij, a\} + P_a(lij, k);$$

and we also have

$$4\begin{bmatrix} ijkl \\ a \end{bmatrix} = \frac{\partial}{\partial x_i}\{jkl, a\} + \frac{\partial}{\partial x_j}\{kli, a\} + \frac{\partial}{\partial x_k}\{lij, a\} + \frac{\partial}{\partial x_l}\{ijk, a\}$$
$$+ P_a(jkl, i) + P_a(kli, j) + P_a(lij, k) + P_a(ijk, l).$$

27. Thus there is a relation, equivalent to

$$\frac{\partial}{\partial x_l}\{ijk, a\} - \frac{\partial}{\partial x_k}\{ijl, a\} = -P_a(ijk, l) + P_a(ijl, k),$$

where, in the quantity $P_a(ijk, l)$, the value is unaltered for interchange of the numbers i, j, k, and no derivatives of higher than the third order occur implicitly: and likewise for the quantity $P_a(ijl, k)$. Thus, from the quantity on the left-hand side of the relation, the aggregate of magnitudes involving fourth-order derivatives disappears.

Another (and equivalent) form for the quantity on the left-hand side can be obtained. We proceed from the relations (§ 23)

$$\{ijk, \mu\} = \frac{\partial}{\partial x_k}\{ij, \mu\} + T(ij, k, \mu) - P(ij, k, \mu),$$

$$\{ijl, \mu\} = \frac{\partial}{\partial x_l}\{ij, \mu\} + T(ij, l, \mu) - P(ij, l, \mu),$$

where

$$T\{ij, m, \mu\} = \frac{1}{3\Omega}\sum_t a_{\mu t}[(ti, jm) + (tj, im)],$$

$$P\{ij, m, \mu\} = \sum_p [\{mi, p\}\{jp, \mu\} + \{mj, p\}\{ip, \mu\}].$$

Thus

$$\begin{aligned}\frac{\partial}{\partial x_l}\{ijk, \mu\} - \frac{\partial}{\partial x_k}\{ijl, \mu\} &\\ = \frac{\partial}{\partial x_l} T(ij, k, \mu) &- \frac{\partial}{\partial x_k} T(ij, l, \mu)\\ &- \frac{\partial}{\partial x_l} P(ij, k, \mu) + \frac{\partial}{\partial x_k} P(ij, l, \mu).\end{aligned}$$

The expressions, in the two lines on the right-hand side, are evaluated separately.

(i) For the expression in the first line, we find that the first term $\frac{\partial}{\partial x_l} T(ij, k, \mu)$ is the sum, over the n values of $t = 1, \ldots, n$, of the quantity $\frac{1}{3\Omega}\Theta_t$, where

$$\begin{aligned}\Theta_t = a_{\mu t}[(ti, jkl) + (tj, ikl)]&\\ - a_{\mu t}\sum_q [\{tl, q\}(iq, jk) - \{il, q\}(tq, jk) &+ \{jl, q\}(kq, ti) - \{kl, q\}(jq, ti)]\\ - a_{\mu t}\sum_q [\{tl, q\}(jq, ik) - \{jl, q\}(tq, ik) &+ \{il, q\}(kq, tj) - \{kl, q\}(iq, tj)]\\ - [(ti, jk) + (tj, ik)]\sum_q [a_{q\mu}\{lq, t\} &+ a_{qt}\{lq, \mu\}]\end{aligned}$$

by using the formula (§ 13) for the parametric derivative of a_{rs}/Ω, and the formula (§ 25) for the parametric derivative of the four-index symbols which occur. A first simplification in $\sum_t \Theta_t$ occurs; for the aggregate

$$\begin{aligned}&- \sum_t \sum_q a_{\mu t}[\{tl, q\}(iq, jk) + \{tl, q\}(jq, ik)]\\ &= - \sum_t \sum_q a_{\mu q}[\{ql, t\}(it, jk) + \{ql, t\}(jt, ik)],\end{aligned}$$

and therefore, owing to the property $(ab, cd) = -(ba, cd)$, it cancels the aggregate

$$- \sum_t \sum_q a_{q\mu}\{lq, t\}[(ti, jk) + (tj, ik)]$$

arising from the last line of Θ_t.

Similarly, for the expression of $\frac{\partial}{\partial x_k} T(ij, l, \mu)$, we obtain it as the sum, taken over the n values of t, of the quantity $\frac{1}{3\Omega}\Phi_t$, where

$$\begin{aligned}\Phi_t = a_{\mu t}\,&[(ti, jlk) + (tj, ilk)]\\ &- a_{\mu t}\sum_q [\{tk, q\}(iq, jl) - \{ik, q\}(tq, jl) + \{jk, q\}(lq, ti) - \{kl, q\}(jq, ti)]\\ &- a_{\mu t}\sum_q [\{tk, q\}(jq, il) - \{jk, q\}(tq, il) + \{ik, q\}(lq, tj) - \{kl, q\}(iq, tj)]\\ &- [(ti, jl) + (tj, il)]\sum_q [a_{q\mu}\{kq, t\} + a_{qt}\{kq, \mu\}],\end{aligned}$$

with properties similar to those of Θ_t.

When the terms are gathered together, various reductions are made. Thus by means of the identity (§ 26)

$$(ij, mkl) + (ij, klm) + (ij, lmk) = 0$$

and the other identities affecting this five-letter symbol, the aggregate of the terms in the quantity

$$\frac{1}{3\Omega}\sum(\Theta_t - \Phi_t)$$

becomes

$$= \frac{1}{3\Omega}\sum_t a_{\mu t}[(it, klj) + (jt, kli)].$$

The remaining terms involve only the four-letter symbols; and we find that, in the same quantity, their aggregate

$$\begin{aligned}= \frac{1}{3\Omega}\sum_t\sum_q a_{qt}\{kq, \mu\}&[ti, jl) + (tj, il)]\\ &- \frac{1}{3\Omega}\sum_t\sum_q a_{qt}\{lq, \mu\}[(ti, jk) + (tj, ik)]\\ &- \frac{1}{3\Omega}\sum_t a_{\mu t}\left(\sum_q U_q\right)\end{aligned}$$

where

$$\begin{aligned}U_q = \{jl, q\}[(kq, ti) - (tq, ik)] &+ \{il, q\}[(kq, tj) - (tq, jk)]\\ &- \{jk, q\}[(lq, ti) - (tq, il)] - \{ik, q\}[(lq, tj) - (tq, jl)].\end{aligned}$$

(ii) For the expression in the second line of the required quantity, we have

$$\frac{\partial}{\partial x_l}P(ij, k, \mu) = \sum_p \Psi_p, \quad \frac{\partial}{\partial x_k}P(ij, l, \mu) = \sum_p X_p,$$

where

$$\begin{aligned}\Psi_p = \{jp, \mu\}&[\{ikl, p\} - T(ik, l, p) + P(ik, l, p)]\\ &+ \{ip, \mu\}[\{jkl, p\} - T(jk, l, p) + P(jk, l, p)]\\ &+ \{ki, p\}[\{jlp, \mu\} - T(jp, l, \mu) + P(jp, l, \mu)]\\ &+ \{kj, p\}[\{ilp, \mu\} - T(ip, l, \mu) + P(ip, l, \mu)],\end{aligned}$$

$$\begin{aligned} X_p = \{jp, \mu\}\,[\{ikl, p\} - T(il, k, p) + P(il, k, p)] \\ + \{ip, \mu\}\,[\{jkl, p\} - T(jl, k, p) + P(jl, k, p)] \\ + \{li, p\}\,[\{jkp, \mu\} - T(jp, k, \mu) + P(jp, k, \mu)] \\ + \{lj, p\}\,[\{ikp, \mu\} - T(ip, k, \mu) + P(ip, k, \mu)]. \end{aligned}$$

In the expression $\sum\limits_p (\Psi_p - X_p)$, the aggregate of the terms involving magnitudes of the type $\{abc, d\}$ is

$$= \sum_p [\{li, p\}\{jkp, \mu\} - \{ki, p\}\{jlp, \mu\} - \{kj, p\}\{ilp, \mu\} + \{lj, p\}\{ikp, \mu\}].$$

In the same expression, the aggregate of the terms arising out of the quantities $T(\alpha\beta, \gamma, \delta)$ is the sum, for the index p, of the aggregate

$$\begin{aligned} \{jp, \mu\}\,[T(il, k, p) - T(ik, l, p)] \\ + \{ip, \mu\}\,[T(jl, k, p) - T(jk, l, p)] \\ + \{li, p\}\,T(jp, k, \mu) - \{ki, p\}\,T(jp, l, \mu) \\ + \{lj, p\}\,T(ip, k, \mu) - \{kj, p\}\,T(ip, l, \mu). \end{aligned}$$

Now

$$T(ab, c, \lambda) = \frac{1}{3\Omega}\sum_t a_{\lambda t}\,[(ta, bc) + (tb, ac)].$$

When this value is substituted, the first line

$$= \frac{1}{\Omega}\sum_t a_{pt}\{jp, \mu\}(ti, lk),$$

after some reduction: and, similarly, the second line

$$= \frac{1}{\Omega}\sum_t a_{pt}\{ip, \mu\}(tj, lk).$$

Similarly, after substitution, the aggregate of the third and fourth lines can be expressed in the form

$$-\frac{1}{3\Omega}\sum_t a_{\mu t} U_p,$$

with the same significance for the symbols U as before.

In the same expression, the aggregate of the terms arising out of the quantities $P(\alpha\beta, \gamma, \delta)$, where

$$P(\alpha\beta, \gamma, \delta) = \sum_q [\{\alpha\gamma, q\}\{\beta q, \delta\} + \{\beta\gamma, q\}\{\alpha q, \delta\}],$$

is equal to

$$\begin{aligned} \{jp, \mu\}\,P(il, k, p) + \{ip, \mu\}\,P(jl, k, p) + \{li, p\}\,P(jp, k, \mu) + \{lj, p\}\,P(ip, k, \mu) \\ - \{jp, \mu\}\,P(ik, l, p) - \{ip, \mu\}\,P\{jk, l, p\} - \{ki, p\}\,P(jp, l, \mu) - \{kj, p\}\,P(ip, l, \mu); \end{aligned}$$

and this aggregate is to be summed for the values of the number p. When the substitution of the values of the various quantities P is made, and the double summation is effected, this aggregate is found to vanish.

(iii) Gathering together the various results in the relation

$$\frac{\partial}{\partial x_l}\{ijk,\mu\}-\frac{\partial}{\partial x_k}\{ijl,\mu\}$$
$$=\frac{1}{3\Omega}\sum_t(\Theta_t-\Phi_t)-\sum_p(\Psi_p-X_p),$$

we note that the quantity $-\frac{1}{3\Omega}\sum_t a_{\mu t}\left(\sum_q U_q\right)$ from the first summation on the right-hand side cancels the term $-\frac{1}{3\Omega}\sum_t a_{\mu t}\left(\sum_p U_p\right)$ from the second summation $\sum_p(\Psi_p-X_p)$ on the same side. We use a symbol $S_\mu(ij, k, l)$, subject to necessary identities

$$S_\mu(ij,k,l)=S_\mu(ji,k,l)=-S_\mu(ij,l,k)=-S_\mu(ji,l,k),$$

and defined by the equation

$$S_\mu(ij,k,l)=3\{jq,\mu\}(ti,lk)+3\{iq,\mu\}(tj,lk)$$
$$+\{kq,\mu\}[(ti,jl)+(tj,il)]-\{lq,\mu\}[(ti,jk)+(tj,ik)]\,;$$

and the final result becomes

$$\frac{\partial}{\partial x_l}\{ijk,\mu\}-\frac{\partial}{\partial x_k}\{ijl,\mu\}$$
$$=\sum_p[\{ilp,\mu\}\{jk,p\}-\{ikp,\mu\}\{jl,p\}-\{jkp,\mu\}\{il,p\}+\{jlp,\mu\}\{ik,p\}]$$
$$+\frac{1}{3\Omega}\sum_t a_{\mu t}[(it,klj)+(jt,kli)]-\frac{1}{3\Omega}\sum_q\sum_t a_{qt}\,S_\mu(ij,k,l).$$

An expression for x_μ'''' along a geodesic.

28. Further, we have had (§ 23) the relation

$$-x_\mu'''=\sum_i\sum_j\sum_k\{ijk,\mu\}x_i'x_j'x_k'\,;$$

and therefore, differentiating once more along the geodesic,

$$-x_\mu''''=\sum_i\sum_j\sum_k\sum_l\frac{\partial}{\partial x_l}\{ijk,\mu\}x_i'x_j'x_k'x_l'$$
$$+\sum_i\sum_j\sum_k\{ijk,\mu\}(x_i''x_j'x_k'+x_i'x_j''x_k'+x_i'x_j'x_k'').$$

As x_i'', x_j'', x_k'', are expressible as homogeneous quadratic functions of the variables x', the right-hand side is a homogeneous quartic function of these variables; we therefore write

$$-x_\mu''''=\sum_i\sum_j\sum_k\sum_l\{ijkl,\mu\}x_i'x_j'x_k'x_l'.$$

When we take the full coefficient of the combination $x_i'x_j'x_k'x_l'$ in the two expressions for x_μ'''', we find

$$\begin{aligned}24\{ijkl, \mu\} &= 6\left[\frac{\partial}{\partial x_i}\{jkl, \mu\} + \frac{\partial}{\partial x_j}\{kli, \mu\} + \frac{\partial}{\partial x_k}\{lij, \mu\} + \frac{\partial}{\partial x_l}\{ijk, \mu\}\right] \\ &\quad - 12\sum_\epsilon [\{il, \epsilon\}\{jk\epsilon, \mu\} + \{jl, \epsilon\}\{ki\epsilon, \mu\} + \{kl, \epsilon\}\{ij\epsilon, \mu\} \\ &\quad + \{jk, \epsilon\}\{il\epsilon, \mu\} + \{ki, \epsilon\}\{jl\epsilon, \mu\} + \{ij, \epsilon\}\{kl\epsilon, \mu\}].\end{aligned}$$

Let the quantity under the sign $\sum_\epsilon$ be denoted by $\theta_\mu(ijkl, \epsilon)$, where θ is unaffected in value by interchange of the numbers i, j, k, l. Also, for the summation in the right-hand side of the first line, let the value obtained in § 27 be substituted; then

$$\begin{aligned}4\begin{bmatrix} ijkl \\ \mu \end{bmatrix} - 4\{ijkl, \mu\} - 2\sum_\epsilon \theta_\mu(ijkl, \epsilon) \\ = P_a(jkl, i) + P_a(kli, j) + P_a(lij, k) + P_a(ijk, l),\end{aligned}$$

where, now, both sides of the relation are unaffected by this interchange of the numbers i, j, k, l. The magnitude of the coefficient $\{ijkl, \mu\}$ in the expression for x_μ'''' is therefore known in terms of magnitudes connected with the amplitude.

Also, we have

$$\frac{\partial}{\partial x_i}\{jkl, \mu\} + \frac{\partial}{\partial x_j}\{kli, \mu\} + \frac{\partial}{\partial x_k}\{lij, \mu\} + \frac{\partial}{\partial x_l}\{ijk, \mu\} = 4\{ijkl, \mu\} + 2\sum_\epsilon \theta_\mu(ijkl, \epsilon).$$

29. The third geodesic-derivative of a point-variable y is given (§ 23) by

$$y''' = \sum_i \sum_j \sum_k \eta_{ijk} x_i' x_j' x_k';$$

and therefore, after another differentiation,

$$\begin{aligned}y'''' &= \sum_i \sum_j \sum_k \sum_l \frac{\partial}{\partial x_l}(\eta_{ijk}) x_i' x_j' x_k' x_l' \\ &\quad + \sum_i \sum_j \sum_k \eta_{ijk}(x_i''x_j'x_k' + x_i'x_j''x_k' + x_i'x_j'x_k'').\end{aligned}$$

When the values of x_i'', x_j'', x_k'', are inserted, the right-hand side becomes a homogeneous quartic function of the variables x'; consequently we may write

$$y'''' = \sum_i \sum_j \sum_k \sum_l \eta_{ijkl} x_i' x_j' x_k' x_l'.$$

Let the full coefficient of the combination $x_i'x_j'x_k'x_l'$ be taken in the two expressions for y''''; we then have

$$\begin{aligned}24\eta_{ijkl} &= 6\left[\frac{\partial}{\partial x_i}\eta_{jkl} + \frac{\partial}{\partial x_j}\eta_{kli} + \frac{\partial}{\partial x_k}\eta_{lij} + \frac{\partial}{\partial x_l}\eta_{ijk}\right] \\ &\quad - 12\sum_\epsilon [\{il, \epsilon\}\eta_{jk\epsilon} + \{jl, \epsilon\}\eta_{ki\epsilon} + \{kl, \epsilon\}\eta_{ij\epsilon} \\ &\quad + \{jk, \epsilon\}\eta_{il\epsilon} + \{ki, \epsilon\}\eta_{jl\epsilon} + \{ij, \epsilon\}\eta_{kl\epsilon}].\end{aligned}$$

The quantity under the sign $\sum\limits_{\epsilon}$ will be denoted by

$$\phi_\epsilon(ijkl),$$

a magnitude unaltered by interchange of the numbers i, j, k, l; and thus

$$4\eta_{ijkl}+2\sum_{\epsilon}\phi_\epsilon(ijkl)=\frac{\partial}{\partial x_i}\eta_{jkl}+\frac{\partial}{\partial x_j}\eta_{kli}+\frac{\partial}{\partial x_k}\eta_{lij}+\frac{\partial}{\partial x_l}\eta_{ijk}.$$

Again, the value of η_{ijk} in terms of known magnitudes connected with the amplitude was given (§ 26) by

$$\eta_{ijk}=\frac{\partial^3 y}{\partial x_i\partial x_j\partial x_k}-\sum_r\frac{\partial y}{\partial x_r}\{ijk, r\}-\sum_\lambda T_\lambda(ijk)-\sum_r\sum_\lambda\frac{\partial y}{\partial x_r}\Phi(ijk, r, \lambda).$$

When this is differentiated with respect to x_l, it gives

$$\begin{aligned}\frac{\partial}{\partial x_l}\eta_{ijk}=\frac{\partial^4 y}{\partial x_i\partial x_j\partial x_k\partial x_l}&-\sum_r\frac{\partial y}{\partial x_r}\left[\frac{\partial}{\partial x_l}\{ijk, r\}\right]-\sum_r\frac{\partial^2 y}{\partial x_r\partial x_l}\{ijk, r\}\\&-\sum_\lambda\frac{\partial}{\partial x_l}T_\lambda(ijk)\\&-\sum_r\sum_\lambda\frac{\partial y}{\partial x_r}\frac{\partial}{\partial x_l}\Phi(ijk, r, \lambda)-\sum_r\sum_\lambda\frac{\partial^2 y}{\partial x_r\partial x_l}\Phi(ijk, r, \lambda).\end{aligned}$$

Now we have

$$\frac{\partial^2 y}{\partial x_r\partial x_l}=\eta_{rl}-\sum_t\frac{\partial y}{\partial x_t}\{rl, t\};$$

and the magnitudes in the third line can be expressed in terms of known quantities, connected with the configuration. Also

$$\frac{\partial}{\partial x_l}T_\lambda(ijk)=\frac{\partial}{\partial x_l}[\eta_{i\lambda}\{jk, \lambda\}+\eta_{j\lambda}\{ki, \lambda\}+\eta_{k\lambda}\{ij, \lambda\}];$$

the derivatives

$$\frac{\partial\eta_{i\lambda}}{\partial x_l},\quad\frac{\partial\eta_{j\lambda}}{\partial x_l},\quad\frac{\partial\eta_{k\lambda}}{\partial x_l},$$

are expressible by $\eta_{il\lambda}$, $\eta_{jl\lambda}$, $\eta_{kl\lambda}$, and magnitudes connected with the configuration. We therefore take

$$\frac{\partial}{\partial x_l}\eta_{ijk}=\frac{\partial^4 y}{\partial x_i\partial x_j\partial x_k\partial x_l}-\sum_r\frac{\partial y}{\partial x_r}\left[\frac{\partial}{\partial x_l}\{ijk, r\}\right]-H(ijk, l),$$

where H is linear in the magnitudes of the types $\dfrac{\partial y}{\partial x_t}$, η_{ab}, η_{abc}, with coefficients that are established magnitudes, and is unaffected by an interchange of the numbers i, j, k.

Similarly for the derivatives of η_{jkl}, η_{kli}, η_{lij}; and therefore

$$
\begin{aligned}
4\eta_{ijkl} + 2\sum_{\epsilon} \phi_{\epsilon}(ijkl) \\
&= 4\frac{\partial^4 y}{\partial x_i \partial x_j \partial x_k \partial x_l} - H(jkl, i) - H(kli, j) - H(lij, k) - H(ijk, l) \\
&\quad - \sum_r \frac{\partial y}{\partial x_r}\left[\frac{\partial}{\partial x_i}\{jkl, r\} + \frac{\partial}{\partial x_j}\{kli, r\} + \frac{\partial}{\partial x_k}\{lij, r\} + \frac{\partial}{\partial x_l}\{ijk, r\}\right] \\
&= 4\frac{\partial^4 y}{\partial x_i \partial x_j \partial x_k \partial x_l} - \overline{H}(ijkl) - \sum_r \frac{\partial y}{\partial x_r}\left[4\{ijkl, r\} + 2\sum_{\epsilon} \theta_r(ijkl, \epsilon)\right],
\end{aligned}
$$

where $\overline{H}$ is a function unaltered by interchanges of the numbers i, j, k, l. Accordingly, the quantity

$$\eta_{ijkl} - \frac{\partial^4 y}{\partial x_i \partial x_j \partial x_k \partial x_l} + \sum_r \frac{\partial y}{\partial x_r}\{ijkl, r\}$$

is expressible in terms of known magnitudes, all of which are of orders of derivation less than four.

CHAPTER III

Curvatures of Geodesics in Free General Amplitudes

Secondary magnitudes.

30. Certain magnitudes, connected with the circular curvature of an amplitudinal geodesic, and denoted by L_{ij} for all values $1, \ldots, n$, of i and j independent of one another, are defined by the relation

$$L_{ij}=\sum Y_m \eta_{ij}{}^{(m)}=\sum Y \eta_{ij},$$

the summation being for the range of the homaloidal plenary space. Sometimes they are called the *secondary magnitudes* ; while they belong to the amplitude, they belong specially to any geodesic in the amplitude, because the direction-cosine Y of the prime normal involves the direction-variables of the geodesic itself. Owing to the equations

$$\sum Y \frac{\partial y}{\partial x_r}=0,$$

for all the n values of r, the definition of L_{ij} may also be taken

$$L_{ij}=\sum Y \frac{\partial^2 y}{\partial x_i \partial x_j};$$

in either form, these secondary magnitudes implicitly involve the direction-variables of the geodesic.

The typical relation, for the circular curvature of a geodesic and for the direction of the prime normal, is

$$\frac{Y}{\rho}=\sum_i \sum_j \eta_{ij} x_i' x_j'.$$

Let this relation be multiplied by its own Y, and let the product be summed for all the plenary range of values of m; because $\sum Y^2=1$, we have

$$\frac{1}{\rho}=\sum Y \frac{Y}{\rho}=\sum_i \sum_j x_i' x_j' \left(\sum Y \eta_{ij}\right)=\sum_i \sum_j L_{ij} x_i' x_j',$$

a first expression for the circular curvature of the geodesic in the direction $x_i', \ldots, x_n'$. But the expression is not, ultimately, a bilinear polynomial in the variables x', because they occur implicitly in the coefficients L_{ij}.

A different expression for the curvature comes as follows. We have

$$\frac{1}{\rho^2}=\sum \left(\frac{Y}{\rho}\right)^2=\left(\sum_i \sum_j \eta_{ij} x_i' x_j'\right)^2=\sum_i \sum_j \sum_k \sum_l Z_{ijkl} x_i' x_j' x_k' x_l',$$

the summation extending over the range $i, j, k, l, =1, \ldots, n$, independently of one another. It is more convenient to take the equation in the form

$$\frac{1}{\rho^2}=\sum_i Z_{iiii}x_i'^4+4\sum_i\sum_l Z_{iiil}x_i'^3x_l'+6\sum_i\sum_l Z_{iill}x_i'^2x_l'^2$$
$$+12\sum_i\sum_k\sum_l Z_{iikl}x_i'^2x_k'x_l'+24\sum_i\sum_j\sum_k\sum_l Z_{ijkl}x_i'x_j'x_k'x_l',$$

where in $\sum_i\sum_l$, the summations are for values of i and l unequal to one another; in $\sum_i\sum_k\sum_l$, the summation is for values of i, k, l, no two of which are to be equal; and in $\sum_i\sum_j\sum_k\sum_l$, the summation is for values of i, j, k, l, no two of which are to be equal: these said values being selected otherwise, unrestrictedly, from the range $1, 2, \ldots, n$. In particular, the values of the coefficients Z, in terms of the coefficients η, are

$$\begin{aligned}
Z_{iiii}&=\sum\eta_{ii}^2,\\
Z_{iiil}&=\sum\eta_{ii}\eta_{il},\\
3Z_{iill}&=\sum\eta_{ii}\eta_{ll}+2\sum\eta_{il}^2,\\
3Z_{iikl}&=\sum\eta_{ii}\eta_{kl}+2\sum\eta_{ik}\eta_{il},\\
3Z_{ijkl}&=\sum(\eta_{ij}\eta_{kl}+\eta_{ik}\eta_{lj}+\eta_{il}\eta_{jk}),
\end{aligned}$$

these summations being taken over the whole of the symbols η corresponding to the m space-variables.

Next, proceeding from the same typical relation connecting the curvature and the direction of the prime normal of the geodesic, and multiplying throughout by $\eta_{\alpha\beta}$, the left-hand sides (after addition) give

$$\frac{1}{\rho}\sum Y\eta_{\alpha\beta}, \quad =\frac{1}{\rho}L_{\alpha\beta};$$

and therefore, for all the values $\alpha, \beta, =1, 2, \ldots, n$, we have

$$\frac{L_{\alpha\beta}}{\rho}=\sum_i\sum_j \eta_{\alpha\beta}\eta_{ij}x_i'x_j'.$$

On the right-hand side, only some of the coefficients of the quantities $x_i'x_j'$ are expressible in terms of the coefficients Z_{ijkl}; and, owing to the earlier value of ρ, this relation implicitly gives $L_{\alpha\beta}$ as an irrational homogeneous function of $x_1', \ldots, x_n'$, of degree zero.

In certain investigations, connected with the later curvatures of an amplitudinal geodesic, these quantities L_{ij} frequently occur in the linear combinations, given in denotation and significance by the relation

$$v_i=\sum_{j=1}^{n} L_{ij}x_j'.$$

These combinations can be expressed simply in terms of derivatives of the curvature, taken with regard to the direction-variables x'.

As $1/\rho^2$ is a rational quartic polynomial in the direction-variables $x_1', \ldots, x_n'$, and usually is not a perfect square of a quadratic polynomial (the exceptional instance arising when the amplitude is primary to its plenary homaloidal space), the quantity $1/\rho$ is homogeneous of the second order in those direction-variables and usually not rational. Now

$$\frac{1}{\rho^2}=\sum_m\left(\sum_i\sum_j \eta_{ij}{}^{(m)}x_i'x_j'\right)^2,$$

and therefore

$$\begin{aligned}\frac{\partial}{\partial x_i'}\left(\frac{1}{\rho^2}\right)&=2\sum_m\left[\left(\sum_i\sum_j \eta_{ij}{}^{(m)}x_i'x_j'\right)\frac{\partial}{\partial x_i'}\left\{\sum_i\sum_j \eta_{ij}{}^{(m)}x_ix_j'\right\}\right]\\ &=4\sum_m\left(\sum_i\sum_j \eta_{ij}{}^{(m)}x_i'x_j'\right)\left(\sum_j \eta_{ij}{}^{(m)}x_j'\right)\\ &=4\sum_m\frac{Y_m}{\rho}\left(\sum_j \eta_{ij}{}^{(m)}x_j'\right)\\ &=\frac{4}{\rho}\sum_j L_{ij}x_j'.\end{aligned}$$

Hence

$$2v_i=2\sum_j L_{ij}x_j'=\frac{\partial}{\partial x_i'}\left(\frac{1}{\rho}\right),$$

where, on the right-hand side, the quantity $1/\rho$ is a non-rational homogeneous function of the direction-variables of the geodesic.

Manifestly

$$\sum_i v_ix_i'=\frac{1}{\rho},$$

whether we use the result for v_i in terms of the quantities L_{ij} or use its expression as a partial derivative of $1/\rho$.

As already stated, the forms

$$v_i=\sum_{j=1}^n L_{ij}x_j', \quad (i=1, \ldots, n),$$

in association with the expression for $1/\rho$, given by

$$\frac{1}{\rho}=\sum_i\sum_j L_{ij}x_i'x_j',$$

and connected with that expression by relations

$$2v_i=\frac{\partial}{\partial x_i'}\left(\frac{1}{\rho}\right),$$

will recur later. Moreover, there is a permanent equation

$$U=\sum_i\sum_j A_{ij}x_i'x_j'=1,$$

arising out of the expression for the arc-element of the amplitude; and we shall require forms, denoted by u_i and bearing the same apparent relation to U as are borne by the forms v_i to $1/\rho$, being defined by the equations

$$u_i = \sum_j A_{ij} x_j', \quad 2u_i = \frac{\partial U}{\partial x_i'},$$

for all the values of $i = 1, \ldots, n$.

These values of $2u_i$ are the actual partial derivatives of U with regard to the direction-variables x'. The values of $2v_i$ are the apparent partial derivatives of $1/\rho$, for no account has been taken of the fact that the magnitudes L_{ij} are implicit functions of the direction-variables; on the other hand, these values of $2v_i$ have proved actually equal to those partial derivatives. The explanation of the elusion is to be found, as follows. We have

$$\frac{L_{kl}}{\rho} = \sum_i \sum_j \eta_{kl} \eta_{ij} x_i' x_j';$$

and therefore

$$\frac{1}{\rho} \frac{\partial L_{kl}}{\partial x_i'} + L_{kl} \frac{\partial}{\partial x_i'} \left(\frac{1}{\rho}\right) = 2 \sum_j \eta_{kl} \eta_{ij} x_j'.$$

Multiply throughout by the combination $x_k' x_l'$, and then take the sum of the products for all the values $k, l, = 1, \ldots, n$, as usual.

The left-hand side becomes

$$\frac{1}{\rho} \sum_k \sum_l x_k' x_l' \frac{\partial L_{kl}}{\partial x_i'} + \frac{1}{\rho} \frac{\partial}{\partial x_i'} \left(\frac{1}{\rho}\right).$$

The right-hand side becomes

$$= 2 \sum_j \left\{ \sum_k \sum_l (\eta_{kl} x_k' x_l') \right\} \eta_{ij} x_j'$$

$$= 2 \sum_j \frac{Y}{\rho} \eta_{ij} x_j'$$

$$= \frac{2}{\rho} \sum_j L_{ij} x_j' = \frac{2}{\rho} v_i = \frac{1}{\rho} \frac{\partial}{\partial x_i'} \left(\frac{1}{\rho}\right).$$

Consequently we have the result

$$\sum_k \sum_l x_k' x_l' \frac{\partial L_{kl}}{\partial x_i'} = 0,$$

for all the values $i = 1, \ldots, n$.

When we proceed from the relation

$$\frac{1}{\rho} = \sum_k \sum_l L_{kl} x_k' x_l',$$

and take into account the implicit occurrence of the direction-variables in L_{kl}, we have

$$\frac{\partial}{\partial x_i'} \left(\frac{1}{\rho}\right) = 2 \sum_l L_{il} x_l' + \sum_k \sum_l x_k' x_l' \frac{\partial L_{kl}}{\partial x_i'}.$$

The second term on the right-hand side is equal to zero, by the foregoing result; the first term is $2v_i$ by definition; and therefore we again find the earlier result, obtained by partial differentiation of $1/\rho$, proper account having been taken of the inferred occurrence of the direction-variables of the geodesic in the magnitudes L_{ij}.

31. Connected with the directions of the tangent and the prime normal, and leading to these quantities u_i and v_i, for $i=1, \ldots, n$, these are sums, symmetric in space-magnitudes, of frequent occurrence. One of these is the quantity $\sum y' \frac{\partial y}{\partial x_r}$, for all values of r: the other is $\sum Y' \frac{\partial y}{\partial x_r}$, likewise for all values of r: the sum, in each expression, extending over the range of the plenary space.

For the former, we have

$$\begin{aligned}\sum y' \frac{\partial y}{\partial x_r} &= \sum \left(\frac{\partial y}{\partial x_1} x_1' + \frac{\partial y}{\partial x_2} x_2' + \ldots + \frac{\partial y}{\partial x_n} x_n' \right) \frac{\partial y}{\partial x_r} \\ &= A_{1r} x_1' + A_{2r} x_2' + \ldots + A_{nr} x_n' \\ &= u_r,\end{aligned}$$

with the significance of the symbols u just assigned.

For the latter, we differentiate the relation

$$\sum Y \frac{\partial y}{\partial x_r} = 0$$

along the geodesic, so that

$$\begin{aligned}\sum Y' \frac{\partial y}{\partial x_r} &= -\sum Y \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) \\ &= -\sum Y \left(\frac{\partial^2 y}{\partial x_1 \, \partial x_r} x_1' + \frac{\partial^2 y}{\partial x_2 \, \partial x_r} x_2' + \ldots + \frac{\partial^2 y}{\partial x_n \, \partial x_r} x_n' \right) \\ &= -(L_{1r} x_1' + L_{2r} x_2' + \ldots + L_{nr} x_n'),\end{aligned}$$

by the definitions of the quantities L_{ij} in § 30; and the value of the right-hand side is $-v_r$. Hence we have

$$\sum y' \frac{\partial y}{\partial x_r} = u_r, \quad \sum Y' \frac{\partial y}{\partial x_r} = -v_r, \quad (r=1, \ldots, n),$$

which are the two results in question.

32. It is convenient, at this stage, to consider the binormal and some of the other principal lines in the orthogonal frame of a geodesic. The n-fold homaloid, tangential to the amplitude, contains n independent directions which, in various aggregates, can constitute a completely orthogonal system within the range of the homaloid and at the same time can supply a set of guiding lines for the homaloid;

and, in the latter service, they must be equivalent to the n parametric lines with spatial direction-cosines proportional to

$$\frac{\partial y_1}{\partial x_r}, \quad \frac{\partial y_2}{\partial x_r}, \quad \frac{\partial y_3}{\partial x_r}, \ldots,$$

for $r=1, \ldots, n$.

An amplitudinal geodesic can be regarded, simultaneously with its geodesic quality in the amplitude, as a curve in the plenary homaloidal space within which the amplitude exists; and when thus regarded, its orthogonal frame is built on its $n+n'$ principal organic lines which themselves constitute a completely orthogonal system. The first of these principal lines in the march of the curve is the tangent to the curve; and this tangent lies in the n-fold homaloid tangential to the amplitude. The second of the principal lines, however, is completely outside that tangential n-fold homaloid; this second principal line is the prime normal of the geodesic, and its direction has been proved to be orthogonal to the homaloid in question, being at right angles to every direction in the homaloid. It is therefore natural to enquire whether any of the succeeding principal lines of the geodesic, and, if any, which of these lines, can have their directions lying within the n-fold tangent homaloid of the amplitude.

Accordingly, consider the osculating plane of a geodesic. As the tangent and the prime normal of the geodesic determine the osculating plane, its space-equations can be taken in the form

$$\| \; \bar{y}-y, \; y', \; Y \; \| = 0,$$

the direction-variables of the tangent being $y_1', y_2', y_3', \ldots$, and those of the prime normal being $Y_1, Y_2, Y_3, \ldots$, as in § 21. Any direction in the plane has spatial direction-cosines

$$a y_m' + \beta Y_m,$$

for all values of m, where a and β are parametric for the aggregate of directions; and therefore any direction through the point, represented by the ratios of the quantities $\bar{y}_m - y_m$ for all values of m, is at right angles to the selected planar direction if

$$\sum (\bar{y}-y)(ay'+\beta Y)=0,$$

and consequently it is at right angles to every such direction, that is, it is orthogonal to the plane, if

$$\sum (\bar{y}-y)y'=0, \quad \sum (\bar{y}-y)Y=0.$$

These two equations combined therefore are the space-equations of the $(n+n'-2)$-fold homaloid which is orthogonal to the osculating plane of the geodesic. The complete homaloid, orthogonal to the osculating plane of any curve, contains all the principal lines of the orthogonal frame of the curve which, in rank, are subsequent to the tangent and the prime normal; and therefore all the principal lines of the geodesic, other than its tangent and its prime normal, lie within this homaloid.

Next, consider the $(n+1)$-fold homaloid H, represented by the space-equations

$$\left\| \bar{y}-y, \ \frac{\partial y}{\partial x_1}, \ \frac{\partial y}{\partial x_2}, \ldots, \ \frac{\partial y}{\partial x_n}, \ Y \right\| = 0 :$$

manifestly, it contains the n-fold homaloid tangential to the amplitude and also the prime normal of the geodesic. Its intersection with the $(n+n'-2)$-fold homaloid, orthogonal to the osculating plane, consists of all the points

$$\bar{y}-y = a_1 \frac{\partial y}{\partial x_1} + a_2 \frac{\partial y}{\partial x_2} + \ldots + a_n \frac{\partial y}{\partial x_n} + aY,$$

(where, as usual, $\bar{y}$ is the typical space-coordinate), the parameters $a_1, \ldots, a_n, a$, of which are such as to satisfy the two equations

$$\sum (\bar{y}-y) y' = 0, \quad \sum (\bar{y}-y) Y = 0.$$

Now we have

$$\sum Y \frac{\partial y}{\partial x_r} = 0, \quad \sum Y^2 = 1,$$

the former holding for all values of $r=1, \ldots, n$; the second equation of the orthogonal homaloid therefore requires the condition

$$a = 0.$$

Again, we have

$$\sum y' \frac{\partial y}{\partial x_r} = \sum \frac{\partial y}{\partial x_r} \left(\frac{\partial y}{\partial x_1} x_1' + \frac{\partial y}{\partial x_2} x_2' + \ldots + \frac{\partial y}{\partial x_n} x_n' \right)$$
$$= A_{1r} x_1' + A_{2r} x_2' + \ldots + A_{nr} x_n' = u_r,$$

in the notation stated in § 31; and $\sum y' Y = 0$; hence the first equation of the orthogonal homaloid requires the condition

$$a_1 u_1 + a_2 u_2 + \ldots + a_n u_n = 0.$$

A return will be made later to the latter condition. The former condition shews that the intersection of the $(n+n'-2)$-fold homaloid with the $(n+1)$-fold homaloid is given by the points

$$\bar{y}-y = a_1 \frac{\partial y}{\partial x_1} + a_2 \frac{\partial y}{\partial x_2} + \ldots + a_n \frac{\partial y}{\partial x_n},$$

with a single relation between the parameters $a_1, \ldots, a_n$: that is, the intersection in question lies in the homaloid

$$\left\| \bar{y}-y, \ \frac{\partial y}{\partial x_1}, \ldots, \ \frac{\partial y}{\partial x_n} \right\| = 0,$$

which is the n-fold homaloid tangential to the amplitude. Now the $(n+n'-2)$-fold homaloid contains all the principal lines of the geodesic, subsequent in rank to the tangent and the prime normal; and therefore we may expect that, in the foregoing intersection, several of such principal lines will be found to exist in the n-fold tangent homaloid.

Binormal of a geodesic: the torsion.

33. In the first place, the $(n+1)$-fold homaloid H obviously contains the tangent to the geodesic, the equations of which are

$$\frac{\bar{y}_1 - y_1}{y_1'} = \frac{\bar{y}_2 - y_2}{y_2'} = \ldots = P;$$

for the coordinates of every point on the line satisfy the equations of the $(n+1)$-fold homaloid. This is equally obvious from the fact that the $(n+1)$-fold homaloid contains the tangent homaloid of the amplitude.

In the next place, the $(n+1)$-fold homaloid H contains a second tangent of the geodesic: that is, it contains the tangent at a small arc-distance ϵ along the curve from the initial point. For the equations of such a tangent are

$$\frac{\bar{y}_1 - (y_1 + \epsilon y_1')}{y_1' + \epsilon y_1''} = \frac{\bar{y}_2 - (y_2 + \epsilon y_2')}{y_2' + \epsilon y_2''} = \ldots = Q,$$

and so the coordinates of any point on the line are

$$\bar{y}_m = y_m + \epsilon y_m' + Q(y_m' + \epsilon y_m''),$$

that is,

$$\begin{aligned}\bar{y}_m - y &= (\epsilon + Q)\, y_m' + \frac{\epsilon Q}{\rho} Y_m \\ &= (\epsilon + Q)\, x_1' \frac{\partial y_m}{\partial x_1} + (\epsilon + Q)\, x_2' \frac{\partial y_m}{\partial x_2} + \ldots + (\epsilon + Q)\, x_n' \frac{\partial y_m}{\partial x_n} + \frac{\epsilon Q}{\rho} Y_m,\end{aligned}$$

which manifestly satisfy the equations of the $(n+1)$-fold homaloid. Thus the $(n+1)$-fold homaloid contains the direction of a second tangent, drawn at a consecutive point, distinct from the tangent at the first point in the osculating plane, and not lying in the tangent homaloid; hence the $(n+1)$-fold homaloid, containing two consecutive tangents and the prime normal, contains the flat determined by those tangents and the prime normal, that is, it contains the osculating flat of the geodesic. The osculating flat at any point of a curve contains the binormal of the curve; and therefore the binormal of the geodesic lies within the $(n+1)$-fold homaloid. The typical spatial direction-cosine of the binormal, being denoted as usual (§ 8) by l_3, must therefore be such as to satisfy equations

$$l_3 = (\lambda_1)_3 \frac{\partial y}{\partial x_1} + (\lambda_2)_3 \frac{\partial y}{\partial x_2} + \ldots + (\lambda_n)_3 \frac{\partial y}{\partial x_n} + (\lambda)_3 Y,$$

for appropriate values of the parameters $(\lambda_1)_3, (\lambda_2)_3, \ldots, (\lambda_n)_3, (\lambda)_3$. But, because the binormal is always at right angles to the prime normal, given as to direction-cosines by the typical quantity Y, we have

$$\sum l_3 Y = 0;$$

and as

$$\sum Y \frac{\partial y}{\partial x_r} = 0$$

for $r = 1, \ldots, n$, it follows that

$$(\lambda)_3 = 0.$$

Consequently

$$l_3 = (\lambda_1)_3 \frac{\partial y}{\partial x_1} + (\lambda_2)_3 \frac{\partial y}{\partial x_2} + \dots + (\lambda_n)_3 \frac{\partial y}{\partial x_n},$$

for appropriate values of $(\lambda_r)_3$, which will be determined later; and therefore the binormal of a geodesic lies in the n-fold homaloid tangential to the amplitude.

It is an immediate inference that, for a surface existing freely in multiple space, the binormal of any superficial geodesic lies in the tangent plane of the surface. When the homaloidal plenary space is triple, so that we have to deal with the customary theory of surfaces due to Gauss, this result follows at once from the consideration that, as the binormal is perpendicular to the prime normal, which itself is at right angles to the tangent plane, the binormal must lie in the tangent plane.

34. It remains to find the values of the parametric coefficients in the expression for l_3, as a typical direction-cosine of the binormal. This typical direction-cosine, in the Frenet equations, is given by the relation

$$\frac{dl_2}{ds} = -\frac{l_1}{\rho_1} + \frac{l_3}{\rho_2},$$

in general; while, for a geodesic, we have

$$l_1 = y', \quad l_2 = Y,$$

all associated with the typical variable y. Hence, using ρ and σ instead of ρ_1 and ρ_2, we have

$$\frac{l_3}{\sigma} = \frac{y'}{\rho} + Y'.$$

Consequently the parameters must be such as to satisfy the set of equations

$$(\lambda_1)_3 \frac{\partial y}{\partial x_1} + \dots\dots + (\lambda_n)_3 \frac{\partial y}{\partial x_n} = \frac{\sigma}{\rho} y' + \sigma Y'.$$

There are only n parameters, while there are $n + n'$ relations: the co-existence of all the relations is ensured by the property that, as the binormal lies in the tangent n-fold homaloid of the amplitude, its direction-cosines are expressible linearly in terms of the direction-parameters of leading lines in the homaloid.

Let the typical equation be multiplied by $\frac{\partial y}{\partial x_r}$, where r has any value $1, \dots, n$; and then let the sum of the products be taken over all the equations. The value of the left-hand side in this sum

$$= A_{1r}(\lambda_1)_3 + A_{2r}(\lambda_2)_3 + \dots + A_{nr}(\lambda_n)_3:$$

while the value of the right-hand side in the same sum

$$= \frac{\sigma}{\rho}\left(\sum y' \frac{\partial y}{\partial x_r}\right) + \sigma\left(\sum Y' \frac{\partial y}{\partial x_r}\right)$$

$$= \sigma\left(\frac{u_r}{\rho} - v_r\right),$$

by the results in § 31. Hence

$$A_{1r}(\lambda_1)_3+A_{2r}(\lambda_2)_3+\ldots+A_{nr}(\lambda_n)_3=\sigma\left(\frac{u_r}{\rho}-v_r\right),$$

an equation holding for $r=1, \ldots, n$, and therefore typical of an aggregate of n equations, determining the n parameters. When these n equations are resolved, we have

$$\begin{aligned}(\lambda_\mu)_3&=\frac{\sigma}{\Omega}\sum_r\left\{a_{\mu r}\left(\frac{u_r}{\rho}-v_r\right)\right\}\\&=\frac{\sigma}{\Omega\rho}\left(\sum_r a_{\mu r}u_r\right)-\frac{\sigma}{\Omega}\left(\sum_r a_{\mu r}v_r\right),\end{aligned}$$

the summation being over the values $r=1, \ldots, n$. Now

$$\begin{aligned}\sum_r a_{\mu r}u_r&=\sum_r a_{\mu r}(A_{1r}x_1'+A_{2r}x_2'+\ldots+A_{rn}x_n')\\&=\Omega x_\mu',\end{aligned}$$

because of the determinantal properties

$$\sum_r a_{\mu r}A_{\mu r}=\Omega, \quad \sum_r a_{\mu r}A_{\lambda r}=0,$$

when, in the latter, λ and μ are different. Thus, finally,

$$(\lambda_\mu)_3=\frac{\sigma}{\rho}x_\mu'-\frac{\sigma}{\Omega}\left(\sum_r a_{\mu r}v_r\right),$$

equations which give the values of the parameters in the expressions for the direction-cosines of the binomial.

One analytical result of importance can be deduced. When these values of $(\lambda_\mu)_3$ are inserted in the expression for l_3, the aggregate of terms involving the variables x' is

$$=\frac{\sigma}{\rho}\left(\sum_\mu\frac{\partial y}{\partial x_\mu}x_\mu'\right)=\frac{\sigma}{\rho}y',$$

and the aggregate of terms involving the magnitudes v_r is

$$-\frac{\sigma}{\Omega}\left\{\sum_\mu\sum_r a_{\mu r}v_r\frac{\partial y}{\partial x_\mu}\right\}.$$

Also we have

$$l_3=\frac{\sigma}{\rho}y'+\sigma Y'.$$

Consequently we have the result

$$Y'=-\frac{1}{\Omega}\sum_\mu\sum_r\left(a_{\mu r}v_r\frac{\partial y}{\partial x_\mu}\right),$$

giving the arc-derivatives of the spatial direction-cosines of the prime normal of the geodesic.

As an immediate corollary, we have the result

$$\sum Y'\frac{\partial y}{\partial x_i}=-\frac{1}{\Omega}\sum_\mu\sum_r a_{\mu r}v_r\left(\sum\frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_i}\right)=-\frac{1}{\Omega}\sum_\mu\sum_r a_{\mu r}A_{\mu i}v_r=-v_i,$$

already obtained in § 31.

35. Further inferences may be noted; they will arise also in other associations. By the foregoing analysis, we have

$$\sum l_3 \frac{\partial y}{\partial x_r} = A_{1r}(\lambda_1)_3 + A_{2r}(\lambda_2)_3 + \dots + A_{nr}(\lambda_n)_3 = \sigma\left(\frac{u_r}{\rho} - v_r\right),$$

for all values $r=1, \dots, n$, and therefore

$$\frac{1}{\sigma}\sum l_3 \frac{\partial y}{\partial x_r} = \frac{u_r}{\rho} - v_r.$$

Consequently for all directions in the amplitude, given by the n equations

$$\frac{u_r}{\rho} - v_r = 0,$$

which are homogeneous in the direction-variables $x_1', \dots, x_n'$, made specific by the permanent relation $\sum\limits_i \sum\limits_j A_{ij} x_i' x_j' = 1$, we have

$$\frac{1}{\sigma}\sum l_3 \frac{\partial y}{\partial x_r} = 0.$$

Now not all the magnitudes $\sum l_3 \frac{\partial y}{\partial x_r}$, for $r=1, \dots, n$, can be zero; the implication would be that the line, with direction-cosines typified by l_3, is orthogonal to the n-fold homaloid touching the amplitude whereas, in fact, it lies within that homaloid. Hence, for all the directions specified, we have

$$\frac{1}{\sigma} = 0:$$

that is, the torsion vanishes. Consequently, through any point in the amplitude, there are directions given by

$$\frac{u_r}{\rho} - v_r = 0, \quad (r = 1, \dots, n),$$

such that the torsion of the geodesic in those directions is zero. We shall find (§ 54) that this property is characteristic of the curves of circular curvature at any point in the amplitude.

Next, it was noted (§ 8) that, for all geodesics in any amplitude, where direction of the prime normal of the geodesic is denoted by Y as the typical direction-cosine, we have

$$\sum Y'^2 = \frac{1}{\rho^2} + \frac{1}{\sigma^2},$$

(the summation on the left-hand side extending over the space-range), as an immediate consequence of the Frenet equation

$$\frac{l_3}{\sigma} = \frac{y'}{\rho} + Y'.$$

It has (p. 77) been proved that

$$Y' = -\frac{1}{\Omega}\sum_\mu \sum_r \left(a_{\mu r} v_r \frac{\partial y}{\partial x_\mu}\right);$$

and therefore, for any geodesic in the amplitude,

$$\begin{aligned}\Omega^2\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) &= \sum_m \left\{\sum_\mu \sum_r \left(a_{\mu r} v_r \frac{\partial y_m}{\partial x_\mu}\right)\right\}^2 \\ &= \sum_m \sum_\lambda \sum_\mu \sum_r \sum_t a_{\mu r} a_{\lambda t} v_r v_t \frac{\partial y}{\partial x_\lambda}\frac{\partial y}{\partial x_\mu} \\ &= \sum_\lambda \sum_\mu \sum_r \sum_t A_{\lambda\mu} a_{\mu r} a_{\lambda t} v_r v_t.\end{aligned}$$

Now $\sum_\mu A_{\lambda\mu} a_{\mu r}$ is equal to zero when λ and r are different, and is equal to Ω when λ and r are the same; and therefore the right-hand side

$$= \Omega \sum_r \sum_t a_{rt} v_r v_t.$$

Consequently

$$\Omega\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) = \sum_r \sum_t a_{rt} v_r v_t,$$

a relation which gives an expression for the torsion when the value of the circular curvature is substituted.

A modified form can be given to the relation, so as to yield an explicit expression for the torsion alone. We have

$$u_i = A_{1i} x_1' + \ldots + A_{1n} x_n',$$

and therefore

$$\sum_i a_{ij} u_i = \Omega x_j'.$$

Hence

$$\sum_i \sum_j a_{ij} u_i u_j = \Omega \sum_j u_j x_j' = \Omega.$$

Hence also, as (§ 30)

$$\frac{1}{\rho} = \sum_j v_j x_j',$$

we have

$$\frac{\Omega}{\rho} = \sum_j v_j \left(\sum_i a_{ij} u_i\right) = \sum_i \sum_j a_{ij} u_i v_j.$$

Consequently

$$\begin{aligned}\frac{\Omega^2}{\sigma^2} &= \Omega \sum_r \sum_t a_{rt} v_r v_t - \left(\sum_i \sum_j a_{ij} u_i v_j\right)^2 \\ &= \left(\sum_i \sum_j a_{ij} u_i u_j\right)\left(\sum_i \sum_j a_{ij} v_i v_j\right) - \left(\sum_i \sum_j a_{ij} u_i v_j\right)^2 \\ &= \sum_i \sum_j \sum_k \sum_l \begin{vmatrix} a_{ik}, & a_{il} \\ a_{jk}, & a_{jl}\end{vmatrix} \begin{vmatrix} u_i, & u_j \\ v_i, & v_j\end{vmatrix} \begin{vmatrix} u_k, & u_l \\ v_k, & v_l\end{vmatrix},\end{aligned}$$

the explicit expression indicated. It can also be stated in the form

$$\frac{\Omega}{\sigma^2}=\begin{vmatrix} A_{11}, & A_{12}, & \dots, & A_{1n}, & u_1, & v_1 \\ A_{21}, & A_{22}, & \dots, & A_{2n}, & u_2, & v_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{n1}, & A_{n2}, & \dots, & A_{nn}, & u_n, & v_n \\ u_1, & u_2, & \dots, & u_n, & 0, & 0 \\ v_1, & v_2, & \dots, & v_n, & 0, & 0 \end{vmatrix}.$$

We may remark, in passing, that a similar equation for the circular curvature exists in the form

$$\begin{vmatrix} 1/\rho, & v_1, & v_2, & \dots, & v_n \\ u_1, & A_{11}, & A_{12}, & \dots, & A_{1n} \\ u_2, & A_{21}, & A_{22}, & \dots, & A_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ u_n, & A_{n1}, & A_{n2}, & \dots, & A_{nn} \end{vmatrix}=0.$$

NOTE. In the course of § 32, it was shewn that, when any line in the n-fold tangent homaloid, which also lies in $(n+n'-2)$-fold homaloid orthogonal to the osculating plane of the geodesic, has its typical spatial direction-cosine given by an expression

$$a_1\frac{\partial y}{\partial x_1}+a_2\frac{\partial y}{\partial x_2}+\dots+a_n\frac{\partial y}{dx_n},$$

where $a_1, \dots, a_n$ are parameters, these parameters must satisfy the condition

$$a_1u_1+a_2u_2+\dots+a_nu_n=0.$$

The binormal is such a line, and its parameters are $(\lambda_\mu)_3$, for $\mu=1, \dots, n$, where

$$(\lambda_\mu)_3=\frac{\sigma}{\rho}x_\mu'-\frac{\sigma}{\Omega}\sum_r a_{\mu r}v_r;$$

and therefore the condition

$$\sum_\mu\left\{\frac{\sigma}{\rho}x_\mu'-\frac{\sigma}{\Omega}\sum_r a_{\mu r}v_r\right\}u_\mu=0$$

should be satisfied. Now

$$\sum_\mu u_\mu x_\mu'=1,$$

$$\sum_r\sum_\mu a_{\mu r}u_\mu v_r=\sum_r\{(\Omega x_r')v_r\}=\frac{\Omega}{\rho};$$

and therefore the required condition is, in fact, satisfied, so far as concerns the expressions for the direction-cosines of the binormal.

Expression for $\frac{d}{ds}\left(\frac{1}{\rho}\right)$ along a geodesic.

36. The earliest Frenet equations, being

$$\frac{dl_1}{ds}=\frac{l_2}{\rho}, \quad \frac{dl_2}{ds}=-\frac{l_1}{\rho}+\frac{l_3}{\sigma},$$

for any curve, become

$$l_1=y', \quad l_2=\rho y'', \quad \frac{l_3}{\sigma}=\frac{y'}{\rho}+\rho' y''+\rho y''',$$

for a geodesic on the surface, while $\rho y''=Y$. Now we have

$$\sum Y l_3=0, \quad \sum Y y'=0\,;$$

and therefore

$$\rho' \sum \frac{Y^2}{\rho}+\rho \sum Y y'''=0,$$

so that

$$\frac{d}{ds}\left(\frac{1}{\rho}\right)=\sum Y y'''.$$

The value of y''' has been obtained (§ 24) in the form

$$y'''=\sum_i \sum_j \sum_k \eta_{ijk} x_i' x_j' x_k',$$

where

$$\begin{aligned}\eta_{ijk}=\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k}&-\sum_r \frac{dy}{\partial x_r}\{ijk, r\}\\ &-\sum_p [\eta_{ip}\{jk, p\}+\eta_{jp}\{ki, p\}+\eta_{kp}\{ij, p\}]\\ &-\sum_r \sum_p \frac{\partial y}{\partial x_r}[\{ip, r\}\{jk, p\}+\{jp, r\}\{ki, p\}+\{kp, r\}\{ij, p\}].\end{aligned}$$

We have

$$\sum Y \frac{\partial y}{\partial x_r}=0, \quad \sum Y \eta_{\lambda\mu}=L_{\lambda\mu},$$

for all values r, λ, μ, $=1, \ldots, n$; and therefore

$$\begin{aligned}\sum Y \eta_{ijk}&=\sum\left(Y \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k}\right)-\sum_p [L_{ip}\{jk, p\}+L_{jp}\{ki, p\}+L_{kp}\{ij, p\}]\\ &=e_{ijk},\end{aligned}$$

on introducing the symbol e_{ijk} to denote the value obtained for $\sum Y \eta_{ijk}$. Hence we have the arc-derivative, taken along the geodesic, of the circular curvature of the geodesic in the form

$$\frac{d}{ds}\left(\frac{1}{\rho}\right)=\sum_i \sum_j \sum_k e_{ijk} x_i' x_j' x_k',$$

with the coefficients e_{ijk} as defined, such coefficients being unaltered by interchange of the integers i, j, k.

The result can be obtained in a different manner, which leads incidentally to expressions for the arc-derivatives of the quantities L_{ij}. From the definitions in § 30, we have

$$L_{ij} = \sum Y\eta_{ij} = \sum Y \frac{\partial^2 y}{\partial x_i \partial x_j};$$

and therefore, on differentiating along the geodesic,

$$\frac{dL_{ij}}{ds} = \sum_k \left\{ x_k' \left(\sum Y \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} \right) \right\} + \sum \left(Y' \frac{\partial^2 y}{\partial x_i \partial x_j} \right).$$

A value of Y' was obtained (§ 34) in the form

$$Y' = -\frac{1}{\Omega} \sum_\mu \sum_r \left\{ a_{\mu r} v_r \frac{\partial y}{\partial x_\mu} \right\};$$

hence

$$\begin{aligned} \sum Y' \frac{\partial^2 y}{\partial x_i \partial x_j} &= -\frac{1}{\Omega} \sum_\mu \sum_r \left\{ a_{\mu r} v_r \left(\sum \frac{\partial y}{\partial x_\mu} \frac{\partial^2 y}{\partial x_i \partial x_j} \right) \right\} \\ &= -\frac{1}{\Omega} \sum_\mu \sum_r a_{\mu r} v_r [ij, \mu] \\ &= -\frac{1}{\Omega} \sum_\mu \sum_r \sum_l a_{\mu r} v_r A_{l\mu} \{ij, l\}, \end{aligned}$$

on the introduction of the Christoffel symbols. As usual,

$$\sum_\mu a_{\mu r} A_{l\mu} = 0, \quad \text{or} \quad \Omega,$$

according as l and r are different, or are the same; thus

$$\sum Y' \frac{\partial^2 y}{\partial x_i \partial x_j} = -\sum_r v_r \{ij, r\},$$

and so

$$\frac{dL_{ij}}{ds} = \sum_k \left\{ x_k' \left(\sum Y \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} \right) \right\} - \sum_r v_r \{ij, r\}.$$

The circular curvature is given by the relation

$$\frac{1}{\rho} = \sum_i \sum_j L_{ij} x_i' x_j';$$

and therefore

$$\begin{aligned} \frac{d}{ds}\left(\frac{1}{\rho}\right) &= \sum_i \sum_j \frac{dL_{ij}}{ds} x_i' x_j' + \sum_\lambda \sum_\mu L_{\lambda\mu} (x_\lambda'' x_\mu' + x_\lambda' x_\mu'') \\ &= \sum_k \left(\sum Y \frac{d^3 y}{\partial x_i \partial x_j \partial x_k} \right) x_i' x_j' x_k' - \sum_i \sum_j \sum_r x_i' x_j' v_r \{ij, r\} \\ &\quad - \sum_\lambda \sum_\mu \sum_p \sum_t L_{\lambda\mu} x_\mu' x_p' x_t' \{pt, \lambda\} \\ &\quad - \sum_\lambda \sum_\mu \sum_p \sum_t L_{\lambda\mu} x_\lambda' x_p' x_t' \{pt, \mu\}. \end{aligned}$$

Also

$$v_r = \sum_q L_{rq} x_q' ;$$

and therefore

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \sum_i \sum_j \sum_k R_{ijk} x_i' x_j' x_k',$$

where

$$R_{ijk} = \sum \left(Y \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} \right) - \sum_r L_{kr}\{ij, r\} - \sum_\lambda L_{i\lambda}\{jk, \lambda\} - \sum_\mu L_{j\mu}\{ki, \mu\}$$

$$= e_{ijk},$$

with the former definition of the magnitudes e_{ijk}: that is,

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \sum_i \sum_j \sum_k e_{ijk} x_i' x_j' x_k',$$

the summation being taken for all the values $i, j, k, = 1, \ldots, n$, independently of one another.

Also, we have

$$\begin{aligned} \frac{dL_{ij}}{ds} &= \sum_k \left\{ \left(\sum Y \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} \right) x_k' \right\} - \sum_r \sum_k L_{rk} x_k' \{ij, r\} \\ &= \sum_k x_k' \left[e_{ijk} + \sum_\mu L_{\mu i}\{jk, \mu\} + \sum_\mu L_{\mu j}\{ik, \mu\} \right], \end{aligned}$$

which is the general expression for the arc-derivative of the magnitude L_{ij} taken along the geodesic.

Just as it is convenient to have partial derivatives of $\sum_i \sum_j A_{ij} x_i' x_j'$ and of $\sum_i \sum_j L_{ij} x_i' x_j'$ with regard to $x_1', x_2', \ldots, x_n'$, denoted by the special symbols u and v, so it is convenient to denote the like derivatives of $\sum_i \sum_j \sum_k e_{ijk} x_i' x_j' x_k'$ by a special symbol. We write

$$w_\mu = \tfrac{1}{3} \frac{\partial}{\partial x_\mu'} \left(\sum_i \sum_j \sum_k e_{ijk} x_i' x_j' x_k' \right),$$

for all the values $\mu = 1, \ldots, n$. Apparently, these quantities $w_1, \ldots, w_n$, are homogeneous quadratic forms in the variables x': the coefficients e_{ijk}, however, involve those variables implicitly, through the occurrence of Y and of magnitudes $L_{\lambda\mu}$ in their values as defined.

37. In subsequent investigations, we shall require the values of arc-derivatives of the quantities u and v along the geodesic.

Differentiating the value of u_r, as given by the relation

$$u_r = \sum_\mu A_{r\mu} x_\mu',$$

we have

$$\frac{du_r}{ds}=\sum_\mu x_\mu'\frac{dA_{r\mu}}{ds}+\sum_p A_{rp}x_p''$$
$$=\sum_\mu\sum_t x_\mu' x_t'\frac{\partial A_{r\mu}}{\partial x_t}-\sum_i\sum_j\sum_p A_{rp}\{ij,\,p\}x_i'x_j'.$$

But, by § 12,

$$\frac{\partial A_{r\mu}}{\partial x_t}=\sum_p[A_{rp}\{\mu t,\,p\}+A_{\mu p}\{rt,\,p\}];$$

and

$$\sum_\mu\sum_t\sum_p A_{rp}\{\mu t,\,p\}x_\mu' x_t'=\sum_i\sum_j\sum_p A_{rp}\{ij,\,p\}x_i'x_j',$$

on changing μ, t, on the left-hand side, into i and j. Hence

$$\frac{du_r}{ds}=\sum_\mu\sum_t\sum_p A_{\mu p}\{rt,\,p\}x_\mu' x_t'$$
$$=\sum_t\sum_p[\{rt,\,p\}x_t']u_p,$$

by the definition of u_p.

As the combination $\sum_t\{rt,\,p\}x_t'$ is of frequent occurrence, we denote it by g_{pr} where convenient, so that

$$g_{pr}=\sum_t\{rt,\,p\}x_t',$$

where it should be noted that, in g_{pr}, the numbers r and p are not interchangeable: and g_{pr} has the same relation to $-x_p''$, the value of which is

$$\sum_r\sum_t\{rt,\,p\}x_r'x_t',$$

as u_i and v_i have to $\sum_i\sum_j A_{ij}x_i'x_j'$ and $\sum_i\sum_j L_{ij}x_i'x_j'$ respectively. Then we have

$$\frac{du_r}{ds}=\sum_p g_{pr}u_p=g_{1r}u_1+g_{2r}u_2+\dots+g_{nr}u_n,$$

which is the value of the derivative of u_r; and the result holds for $r=1,\dots,n$.

We proceed similarly to obtain the derivative of v_r, given by

$$v_r=\sum_\mu L_{r\mu}x_\mu'.$$

On differentiating along the geodesic, we have

$$\frac{dv_r}{ds}=\sum_\mu x_\mu'\frac{dL_{r\mu}}{ds}+\sum_p L_{rp}x_p'',$$

which, on substitution of the derivatives of the successive quantities $L_{r\mu}$, becomes

$$\frac{dv_r}{ds}=\sum_\mu\sum_k e_{r\mu k}x_\mu' x_k'+\sum_\mu x_\mu'\sum_k\sum_t[x_k'L_{tr}\{k\mu,\,t\}+x_k'L_{t\mu}\{kr,\,t\}]$$
$$-\sum_i\sum_j\sum_p L_{rp}\{ij,\,p\}x_i'x_j'.$$

Now

$$\sum_i \sum_j \sum_p L_{rp}\{ij, p\} x_i' x_j' = \sum_\mu \sum_k \sum_p L_{rp}\{k\mu, p\} x_\mu' x_k'$$
$$= \sum_\mu \sum_k \sum_t L_{tr}\{k\mu, t\} x_\mu' x_k',$$

on first changing i and j into μ and k, and afterwards changing p into t; and therefore

$$\frac{dv_r}{ds} = \sum_\mu \sum_k e_{r\mu k} x_\mu' x_k' + \sum_\mu \sum_k \sum_t L_{t\mu}\{kr, t\} x_\mu' x_k'.$$

Also

$$\sum_\mu \sum_k e_{r\mu k} x_\mu' x_k' = \tfrac{1}{3} \frac{\partial}{\partial x_r'} \left(\sum_a \sum_b \sum_c e_{abc} x_a' x_b' x_c' \right) = w_r,$$

$$\sum_\mu L_{t\mu} x_\mu' = v_t, \quad \sum_k \{kr, t\} x_k' = g_{tr};$$

and therefore

$$\frac{dv_r}{ds} - w_r + \sum_t g_{tr} v_t$$
$$= w_r + g_{1r} v_1 + g_{2r} v_2 + \dots + g_{nr} v_n,$$

which is the value of the derivative of v_r. The result holds for $r = 1, \dots, n$.

38. Further, using the value

$$Y' = -\frac{1}{\Omega} \sum_t \sum_\mu a_{t\mu} v_t \frac{\partial y}{\partial x_\mu},$$

we have

$$\sum Y' \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) = -\frac{1}{\Omega} \sum_t \sum_\mu a_{t\mu} v_t \sum \left\{ \frac{\partial y}{\partial x_\mu} \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) \right\}.$$

Now

$$\sum \frac{\partial y}{\partial x_\mu} \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) = \sum \sum_i \frac{\partial y}{\partial x_\mu} \frac{\partial^2 y}{\partial x_r \partial x_i} x_i'$$
$$= \sum_i x_i' [ri, \mu]$$
$$= \sum_i \sum_k x_i' A_{k\mu} \{ri, k\}$$
$$= \sum_k g_{kr} A_{k\mu};$$

and therefore

$$\sum Y' \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) = -\frac{1}{\Omega} \sum_t \sum_\mu \sum_k g_{kr} A_{k\mu} a_{t\mu} v_t.$$

As usual, we have

$$\sum_\mu A_{k\mu} a_{t\mu} = 0, \text{ or } \Omega,$$

according as k and t are different or are the same; hence

$$\sum Y' \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) = -\sum g_{tr} v_t.$$

Earlier (§ 31), the relation

$$\sum Y' \frac{\partial y}{\partial x_r} = -v_r$$

was established; therefore

$$\sum Y'' \frac{\partial y}{\partial x_r} + \sum Y' \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) = -\frac{dv_r}{ds} = -w_r - \sum_t g_{tr} v_t,$$

and thus there follows the result

$$\sum Y'' \frac{\partial y}{\partial x_r} = -w_r.$$

Moreover, when second derivatives along the geodesic arc are effected on the relation $\sum Y \frac{\partial y}{\partial x_r} = 0$, we have

$$\sum \left\{ Y \frac{d^2}{ds^2}\left(\frac{\partial y}{\partial x_r}\right)\right\} + 2\sum \left\{ Y' \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right)\right\} + \sum Y'' \frac{\partial y}{\partial x_r} = 0;$$

and therefore, on substituting the values of the second and the third terms, it follows that

$$\sum \left\{ Y \frac{d^2}{ds^2}\left(\frac{\partial y}{\partial x_r}\right)\right\} = w_r + \sum_t g_{tr} v_t.$$

This result can be otherwise established, by evaluating

$$\sum Y \left[\sum_i \frac{\partial^2 y}{\partial x_r \, \partial x_i} x_i'' + \sum_i \sum_j \left(\frac{\partial^3 y}{\partial x_r \, \partial x_i \, \partial x_j} x_i' x_j' \right) \right].$$

The foregoing relation

$$\sum Y'' \frac{\partial y}{\partial x_r} = -w_r$$

can be deduced, by constructing an explicit expression for Y'' in terms of magnitudes belonging to the amplitude. Because

$$Y' = -\frac{1}{\Omega} \sum_i \sum_j a_{ij} v_i \frac{\partial y}{\partial x_j},$$

we have

$$Y'' = -\sum_i \sum_j \frac{d}{ds}\left(\frac{a_{ij}}{\Omega}\right) v_i \frac{\partial y}{\partial x_j} - \frac{1}{\Omega} \sum_i \sum_j \sum_k a_{ij} v_i x_k' \frac{\partial^2 y}{\partial x_j \, \partial x_k}$$
$$- \frac{1}{\Omega} \sum_i \sum_j \left\{ a_{ij} \frac{\partial y}{\partial x_j} \left(w_i + \sum_t g_{ti} v_t \right) \right\}.$$

Now, by § 13,

$$\begin{aligned} \frac{d}{ds}\left(\frac{a_{ij}}{\Omega}\right) &= \sum_k x_k' \frac{\partial}{\partial x_k}\left(\frac{a_{ij}}{\Omega}\right) \\ &= -\frac{1}{\Omega} \sum_k \sum_p x_k' \left[a_{pi} \{kp, j\} + a_{pj} \{kp, i\} \right] \\ &= -\frac{1}{\Omega} \sum_p (a_{ip} g_{jp} + a_{jp} g_{ip}), \end{aligned}$$

so that the first term in the expression for Y'' becomes

$$\frac{1}{\Omega}\sum_i \sum_j \sum_p (a_{ip}g_{jp}+a_{jp}g_{ip}) v_i \frac{\partial y}{\partial x_j}.$$

In the second term of that expression, we substitute for $\dfrac{\partial^2 y}{\partial x_j \partial x_k}$ from the relation

$$\frac{\partial^2 y}{\partial x_j\, \partial x_k}=\eta_{jk}+\sum_r \{jk, r\} \frac{\partial y}{\partial x_r}.$$

That second term then consists of two parts. The first is

$$-\frac{1}{\Omega}\sum_i \sum_j \sum_k a_{ij} v_i x_k' \eta_{jk};$$

while the other part

$$\begin{aligned} &= -\frac{1}{\Omega}\sum_i \sum_j \sum_k \sum_r a_{ij} v_i x_k' \{jk, r\} \frac{\partial y}{\partial x_r} \\ &= -\frac{1}{\Omega}\sum_i \sum_j \sum_r a_{ij} v_i g_{rj} \frac{\partial y}{\partial x_r} \\ &= -\frac{1}{\Omega}\sum_i \sum_j \sum_p a_{ip} g_{jp} v_i \frac{\partial y}{\partial x_j}. \end{aligned}$$

Thus the second term in the expression for Y''

$$= -\frac{1}{\Omega}\sum_i \sum_j \sum_k a_{ij} v_i x_k' \eta_{jk} - \frac{1}{\Omega}\sum_i \sum_j \sum_p a_{ip} g_{jp} v_i \frac{\partial y}{\partial x_j}.$$

The third term in the expression for Y''

$$\begin{aligned} &= -\frac{1}{\Omega}\sum_i \sum_j a_{ij} w_i \frac{\partial y}{\partial x_j} - \frac{1}{\Omega}\sum_i \sum_j \sum_t a_{tj} g_{ti} v_t \frac{\partial y}{\partial x_j} \\ &= -\frac{1}{\Omega}\sum_i \sum_j a_{ij} w_i \frac{\partial y}{\partial x_j} - \frac{1}{\Omega}\sum_i \sum_j \sum_p a_{jp} g_{ip} v_i \frac{\partial y}{\partial x_j}. \end{aligned}$$

Gathering together the three terms thus modified, we have an expression for Y'' in the form

$$Y'' = -\frac{1}{\Omega}\sum_i \sum_j a_{ij} w_i \frac{\partial y}{\partial x_j} - \frac{1}{\Omega}\sum_i \sum_j \sum_k a_{ij} v_i x_k' \eta_{jk}.$$

The cited relation follows immediately. For all values of i, j, k, we have

$$\sum \eta_{ij} \frac{\partial y}{\partial x_k}=0;$$

and therefore

$$\sum Y'' \frac{\partial y}{\partial x_r} = -\frac{1}{\Omega}\sum_i \sum_j a_{ij} w_i A_{jr} = -w_r,$$

using the customary properties of the first minors of Ω.

A modified form of the second term in Y'' will be obtained in § 40; and the

relation between the two forms of this second term will result from later equations to be established (§ 44) concerning the magnitudes η_{jk}.

Trinormal of a geodesic: the tilt.

39. We next proceed to find the direction of the trinormal of a geodesic of the amplitude and the magnitude of the tilt.

For the purpose, we return to the $(n+1)$-fold homaloid H

$$\left\| \bar{y}-y, \quad \frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n}, \quad Y \right\| = 0.$$

It has been shewn to contain the tangent at the initial point O and the osculating plane of the geodesic at O in the direction $x_1', \ldots, x_n'$. It was also shewn to contain the tangent at a point contiguous to O, by noting that the equations of such a tangent satisfy the equations of the homaloid; and, for this purpose, the small arc to the contiguous point was, in magnitude, retained to the first order; and consequently the $(n+1)$-fold homaloid contains the osculating flat of the amplitudinal geodesic. We now proceed to prove that the $(n+1)$-fold homaloid contains the tangent to the geodesic at a next consecutive point, thus containing a direction not lying in the osculating flat: and, for the purpose, it is sufficient to take the tangent at a point contiguous to O, now retaining the second power of the small arc. The equations of this tangent now are

$$\frac{\bar{y}_1-(y_1+\epsilon y_1'+\frac{1}{2}\epsilon^2 y_1'')}{y_1'+\epsilon y_1''+\frac{1}{2}\epsilon^2 y_1'''} = \frac{\bar{y}_2-(y_2+\epsilon y_2'+\frac{1}{2}\epsilon^2 y_2'')}{y_2'+\epsilon y_2''+\frac{1}{2}\epsilon^2 y_2'''} = \ldots = Q,$$

where Q is parametric along the tangent; and therefore any point on the tangent is given, as to its coordinates, by the typical relation

$$\bar{y}-y=y'(\epsilon+Q)+y''(\tfrac{1}{2}\epsilon^2+\epsilon Q)+y'''\tfrac{1}{2}\epsilon^2 Q.$$

It has been proved (§ 33) that the binormal of the geodesic lies in the n-fold homaloid touching the amplitude, its direction-cosines being such that

$$l_3=(\lambda_1)_3\frac{\partial y}{\partial x_1}+\ldots+(\lambda_n)_3\frac{\partial y}{\partial x_n},$$

the coefficients $(\lambda_i)_3$ being determinate. For the geodesic, we have

$$y'=\sum_i \frac{\partial y}{\partial x_i}x_i', \quad \rho y''=Y,$$

and, as

$$\frac{l_3}{\sigma\rho}=y'''+\frac{\rho'}{\rho}y''+\frac{y'}{\rho^2}$$

always, so that

$$y'''=\frac{l_3}{\sigma\rho}-\frac{\rho'}{\rho^2}Y-\frac{y'}{\rho^2},$$

we have

$$y''' = -Y\frac{\rho'}{\rho^2} + \sum_i \frac{\partial y}{\partial x_i}\left\{\frac{1}{\sigma\rho}(\lambda_i)_3 - \frac{1}{\rho^2}x_i'\right\}.$$

Thus the typical coordinate of any point on the tangent acquires the form

$$\bar{y} - y = \frac{Y}{\rho}\left(\tfrac{1}{2}\epsilon^2 + \epsilon Q - \tfrac{1}{2}\epsilon^2 Q\frac{\rho'}{\rho}\right)$$
$$+ \sum_i \frac{\partial y}{\partial x_i}\left[(\epsilon + Q)x_i' + \tfrac{1}{2}\epsilon^2 Q\left\{\frac{1}{\sigma\rho}(\lambda_i)_3 - \frac{1}{\rho^2}x_i'\right\}\right];$$

and therefore the coordinates of any point on this new tangent satisfy the equations of the $(n+1)$-fold homaloid H: that is, the tangent lies in the $(n+1)$-fold homaloid H.

This tangent at a second consecutive point of the geodesic, and the osculating flat at the initial point O, when they are combined, determine the osculating block of the amplitudinal geodesic at O, which therefore lies in the $(n+1)$-fold homaloid H because it contains the new tangent and the osculating flat at O. This osculating block contains the trinormal of the geodesic at O; and consequently the trinormal of the geodesic lies in the $(n+1)$-fold homaloid. Denoting a typical direction-cosine of the trinormal by l_4, in accordance with the notation used (§ 7) for the Frenet equations, we must have l_4 expressible in terms of the direction-parameters of the $(n+1)$-fold homaloid, so that values of parameters $\beta_1, \ldots, \beta_n, \beta$, must exist such that

$$l_4 = \beta_1\frac{\partial y}{\partial x_1} + \ldots + \beta_n\frac{\partial y}{\partial x_n} + \beta Y.$$

Again, the trinormal is perpendicular to the three principal lines of the amplitudinal geodesic which are of earlier rank, viz. the tangent, the prime normal, and the binormal, so that we must have the three relations

$$\sum y' l_4 = 0, \quad \sum Y l_4 = 0, \quad \sum l_3 l_4 = 0,$$

satisfied by the parameters in l_4.

In the first place, as we have

$$\sum Y\frac{\partial y}{\partial x_r} = 0$$

for all the n values of r, the relation $\sum Y l_4 = 0$ gives the condition

$$\beta = 0.$$

Thus the typical direction-cosine of the trinormal is given by

$$l_4 = \beta_1\frac{\partial y}{\partial x_1} + \ldots + \beta_n\frac{\partial y}{\partial x_n},$$

where the magnitudes β are parametric: that is, the typical direction-cosine of

the trinormal is expressible linearly in terms of the direction-parameters of the n-fold homaloid of the amplitudinal geodesic. We therefore infer the property that the trinormal of an amplitudinal geodesic lies in the n-fold homaloid which is tangential to the amplitude.

In the next place, as we have

$$y' = \sum x_i' \frac{\partial y}{\partial x_i}, \quad \sum y' \frac{\partial y}{\partial x_i} = u_i,$$

the condition $\sum y' l_4 = 0$ to be satisfied by the parameters of l_4 becomes

$$\beta_1 u_1 + \ldots + \beta_n u_n = 0.$$

In the third place, we have

$$\frac{l_3}{\sigma} = \frac{y'}{\rho} + Y',$$

and $\sum y' l_4 = 0$, so that the condition $\sum l_3 l_4 = 0$ becomes

$$\sum Y' l_4 = 0,$$

that is,

$$\sum_i \beta_i \left(\sum Y' \frac{\partial y}{\partial x_i} \right) = 0;$$

and therefore, by an earlier result (§ 31), this condition becomes

$$\beta_1 v_1 + \ldots + \beta_n v_n = 0.$$

These two conditions, however, merely secure that the line, specified by the parameters $\beta_1, \ldots, \beta_n$, is at right angles to the tangent and to the binormal; its specific association with the curvatures still has to be secured. As regards the typical direction-cosine of the trinormal of the amplitudinal geodesic, it is given by the modified form (§ 8) of the Frenet equation

$$\frac{l_4}{\sigma\tau} = y' \frac{d}{ds}\left(\frac{1}{\rho}\right) + Y\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) - l_3 \frac{d}{ds}\left(\frac{1}{\sigma}\right) + Y'',$$

which is general for all such geodesics; and now it is known to be given by

$$l_4 = \beta_1 \frac{\partial y}{\partial x_1} + \ldots + \beta_n \frac{\partial y}{\partial x_n}$$

for our immediate representation, with the duly determinate values of β. We therefore have to find the parameters β, such that the relation

$$\frac{1}{\sigma\tau}\left(\beta_1 \frac{\partial y}{\partial x_1} + \ldots + \beta_n \frac{\partial y}{\partial x_n}\right) = y' \frac{d}{ds}\left(\frac{1}{\rho}\right) + Y\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) - l_3 \frac{d}{ds}\left(\frac{1}{\sigma}\right) + Y''$$

is satisfied for all the different space-coordinates and the associated magnitudes l_3, Y, Y''.

The form of the relation, so far as the left-hand side is concerned, is justified by the theorem that the trinormal lies in the n-fold homaloid that touches the amplitude; yet the expression on the right-hand side contains the typical direction-cosine of the prime normal which is orthogonal to the homaloid. The explanation of the apparent paradox is that the later occurrence of Y'' also is a corrective of this orthogonality to the tangent homaloid, as will appear from a later formula. Meanwhile, we note that, on multiplying the equation by Y, adding for all the products, and using the relations

$$\sum Y\frac{\partial y}{\partial x_r}=0, \quad \sum Yy'=0, \quad \sum Yl_3=0,$$

we have

$$\sum YY''+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)=0,$$

which ultimately is an old relation: for $\sum YY'=0$, and therefore

$$-\sum YY''=\sum Y'^2=\frac{1}{\rho^2}+\frac{1}{\sigma^2},$$

by the customary relation (§ 8).

To determine the actual values of $\beta_1, \ldots, \beta_n$, we multiply the relation throughout by $\dfrac{\partial y}{\partial x_r}$, and sum the product over the spatial dimension-range. Then, as

$$\sum y'\frac{\partial y}{\partial x_r}=u_r, \quad \sum Y\frac{\partial y}{\partial x_r}=0, \quad \sum Y''\frac{\partial y}{\partial x_r}=-w_r,$$

$$\sum l_3\frac{\partial y}{\partial x_r}=\sigma\left(\frac{u_r}{\rho}-v_r\right),$$

by relations already obtained, we have

$$\begin{aligned}\frac{1}{\sigma\tau}\sum_i \beta_i A_{ir}&=u_r\frac{d}{ds}\left(\frac{1}{\rho}\right)-\sigma\left(\frac{u_r}{\rho}-v_r\right)\frac{d}{ds}\left(\frac{1}{\sigma}\right)-w_r\\ &=\frac{1}{\sigma}u_r\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\frac{\sigma'}{\sigma}v_r-w_r,\end{aligned}$$

or

$$\frac{1}{\tau}\sum_i \beta_i A_{ir}=u_r\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\sigma' v_r-\sigma w_r.$$

This relation holds for $r=1, \ldots, n$; and therefore it gives n equations linear in the n parameters β. When they are resolved for these parameters we find

$$\frac{\Omega}{\tau}\beta_\mu=\left(\sum_r a_{\mu r}u_r\right)\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\sigma'\sum_r a_{\mu r}v_r-\sigma\sum_r a_{\mu r}w_r.$$

Now we have

$$\sum_r a_{\mu r}u_r=\Omega x_\mu';$$

and therefore

$$\frac{1}{\tau}\beta_\mu = x_\mu' \frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - \frac{\sigma'}{\Omega}\sum_r a_{\mu r} v_r - \frac{\sigma}{\Omega}\sum_r a_{\mu r} w_r,$$

so that the values of the quantities β can be regarded as known.

Let these values be substituted in the expression for l_4, as postulated initially in the form

$$l_4 = \beta_1 \frac{\partial y}{\partial x_1} + \ldots + \beta_n \frac{\partial y}{\partial x_n}.$$

We have

$$\sum_\mu x_\mu' \frac{\partial y}{\partial x_\mu} = y',$$

$$\sum_r \sum_\mu a_{\mu r} v_r \frac{\partial y}{\partial x_\mu} = -\Omega Y' = -\Omega\left(\frac{l_3}{\sigma} - \frac{y'}{\rho}\right);$$

then we have

$$\frac{l_4}{\tau} = y' \frac{d}{ds}\left(\frac{\sigma}{\rho}\right) + \sigma'\left(\frac{l_3}{\sigma} - \frac{y'}{\rho}\right) - \frac{\sigma}{\Omega}\sum_r \sum_\mu a_{\mu r} w_r \frac{\partial y}{\partial x_\mu},$$

that is,

$$\frac{l_4}{\sigma\tau} = y' \frac{d}{ds}\left(\frac{1}{\rho}\right) - l_3 \frac{d}{ds}\left(\frac{1}{\sigma}\right) - \frac{1}{\Omega}\sum_r \sum_\mu a_{\mu r} w_r \frac{\partial y}{\partial x_\mu}.$$

40. One inference is immediate. When the former general expression and the later specific expression for l_4 are compared, we have

$$Y'' + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) Y = -\frac{1}{\Omega}\sum_r \sum_\mu a_{\mu r} w_r \frac{\partial y}{\partial x_\mu},$$

an expression for Y'' similar to the expression for Y' in the form

$$Y' = -\frac{1}{\Omega}\sum_r \sum_\mu a_{\mu r} v_r \frac{\partial y}{\partial x_\mu}.$$

Now $\sum YY' = 0$ because $\sum Y^2 = 1$; hence multiplying the left-hand sides of those equations together, and adding for all the products, we have

$$\begin{aligned}\sum Y'Y'' &= \frac{1}{\Omega^2}\sum_r \sum_p \sum_\lambda \sum_\mu a_{\mu r} a_{\lambda p} w_r v_p \left(\sum \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\lambda}\right) \\ &= \frac{1}{\Omega^2}\sum_r \sum_p \sum_\lambda \sum_\mu A_{\mu\lambda} a_{\mu r} a_{\lambda p} w_r v_p.\end{aligned}$$

As usual, $\sum_\mu A_{\mu\lambda} a_{\mu r} = 0$ if λ and r are different, and $= \Omega$ if λ and r are the same; and therefore

$$\sum Y'Y'' = \frac{1}{\Omega}\sum_r \sum_p a_{rp} v_p w_r.$$

Moreover,

$$\sum Y'^2 = \frac{1}{\rho^2} + \frac{1}{\sigma^2},$$

so that

$$\sum Y' Y'' = -\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right);$$

and therefore we have

$$\sum_r \sum_p a_{rp} v_p w_r = -\Omega\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right).$$

Further, a comparison of the foregoing value of Y'' with the earlier value in § 38 gives the relation

$$\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) Y = \frac{1}{\Omega} \sum_i \sum_j \sum_k a_{ij} v_i x_k' \eta_{jk};$$

and when a later value (§ 44) has been obtained for η_{jk} in terms of direction-cosines of principal lines of the geodesic, the relation will provide some covariantive properties of the amplitude. Meanwhile a critical verification may be noted; for, on multiplying by Y and adding all the products, we have

$$\begin{aligned}\frac{1}{\rho^2}+\frac{1}{\sigma^2} &= \frac{1}{\Omega} \sum_i \sum_j \sum_k a_{ij} v_i x_k' \left(\sum Y \eta_{jk}\right)\\ &= \frac{1}{\Omega} \sum_i \sum_j \sum_k a_{ij} v_i x_k' L_{jk} = \frac{1}{\Omega} \sum_i \sum_j a_{ij} v_i v_j,\end{aligned}$$

a known result.

It appeared (§ 39) that the parameters $\beta_1, \ldots, \beta_n$, in the expression for l_4, must satisfy the two conditions

$$\sum_\mu \beta_\mu u_\mu = 0, \quad \sum_\mu \beta_\mu v_\mu = 0.$$

The verification, that the obtained values do satisfy the conditions, is simple. For the first of them, we have

$$\frac{1}{\tau}\sum_\mu \beta_\mu u_\mu = \left(\sum_\mu u_\mu x_\mu'\right)\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - \frac{\sigma'}{\Omega}\sum_r \left\{v_r\left(\sum_\mu a_{\mu r} u_\mu\right)\right\} - \frac{\sigma}{\Omega}\sum_r \left\{w_r\left(\sum_\mu a_{\mu r} u_\mu\right)\right\};$$

and

$$\sum_\mu u_\mu x_\mu' = 1, \quad \sum_\mu a_{\mu r} u_\mu = \Omega x_r',$$

so that

$$\sum_r \left\{v_r\left(\sum_\mu a_{\mu r} u_\mu\right)\right\} = \Omega \sum_r v_r x_r' = \frac{\Omega}{\rho},$$

$$\sum_r \left\{w_r\left(\sum_\mu a_{\mu r} u_\mu\right)\right\} = \Omega \sum_r w_r x_r' = \Omega \frac{d}{ds}\left(\frac{1}{\rho}\right).$$

Let these values be substituted; then

$$\sum_\mu \beta_\mu u_\mu = 0.$$

For the second condition, we have

$$\frac{1}{\tau}\sum_\mu \beta_\mu v_\mu = \left(\sum_\mu v_\mu x_\mu'\right)\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - \frac{\sigma'}{\Omega}\sum_r\sum_\mu a_{\mu r} v_r v_\mu - \frac{\sigma}{\Omega}\sum_r\sum_\mu a_{\mu r} v_\mu w_r\,;$$

and

$$\sum v_\mu x_\mu' = \frac{1}{\rho},$$

while (§ 35)

$$\sum_r\sum_\mu a_{\mu r} v_r v_\mu = \Omega\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right),$$

and (p. 93)

$$\sum_r\sum_\mu a_{\mu r} v_\mu w_r = -\Omega\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right).$$

Let these values be substituted; then

$$\sum_\mu \beta_\mu v_\mu = 0.$$

Accordingly, the two required conditions are satisfied by the values of the parameters β.

As a last inference at this stage, we form an expression giving the magnitude of the tilt $1/\tau$.

The typical direction-cosine of the trinormal is given (§ 39) by the equation

$$\frac{l_4}{\sigma\tau} - y'\frac{d}{ds}\left(\frac{1}{\rho}\right) - l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right) = -\frac{1}{\Omega}\sum_r\sum_\mu a_{\mu r} w_r \frac{\partial y}{\partial x_\mu}.$$

Let both sides of the equation be squared, and the sums be taken for the dimension-range. We have

$$\sum l_4^2 = 1,\quad \sum y'^2 = 1,\quad \sum l_3^2 = 1,\quad \sum l_4 y' = 0,\quad \sum l_4 l_3 = 0,\quad \sum y' l_3 = 0\,;$$

so that the sum of the left-hand sides

$$= \frac{1}{\sigma^2\tau^2} + \frac{\rho'^2}{\rho^4} + \frac{\sigma'^2}{\sigma^4}.$$

The sum of the right-hand sides

$$\begin{aligned}
&= \frac{1}{\Omega^2}\sum_r\sum_p\sum_\lambda\sum_\mu \left\{a_{\mu r} a_{\lambda p} w_r w_p \left(\sum \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\lambda}\right)\right\} \\
&= \frac{1}{\Omega^2}\sum_r\sum_p\sum_\lambda\sum_\mu A_{\mu\lambda} a_{\mu r} a_{\lambda p} w_r w_p \\
&= \frac{1}{\Omega}\sum_r\sum_p a_{rp} w_r w_p\,;
\end{aligned}$$

and therefore

$$\sum_r\sum_p a_{rp} w_r w_p = \Omega\left(\frac{1}{\sigma^2\tau^2} + \frac{\rho'^2}{\rho^4} + \frac{\sigma'^2}{\sigma^4}\right).$$

For comparison, we may record the results of like form in the bilinear combinations of u_r, v_r, w_r: in addition to the result just stated, they are

$$\sum_r \sum_p a_{rp} u_r u_p = \Omega,$$

$$\sum_r \sum_p a_{rp} u_r v_p = \frac{\Omega}{\rho},$$

$$\sum_r \sum_p a_{rp} u_r w_p = \Omega \frac{d}{ds}\left(\frac{1}{\rho}\right),$$

$$\sum_r \sum_p a_{rp} v_r v_p = \Omega \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right),$$

$$\sum_r \sum_p a_{rp} v_r w_p = -\Omega \left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right).$$

Expression for $\dfrac{d^2}{ds^2}\left(\dfrac{1}{\rho}\right)$.

41. The first arc-derivative of the circular curvature of a geodesic has been obtained in the form

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \sum_i \sum_j \sum_k e_{ijk} x_i' x_j' x_k',$$

where the expression of the coefficients e_{ijk}, in terms of earlier magnitudes, is (§ 36)

$$e_{ijk} = \sum \left(Y \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k}\right) - \sum_p [L_{ip}\{jk, p\} + L_{jp}\{ki, p\} + L_{kp}\{ij, p\}],$$

and where, for the aggregated product $x_i' x_j' x_k'$, the full coefficient is $6e_{ijk}$. For the determination of the parameters in the spatial direction-cosines of the quadri-normal, we shall require an expression for the second arc-derivative of the circular curvature.

From the foregoing formula, we have

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) = \sum_i \sum_j \sum_k \frac{de_{ijk}}{ds} x_i' x_j' x_k' + \sum \sum \sum e_{ijk} (x_i'' x_j' x_k' + x_i' x_j'' x_k' + x_i' x_j' x_k'');$$

and it is not difficult to see that, when the first derivatives of the coefficients e are substituted in the first summation and when the values of x_i'', x_j'', x_k'', have been substituted in the second summation, the whole right-hand side becomes a homogeneous quartic function of the variables $x_1', \dots, x_n'$. We shall write

$$T = \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) = \sum_i \sum_j \sum_k \sum_l f_{ijkl} x_i' x_j' x_k' x_l',$$

in which the full coefficient of the aggregated product $x_i' x_j' x_k' x_l'$ is $24 f_{ijkl}$; and it will be found convenient to write

$$t_r = \tfrac{1}{4} \frac{\partial}{\partial x_r'} \left\{\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)\right\} = \tfrac{1}{4} \frac{\partial T}{\partial x_r'},$$

so that, for all values 1, ..., n, the quantity t_r denotes a homogeneous cubic function of the variables x_1', ..., x_n'.

Taking the foregoing value of e_{ijk} in the form

$$e_{ijk} = \sum \left(Y \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} \right) - \sum_p \Theta_p,$$

we have

$$\frac{de_{ijk}}{ds} = \sum_l \sum \left(Y \frac{\partial^4 y}{\partial x_i \partial x_j \partial x_k \partial x_l} \right) x_l' - \sum \left(Y' \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} \right) - \sum_p \frac{d\Theta_p}{ds}.$$

The value of Y' has been obtained in the form

$$Y' = -\frac{1}{\Omega} \sum_r \sum_\mu a_{r\mu} v_\mu \frac{\partial y}{\partial x_r},$$

so that

$$\sum Y' \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = -\frac{1}{\Omega} \sum_r \sum_\mu \left(\sum \frac{\partial y}{\partial x_r} \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} \right) a_{r\mu} v_\mu$$

$$= -\frac{1}{\Omega} \sum_r \sum_\mu a_{r\mu} v_\mu \sum_\theta A_{r\theta} \begin{bmatrix} ijk \\ \theta \end{bmatrix},$$

on using the symbol defined in § 22. But

$$\sum_r a_{r\mu} A_{r\theta} = 0, \text{ or } \Omega,$$

according as μ and θ are different, or are the same; hence

$$\sum Y' \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = -\sum_\mu v_\mu \begin{bmatrix} ijk \\ \mu \end{bmatrix},$$

which is a linear expression in the variables x_1', ..., x_n'.

Further, as regards the quantities $\dfrac{d\Theta_p}{ds}$, we have

$$\frac{dL_{ab}}{ds} = \sum_c x_c' \left[e_{abc} + \sum_\theta (L_{a\theta} \{bc, \theta\} + L_{b\theta} \{ac, \theta\}) \right],$$

$$\frac{d}{ds} \{\alpha\beta, \mu\} = \sum_q \left[\frac{\partial}{\partial x_q} \{\alpha\beta, \mu\} \right] x_q'$$

$$= \sum_q x_q' \left\{ \begin{bmatrix} \alpha\beta q \\ \mu \end{bmatrix} - \sum_p \{\alpha\beta, p\} \{pq, \mu\} + \frac{1}{\Omega} \sum_r a_{r\mu} \left(\sum \eta_{rq} \eta_{\alpha\beta} \right) \right\};$$

and the substitution of such values gives $\dfrac{d\Theta_p}{ds}$ as an expression which formally is linear in the variables x_1', ..., x_n'.

Consequently, when the full values of the arc-derivatives of e_{ijk} are substituted, we obtain the stated quartic form for the magnitude denoted by T.

42. The full typical term in T is

$$24 f_{ijkl} x_i' x_j' x_k' x_l'$$

so that the full typical term in $\dfrac{\partial T}{\partial x_r'}$ is

$$24 f_{rjkl} x_j' x_k' x_l'.$$

The full typical term in $\sum_j \sum_k \sum_l f_{rjkl} x_j' x_k' x_l'$ is

$$6 f_{rjkl} x_j' x_k' x_l' ;$$

and therefore we have

$$\tfrac{1}{4} \frac{\partial T}{\partial x_r'} = \sum_j \sum_k \sum_l f_{rjkl} x_j' x_k' x_l',$$

so that

$$t_r = \sum_j \sum_k \sum_l f_{rjkl} x_j' x_k' x_l'.$$

The value of the quantity w_r, of § 36, is

$$w_r = \sum_\alpha \sum_\beta e_{r\alpha\beta} x_\alpha' x_\beta',$$

and therefore

$$\frac{dw_r}{ds} = \sum_\alpha \sum_\beta \frac{de_{r\alpha\beta}}{ds} x_\alpha' x_\beta' + \sum_\alpha \sum_\beta e_{r\alpha\beta} (x_\alpha'' x_\beta' + x_\alpha' x_\beta'').$$

As above, we have

$$T = \sum_\alpha \sum_\beta \sum_\gamma \frac{de_{\alpha\beta\gamma}}{ds} x_\alpha' x_\beta' x_\gamma'$$
$$+ \sum_\alpha \sum_\beta \sum_\gamma e_{\alpha\beta\gamma} (x_\alpha'' x_\beta' x_\gamma' + x_\alpha' x_\beta'' x_\gamma' + x_\alpha' x_\beta' x_\gamma'').$$

In forming the partial derivative of T with regard to any variable x_r', account must be taken of (i), the numerical coefficient of each term in the various summations so as to secure the relation $t_r = \frac{1}{4} \dfrac{\partial T}{\partial x_r'}$; next, (ii), the occurrence of x_r' in the term so that, in the typical term of the first summation, it will arise out of $\dfrac{de_{\alpha\beta\gamma}}{ds}$, and likewise in the part of each term of the second triple summation other than the quantities x''; and (iii), the occurrence of x_r' in the quantities x_α''. In particular, for (iii), we take the value $-\sum \sum x_p' x_q' \{pq, \alpha\}$ in place of x_α'' in the summation. Thus we shall have two kinds of contributions to t_r. One consists of the two double summations

$$\sum_\beta \sum_\gamma \frac{de_{r\beta\gamma}}{ds} x_\beta' x_\gamma' + \sum_\beta \sum_\gamma e_{r\beta\gamma} (x_\beta'' x_\gamma' + x_\beta' x_\gamma''),$$

that is, it is equal to

$$\frac{dw_r}{ds}.$$

The other consists of the whole coefficient of x_r' in $\sum_\alpha \sum_\beta \sum_\gamma e_{\alpha\beta\gamma} x_\alpha'' x_\beta' x_\gamma'$ as arising from the terms in x_α'', that is, in

$$-\sum_\alpha \sum_\beta \sum_\gamma \sum_r \sum_\mu e_{\alpha\beta\gamma} x_\beta' x_\gamma' x_r' x_\mu' \{r\mu, a\},$$

and therefore it is

$$= -\sum_\alpha \sum_\beta \sum_\gamma \sum_\mu e_{\alpha\beta\gamma} x_\beta' x_\gamma' x_\mu' \{r\mu, a\}.$$

But we have

$$\sum_\beta \sum_\gamma e_{\alpha\beta\gamma} x_\beta' x_\gamma' = w_\alpha, \qquad \sum_\mu x_\mu' \{r\mu, a\} = g_{\alpha r};$$

and therefore this contribution to the value of t_r

$$= -\sum_\alpha g_{\alpha r} w_\alpha .$$

Consequently, we have

$$t_r = \frac{dw_r}{ds} - \sum g_{\alpha r} w_\alpha ;$$

and therefore

$$\frac{dw_r}{ds} = t_r + g_{1r} w_1 + g_{2r} w_2 + \dots + g_{nr} w_n ,$$

for all values of $r = 1, \dots , n$, where

$$t_r = \tfrac{1}{4} \frac{\partial}{\partial x_r'} \left\{ \frac{d^2}{ds^2} \left(\frac{1}{\rho} \right) \right\} .$$

The result is similar to the corresponding result

$$\frac{dv_r}{ds} = w_r + g_{1r} v_1 + g_{2r} v_2 + \dots + g_{nr} v_n ,$$

obtained in § 37.

NOTE. The main purpose of the preceding investigation is the determination of the value of $\frac{dw_r}{ds}$. A fuller discussion, raising other issues also, will be found in a different method of proceeding to the value of $\frac{d^2}{ds}\left(\frac{1}{\rho}\right)$ in connection with domainal geodesics (§ 287).

43. Returning to the expression for Y'' in the form (§ 40)

$$Y'' + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} \right) Y = -\frac{1}{\Omega} \sum_\mu \sum_l a_{\mu l} w_l \frac{\partial y}{\partial x_\mu} ,$$

we multiply throughout by

$$\frac{d}{ds} \left(\frac{\partial y}{\partial x_r} \right)$$

and sum the products for all the space-coordinates y and quantities connected with them. Now

$$\sum Y \frac{d}{ds} \left(\frac{\partial y}{d x_r} \right) = \sum_k x_k' \left(\sum Y \frac{\partial^2 y}{\partial x_r \partial x_k} \right)$$

$$= \sum_k x_k' L_{rk} = v_r ;$$

and
$$\sum \frac{\partial y}{\partial x_\mu}\frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right)=\sum_k x_k'\left(\sum \frac{\partial y}{\partial x_\mu}\frac{\partial^2 y}{\partial x_r \partial x_k}\right)$$
$$=\sum_k x_k'[rk,\mu]$$
$$=\sum_k \sum_\theta A_{\theta\mu}\{rk,\theta\}x_k'$$
$$=\sum_\theta g_{\theta r}A_{\theta\mu};$$
and therefore
$$\sum Y''\frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right)+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)v_r=-\frac{1}{\Omega}\sum_\mu\sum_l\sum_\theta a_{\mu l}A_{\theta\mu}g_{\theta r}w_l.$$
As usual,
$$\sum_\mu a_{\mu l}A_{\theta\mu}=0,\quad \text{or}\quad \Omega,$$
according as l and θ are different, or are the same; hence the right-hand side
$$=-\sum_l g_{lr}w_l,$$
and therefore
$$\sum Y''\frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right)=-\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)v_r-\sum_l g_{lr}w_l.$$
But we had
$$\sum Y''\frac{\partial y}{\partial x_r}=-w_r;$$
and therefore
$$\sum Y'''\frac{\partial y}{\partial x_r}+\sum Y''\frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right)=-\frac{dw_r}{ds}$$
$$=-t_r-\sum_l g_{lr}w_l.$$
Consequently
$$\sum Y'''\frac{\partial y}{\partial x_r}=-t_r+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)v_r,$$
for all values $r=1,\ldots,n$.

This result can be obtained also as follows.

When the original equation for Y'' is differentiated along the geodesic arc, we find
$$Y'''+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)Y'-2\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)Y$$
$$=-\sum_\mu\sum_l\frac{d}{ds}\left(\frac{a_{\mu l}}{\Omega}\right)w_l\frac{\partial y}{\partial x_\mu}-\frac{1}{\Omega}\sum_l\sum_\mu a_{l\mu}\left(t_l+\sum_\lambda g_{\lambda l}w_\lambda\right)\frac{\partial y}{\partial x_\mu}$$
$$-\frac{1}{\Omega}\sum_l\sum_\mu a_{l\mu}w_l\frac{d}{ds}\left(\frac{\partial y}{\partial x_\mu}\right),$$
where the first term, on using the results of § 13, becomes
$$=\frac{1}{\Omega}\sum_\mu\sum_l\sum_q\sum_k x_k'[a_{ql}\{kq,\mu\}+a_{q\mu}\{kq,l\}]w_l\frac{\partial y}{\partial x_\mu}$$
$$=\frac{1}{\Omega}\sum_\mu\sum_l\sum_q(a_{ql}g_{\mu q}+a_{q\mu}g_{lq})w_l\frac{\partial y}{\partial x_\mu}.$$

Now multiply throughout by $\dfrac{\partial y}{\partial x_r}$ and add all the products through the range of the plenary homaloidal space. On the left-hand side, we have

$$\sum Y' \frac{\partial y}{\partial x_r} = -v_r, \quad \sum Y \frac{\partial y}{\partial x_r} = 0.$$

On the right-hand side, the modified first term gives

$$\frac{1}{\Omega} \sum_\mu \sum_l \sum_q (a_{ql} g_{\mu q} + a_{q\mu} g_{lq}) w_l A_{\mu r}$$
$$= \frac{1}{\Omega} \sum_\mu \sum_l \sum_q w_l A_{\mu r} a_{ql} g_{\mu q} + \sum_l g_{lr} w_l .$$

The second term gives

$$-\frac{1}{\Omega} \sum_l \sum_\mu a_{l\mu} \left(t_l + \sum_\lambda g_{\lambda l} w_\lambda\right) A_{\mu r} = -\left(t_r + \sum_\lambda g_{\lambda r} w_\lambda\right).$$

The third term gives

$$-\frac{1}{\Omega} \sum_l \sum_\mu \sum_\theta a_{l\mu} w_l g_{\theta\mu} A_{\theta r} = -\frac{1}{\Omega} \sum_\mu \sum_l \sum_q w_l A_{\mu r} a_{ql} g_{\mu q} .$$

After substitution and re-arrangement, the result provides the value of $\sum Y''' \dfrac{\partial y}{\partial x_r}$ as already obtained.

Quartinormal of a geodesic; the coil.

44. The curvature of an amplitudinal geodesic, next in rank after the circular curvature, the torsion, and the tilt, may be called the *coil*, being the fourth in this succession of curvatures. It is associated, analytically, with the next principal line of the geodesic which is the quartinormal, and is denoted by $1/\kappa$; and the typical spatial direction-cosine of that principal line is denoted by l_5 in the Frenet system of equations for the curve. To obtain the direction-cosines of this quartinormal and the magnitude of the coil of the geodesic, we proceed as before from the $(n+1)$-fold homaloid H represented by the equations

$$\left\| \; \bar{y} - y, \quad \frac{\partial y}{\partial x_1}, \ldots, \quad \frac{\partial y}{\partial x_n}, \quad Y \; \right\| = 0.$$

We have seen that the tangents at a first point, at a second point, and at a third point—or, what is the equivalent in the analytical limit, the tangent up to the second order of small quantities (inclusive) at the first consecutive point—all lie within this $(n+1)$-fold homaloid and, as has been proved, within the n-fold homaloid touching the amplitude. As for the earlier instances of the binormal and the trinormal, the $(n+1)$-fold homaloid H is shewn to contain the tangent at a fourth point: or (as the analytical equivalent) to contain the tangent at the

first consecutive point up to the third order of small quantities. Such a tangent is represented by equations

$$\frac{\bar{y}_1-(y_1+\epsilon y_1'+\frac{1}{2}\epsilon^2 y_1''+\frac{1}{6}\epsilon^3 y_1''')}{y_1'+\epsilon y_1''+\frac{1}{2}\epsilon^2 y_1'''+\frac{1}{6}\epsilon^3 y_1''''}=\ldots\ldots=R,$$

where R is parametric along the line, or typically by

$$\bar{y}-y=\epsilon y'+\tfrac{1}{2}\epsilon^2 y''+\tfrac{1}{6}\epsilon^3 y'''+R(y'+\epsilon y''+\tfrac{1}{2}\epsilon^2 y'''+\tfrac{1}{6}\epsilon^3 y'''').$$

Now we have

$$y''=\frac{1}{\rho}Y,$$

$$y'''=\frac{1}{\rho\sigma}l_3-\frac{\rho'}{\rho^2}Y+\frac{1}{\rho^2}y',$$

$$y''''=\frac{1}{\rho\sigma\tau}l_4-\frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)l_3$$
$$+\left(-\frac{\rho''}{\rho^2}+2\frac{\rho'^2}{\rho^3}-\frac{1}{\rho^3}-\frac{1}{\rho\sigma^2}\right)Y+3\frac{\rho'}{\rho^3}y',$$

as immediate derivatives from the Frenet equations applied to the amplitudinal geodesic. We have seen that y', l_3, l_4, are homogeneous linear combinations of the quantities $\dfrac{\partial y}{\partial x_r}$ in the forms

$$y'=\sum_r x_r'\frac{\partial y}{\partial x_r},\qquad l_3=\sum_r \alpha_r\frac{\partial y}{\partial x_r},\qquad l_4=\sum_r \beta_r\frac{\partial y}{\partial x_r};$$

and therefore any point on the foregoing new tangent has its space-coordinates represented by the typical equation

$$\bar{y}-y=\theta_1\frac{\partial y}{\partial x_1}+\ldots\ldots+\theta_n\frac{\partial y}{\partial x_n}+\theta Y,$$

where $\theta_1, \ldots, \theta_n, \theta$, are magnitudes free from the direction-magnitudes of the lines in the $(n+1)$-fold homaloid H. This line is external to the osculating block; being the next tangent in closeness of contact along the curve, it lies in the five-fold osculating homaloid of the geodesic, which contains the osculating block and this tangent, and the guiding lines of which are those of the block and this tangent, all included in the $(n+1)$-fold homaloid H. This $(n+1)$-fold homaloid therefore contains the five-fold osculating homaloid of the geodesic: that is, it contains the quartinormal of the geodesic. Consequently, the typical direction-cosine l_5 of the quartinormal can be expressed in the form

$$l_5=\gamma_1\frac{\partial y}{\partial x_1}+\ldots\ldots+\gamma_n\frac{\partial y}{\partial x_n}+\gamma Y,$$

where $\gamma_1, \ldots, \gamma_n, \gamma$, are parameters.

But all the principal lines of the geodesic constitute an orthogonal system, and the prime normal and the quartinormal are at right angles. Thus $\sum l_5 Y=0$, that is, $\gamma=0$; and so the typical direction-cosine l_5 can be expressed in the form

$$l_5=\gamma_1\frac{\partial y}{\partial x_1}+\gamma_2\frac{\partial y}{\partial x_2}+\ldots+\gamma_n\frac{\partial y}{\partial x_n},$$

involving the n parametric magnitudes γ proper to the line. Thus l_5 is expressible in terms of the direction-variables of the n-fold homaloid which is tangential to the amplitudinal geodesic; and we therefore infer that this tangent n-fold homaloid contains the quartinormal of the geodesic, just as it contains the tangent, the binormal, and the trinormal.

Moreover, the quartinormal is at right angles to each of these three principal lines, so that we must have the three relations

$$\sum y' l_5=0, \quad \sum l_3 l_5=0, \quad \sum l_4 l_5=0.$$

The condition $\sum y' l_5=0$ is

$$\sum\left\{y'\left(\sum_r \gamma_r\frac{\partial y}{\partial x_r}\right)\right\}=0,$$

the summation being taken over the dimensions of the plenary space; also (§ 31)

$$\sum y'\frac{\partial y}{\partial x_r}=u_r,$$

so that the first condition leads to the relation

$$\sum_r \gamma_r u_r=0.$$

The second condition $\sum l_3 l_5=0$ is of the form

$$\sum\left\{\left(\sum_k a_k\frac{\partial y}{\partial x_k}\right)\left(\sum_r \gamma_r\frac{\partial y}{\partial x_r}\right)\right\}=0,$$

with the same summation as before; also

$$\sum\left\{\left(\sum_k a_k\frac{\partial y}{\partial x_k}\right)\frac{\partial y}{\partial x_r}\right\}=\sum_k A_{rk}a_k,$$

or having regard to the values of the parameters a for the binormal (§ 34), the right-hand side is

$$\sigma\left(\frac{u_r}{\rho}-v_r\right),$$

so that the condition becomes

$$\frac{\sigma}{\rho}\sum_r \gamma_r u_r-\sigma\sum_r \gamma_r v_r=0:$$

that is, when account is taken of the earlier condition, the second condition can be taken in the form

$$\sum_r \gamma_r v_r = 0.$$

The third condition $\sum l_4 l_5 = 0$ becomes

$$\sum \left\{ \left(\sum_k \beta_k \frac{\partial y}{\partial x_k} \right) \left(\sum_r \gamma_r \frac{\partial y}{\partial x_r} \right) \right\} = 0,$$

again with the same space-summation ; also

$$\begin{aligned} \sum \left\{ \left(\sum_k \beta_k \frac{\partial y}{\partial x_k} \right) \frac{\partial y}{\partial x_r} \right\} &= \sum_k A_{rk} \beta_k \\ &= \left\{ \tau \frac{d}{ds} \left(\frac{\sigma}{\rho} \right) \right\} u_r - \sigma' \tau v_r - \sigma \tau w_r, \end{aligned}$$

so that the condition becomes

$$\left\{ \tau \frac{d}{ds} \left(\frac{\sigma}{\rho} \right) \right\} \left(\sum_r \gamma_r u_r \right) - \sigma' \tau \left(\sum_r \gamma_r v_r \right) - \sigma \tau \left(\sum_r \gamma_r w_r \right) = 0 :$$

that is, when account is taken of the two earlier conditions, the third condition can be taken in the form

$$\sum_r \gamma_r w_r = 0.$$

Thus the parameters γ in the expression of the typical direction-cosine l_5 of the quartinormal must satisfy the three conditions

$$\sum_r \gamma_r u_r = 0, \quad \sum_r \gamma_r v_r = 0, \quad \sum_r \gamma_r w_r = 0.$$

The conditions will not be used to determine ratios of the parameters ; but after the parameters have been determined, we shall prove that the three conditions are satisfied by the values then obtained.

45. To determine the parameters in the postulated expression for l_5, which is

$$l_5 = \sum_i \gamma_i \frac{\partial y}{\partial x_i},$$

we proceed from the relation (§ 39)

$$\frac{l_4}{\sigma\tau} = y' \frac{d}{ds} \left(\frac{1}{\rho} \right) + Y \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} \right) - l_3 \frac{d}{ds} \left(\frac{1}{\sigma} \right) + Y''.$$

There is an analytical advantage in having as many terms as possible involving Y and its derivatives on the right-hand side ; we therefore substitute

$$\frac{\sigma}{\rho} y' + \sigma Y'$$

for l_3, and, after a slight re-arrangement, we find

$$l_4 = y'\tau \frac{d}{ds}\left(\frac{\sigma}{\rho}\right) + Y\sigma\tau\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) + Y'\sigma'\tau + Y''\sigma\tau.$$

The Frenet equation, determining l_5, is

$$\frac{dl_4}{ds} = \frac{l_5}{\kappa} - \frac{l_3}{\tau} = \frac{l_5}{\kappa} - \frac{\sigma}{\tau}Y' - \frac{\sigma}{\rho\tau}y';$$

hence, after substituting the last expression for l_4, and gathering terms,

$$\frac{1}{\kappa}l_5 = Py' + QY + RY' + SY'' + \sigma\tau Y''',$$

where

$$P = \frac{\sigma}{\rho\tau} + \frac{d}{ds}\left\{\tau\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\},$$

$$Q = \frac{\tau}{\rho}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) + \frac{d}{ds}\left\{\sigma\tau\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)\right\},$$

$$R = \frac{d}{ds}(\sigma'\tau) + \sigma\tau\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right),$$

$$S = \frac{1}{\sigma}\frac{d}{ds}(\sigma^2\tau).$$

Now, by earlier results, we have

$$\sum y'\frac{\partial y}{\partial x_r} = u_r, \quad \sum Y\frac{\partial y}{\partial x_r} = 0, \quad \sum Y'\frac{\partial y}{\partial x_r} = -v_r, \quad \sum Y''\frac{\partial y}{\partial x_r} = -w_r,$$

$$\sum Y'''\frac{\partial y}{\partial x_r} = -t_r + \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)v_r;$$

and with the postulated value of l_5, we have

$$\sum l_5\frac{\partial y}{\partial x_r} = \sum_\mu A_{\mu r}\gamma_\mu.$$

Hence on multiplying the equation for l_5 by $\frac{\partial y}{\partial x_r}$, and adding the products for all the spatial magnitudes, it follows that

$$\frac{1}{\kappa}\sum_\mu A_{\mu r}\gamma_\mu = Pu_r - Rv_r - Sw_r + \sigma\tau\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)v_r - \sigma\tau t_r$$

$$= Pu_r - \left\{\frac{\sigma}{\tau} + \frac{d}{ds}(\sigma'\tau)\right\}v_r - Sw_r - \sigma\tau t_r,$$

holding for $r = 1, \ldots, n$. Let these linear equations be resolved for the quantities γ; the result is

$$\frac{1}{\kappa}\Omega\gamma_i = P\sum_r a_{ri}u_r - \left\{\frac{\sigma}{\tau}+\frac{d}{ds}(\sigma'\tau)\right\}\sum_r a_{ri}v_r - S\sum_r a_{ri}w_r - \sigma\tau\sum_r a_{ri}t_r,$$

for $i=1, \ldots, n$; and the first term simplifies, because $\sum_r a_{ri}u_r=\Omega x_i'$. When these values of the quantities γ are substituted in the parametric expression for l_5, the result is

$$\frac{1}{\kappa}\Omega l_5=\Omega P y' - \left\{\frac{\sigma}{\tau}+\frac{d}{ds}(\sigma'\tau)\right\} \sum_i \sum_r \left(a_{ri}v_r\frac{\partial y}{\partial x_i}\right)$$
$$- S\sum_i \sum_r \left(a_{ri}w_r\frac{\partial y}{\partial x_i}\right) - \sigma\tau \sum_i \sum_r \left(a_{ri}t_r\frac{\partial y}{\partial x_i}\right).$$

But (§ 34)

$$\sum_i \sum_r \left(a_{ri}v_r\frac{\partial y}{\partial x_i}\right) = -\Omega Y',$$

and (§ 38)

$$\sum_i \sum_r \left(a_{ri}w_r\frac{\partial y}{\partial x_i}\right) = -\Omega\left\{Y''+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)Y\right\};$$

and therefore

$$\frac{1}{\kappa}l_5=Py'+\left\{\frac{\sigma}{\tau}+\frac{d}{ds}(\sigma'\tau)\right\}Y'+S\left\{Y''+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)Y\right\}-\frac{1}{\Omega}\sigma\tau\sum_i \sum_r \left(a_{ri}t_r\frac{\partial y}{\partial x_i}\right).$$

Of this form, we make two separate uses. In the first place, when it is compared with the form

$$\frac{1}{\kappa}l_5=Py'+QY+RY'+SY''+\sigma\tau Y''',$$

we infer the relation

$$\sigma\tau Y'''+\sigma\tau\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)Y'+\left\{Q-S\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)\right\}Y=-\frac{1}{\Omega}\sigma\tau\sum_i \sum_r \left(a_{ri}t_r\frac{\partial y}{\partial x_i}\right),$$

which, on actual substitution of the values of Q and S in the coefficient of Y, leads to the modified relation

$$Y'''+Y'\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)-3Y\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)=-\frac{1}{\Omega}\sum_i \sum_r \left(a_{ri}t_r\frac{\partial y}{\partial x_i}\right).$$

The other use results from transformation of the right-hand side. For this purpose, we use the relations

$$Y'=\frac{l_3}{\sigma}-\frac{y'}{\rho},$$

$$Y''+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)Y=\frac{l_4}{\sigma\tau}-y'\frac{d}{ds}\left(\frac{1}{\rho}\right)+l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right);$$

when these are inserted, and the coefficients are collected, we find

$$\frac{l_5}{\sigma\tau\kappa}=y'\left\{\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)-\frac{1}{\rho}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)\right\}-3Y\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)$$
$$-l_3\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)\right\}-l_4\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+Y''',$$

where, except in the last term, the coefficients, appertaining to the equation as a typical equation, are the typical direction-cosines of the principal lines that are antecedent to the quartinormal. In this respect, the form of the equation for l_5 is similar to that of the equation for l_4, as used in the preceding investigation, namely,

$$\frac{l_4}{\sigma\tau}=y'\frac{d}{ds}\left(\frac{1}{\rho}\right)+Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)-l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right)+Y'' ;$$

indeed, the l_5-equation is deducible immediately by direct differentiation of this l_4-equation.

46. Some inferences, in the form of expressions for covariants of the whole system of forms, must be made from the relation

$$Y'''+Y'\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)-3Y\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)=-\frac{1}{\Omega}\sum_i\sum_r\left(a_{ri}t_r\frac{\partial y}{\partial x_i}\right),$$

before an expression can be constructed for the coil $1/\kappa$ of the amplitudinal geodesic.

In the first place, there are some universal relations * for any geodesic. We have, always,

$$\sum Yy'=0 ;$$

and therefore

$$\sum Y'y'=-\sum Yy''=-\frac{1}{\rho}.$$

Hence

$$\begin{aligned}\sum Y''y'&=-\frac{d}{ds}\left(\frac{1}{\rho}\right)-\sum Y'y''\\&=-\frac{d}{ds}\left(\frac{1}{\rho}\right)-\frac{1}{\rho}\sum Y'Y=-\frac{d}{ds}\left(\frac{1}{\rho}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\sum Y'''y'&=-\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)-\sum Y''y''\\&=-\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)-\frac{1}{\rho}\sum YY''\\&=-\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)+\frac{1}{\rho}\sum Y'^2=-\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)+\frac{1}{\rho}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right).\end{aligned}$$

Next, we have had the relations

$$\sum Y'^2=\frac{1}{\rho^2}+\frac{1}{\sigma^2},$$

$$\sum Y''^2=\frac{1}{\sigma^2\tau^2}+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)^2+\frac{\rho'^2}{\rho^4}+\frac{\sigma'^2}{\sigma^4}.$$

* That is, they hold for every amplitude; they are derivable from Frenet's equations, with (§ 8) the formal assumption $l_2=Y$.

From the first of these, it follows that

$$\sum Y'Y'' = -\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right).$$

Hence, by another differentiation along the geodesic arc,

$$\sum Y'Y''' + \sum Y''^2 = -\left(\frac{\rho''}{\rho^3}+\frac{\sigma''}{\sigma^3}\right) + 3\left(\frac{\rho'^2}{\rho^4}+\frac{\sigma'^2}{\sigma^4}\right),$$

and therefore

$$\begin{aligned}\sum Y'Y''' &= -\frac{1}{\sigma^2\tau^2} - \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)^2 - \left(\frac{\rho''}{\rho^3}+\frac{\sigma''}{\sigma^3}\right) + 2\left(\frac{\rho'^2}{\rho^4}+\frac{\sigma'^2}{\sigma^4}\right)\\ &= -\frac{1}{\sigma^2\tau^2} - \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)^2 + \frac{1}{\rho}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) + \frac{1}{\sigma}\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right).\end{aligned}$$

Also, we have

$$\sum YY' = 0,$$

and therefore

$$\sum YY'' = -\sum Y'^2 = -\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right);$$

and, also, by another differentiation,

$$\begin{aligned}\sum YY''' &= -\sum Y'Y'' + 2\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)\\ &= 3\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right).\end{aligned}$$

As a final result for the present set of relations, we can obtain an expression for $\sum Y''Y'''$, by differentiating the expression for $\sum Y''^2$; it will be required later (p. 108).

Further, we consider some inferences more special to the particular relation

$$Y''' + Y'\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) - 3Y\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right) = -\frac{1}{\Omega}\sum_i\sum_r\left(a_{ri}t_r\frac{\partial y}{\partial x_i}\right).$$

First, multiply by y', and add for all the space-dimensions. Because

$$\sum y'\frac{\partial y}{\partial x_i} = u_i,$$

we have

$$\begin{aligned}-\frac{1}{\Omega}\sum_i\sum_r\left\{a_{ri}t_r\left(\sum y'\frac{\partial y}{\partial x_i}\right)\right\} &= -\frac{1}{\Omega}\sum_i\sum_r a_{ri}t_r u_i\\ &= -\sum_r t_r x_r' = -\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right):\end{aligned}$$

and so

$$\sum(y'Y''') + \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)\sum(Y'y') - 3\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)\sum(Yy') = -\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right),$$

a condition satisfied identically when the foregoing values of $\sum y' Y'''$, $\sum Y' y'$, $\sum Y y'$, are inserted.

Next, multiply by Y, and add for all the space-dimensions. Because

$$\sum Y \frac{\partial y}{\partial x_i} = 0, \quad \sum Y Y' = 0, \quad \sum Y^2 = 1,$$

the relation becomes

$$\sum Y Y''' - 3\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right) = 0,$$

a condition satisfied identically.

Next, multiply by Y', and add for all the space-dimensions. Because (§ 31)

$$\sum_i Y' \frac{\partial y}{\partial x_i} = -v_i, \quad \sum Y Y' = 0, \quad \sum Y'^2 = \frac{1}{\rho^2} + \frac{1}{\sigma^2},$$

the resulting relation becomes

$$\sum Y' Y''' + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^2 = \frac{1}{\Omega} \sum_i \sum_r a_{ri} t_r v_i ;$$

and therefore, on the substitution of the value of $\sum Y' Y'''$, we have

$$\frac{1}{\Omega} \sum_i \sum_r a_{ri} t_r v_i = -\frac{1}{\sigma^2 \tau^2} + \frac{1}{\rho} \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) + \frac{1}{\sigma} \frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right),$$

thus providing the value of a covariant of the system of forms.

Once more, multiply by Y'', and add for all the space-dimensions. Because (§ 32)

$$\sum Y'' \frac{\partial y}{\partial x_i} = -w_i,$$

$$\sum Y Y'' = -\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right), \quad \sum Y' Y'' = -\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right),$$

the relation becomes

$$\sum Y'' Y''' + 2\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right)\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) = \frac{1}{\Omega} \sum_i \sum_r a_{ri} t_r w_i ;$$

and because the differentiation of the expression for $\sum Y''^2$ gives

$$\sum Y'' Y''' = \frac{1}{\sigma\tau} \frac{d}{ds}\left(\frac{1}{\sigma\tau}\right) - 2\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right)\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) + \frac{\rho'\rho''}{\rho^4} - 2\frac{\rho'^2}{\rho^5} + \frac{\sigma'\sigma''}{\sigma^4} - 2\frac{\sigma'^2}{\sigma^5},$$

the relation now becomes

$$\frac{1}{\Omega} \sum_i \sum_r a_{ri} t_r w_i = \frac{1}{\sigma\tau} \frac{d}{ds}\left(\frac{1}{\sigma\tau}\right) - \frac{\rho'}{\rho^2} \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) - \frac{\sigma'}{\sigma^2} \frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right),$$

thus providing the value of another covariant in the system of simultaneous concomitants.

For the last among this group of concomitants, viz.

$$\frac{1}{\Omega}\sum_i \sum_r a_{ri} t_r t_i,$$

an expression in terms of the curvatures will be obtained (§ 49): the expression involves the coil of the geodesic.

47. We now proceed to shew that the three conditions (p. 103)

$$\sum_i \gamma_i u_i = 0, \quad \sum_i \gamma_i v_i = 0, \quad \sum_i \gamma_i w_i = 0,$$

to which the parameters γ must conform, are satisfied in fact by the values of these parameters

$$\frac{\Omega}{\kappa}\gamma_i = P\sum_r (a_{ri}u_r) - \left\{\frac{\sigma}{\tau} + \frac{d}{ds}(\sigma'\tau)\right\}\sum_r (a_{ri}v_r) - S\sum_r (a_{ri}w_r) - \sigma\tau\sum_r (a_{ri}t_r)$$

which were obtained in § 45.

To establish the first of the three conditions, we note the results

$$\sum_r \sum_i a_{ri} u_r u_i = \Omega \sum_r x_r' u_r = \Omega,$$

$$\sum_r \sum_i a_{ri} v_r u_i = \Omega \sum_r x_r' v_r = \frac{\Omega}{\rho},$$

$$\sum_r \sum_i a_{ri} w_r u_i = \Omega \sum_r x_r' w_r = \Omega \frac{d}{ds}\left(\frac{1}{\rho}\right),$$

$$\sum_r \sum_i a_{ri} t_r u_i = \Omega \sum_r x_r' t_r = \Omega \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right);$$

and therefore

$$\frac{1}{\kappa}\sum_i \gamma_i u_i = P - \frac{1}{\rho}\left\{\frac{\sigma}{\tau} + \frac{d}{ds}(\sigma'\tau)\right\} - S\frac{d}{ds}\left(\frac{1}{\rho}\right) - \sigma\tau\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right).$$

Now

$$\begin{aligned} P &= \frac{\sigma}{\rho\tau} + \frac{d}{ds}\left\{\tau\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\} \\ &= \frac{\sigma}{\rho\tau} + (\sigma'\tau' + \sigma''\tau)\frac{1}{\rho} + (2\sigma'\tau + \sigma\tau')\frac{d}{ds}\left(\frac{1}{\rho}\right) + \sigma''\tau\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right), \\ S &= 2\sigma'\tau + \sigma\tau'; \end{aligned}$$

when these values are substituted, the right-hand side vanishes identically; consequently

$$\sum_i \gamma_i u_i = 0,$$

and the first condition is satisfied.

To establish the second of the three conditions, we note the results

$$\sum_r \sum_i a_{ri} u_r v_i = \frac{\Omega}{\rho},$$

$$\sum_r \sum_i a_{ri} v_r v_i = \Omega\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right),$$

$$\sum_r \sum_i a_{ri} w_r v_i = -\Omega\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right),$$

$$\sum_r \sum_i a_{ri} t_r v_i = \Omega\left[-\frac{1}{\sigma^2\tau^2}+\frac{1}{\rho}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)+\frac{1}{\sigma}\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)\right];$$

hence

$$\frac{1}{\kappa}\sum_i \gamma_i v_i = P\frac{1}{\rho}-\left\{\frac{\sigma}{\tau}+\frac{d}{ds}(\sigma^2\tau)\right\}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)+(2\sigma'\tau+\sigma\tau')\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)$$
$$+\frac{1}{\sigma\tau}-\frac{\sigma\tau}{\rho}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)-\tau\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right).$$

When the value of P is substituted, the right-hand side vanishes identically; consequently

$$\sum_i \gamma_i v_i = 0,$$

and the second condition is satisfied.

To establish the remaining condition, we note the results

$$\sum_r \sum_i a_{ri} u_r w_i = \Omega\frac{d}{ds}\left(\frac{1}{\rho}\right) = -\Omega\frac{\rho'}{\rho^2},$$

$$\sum_r \sum_i a_{ri} v_r w_i = -\Omega\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right),$$

$$\sum_r \sum_i a_{ri} w_r w_i = \Omega\left(\frac{1}{\sigma^2\tau^2}+\frac{\rho'^2}{\rho^4}+\frac{\sigma'^2}{\sigma^4}\right),$$

$$\sum_r \sum_i a_{ri} t_r w_i = \Omega\left\{\frac{1}{\sigma\tau}\frac{d}{ds}\left(\frac{1}{\sigma\tau}\right)-\frac{\rho'}{\rho^2}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)-\frac{\sigma'}{\sigma^2}\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)\right\};$$

hence

$$\frac{1}{\kappa}\sum_i \gamma_i w_i = -P\frac{\rho'}{\rho^2}+\left\{\frac{\sigma}{\tau}+\frac{d}{ds}(\sigma'\tau)\right\}\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)$$
$$-(2\sigma'\tau+\sigma\tau')\left(\frac{1}{\sigma^2\tau^2}+\frac{\rho'^2}{\rho^4}+\frac{\sigma'^2}{\sigma^4}\right)$$
$$-\frac{d}{ds}\left(\frac{1}{\sigma\tau}\right)+\frac{\rho'\sigma\tau}{\rho^2}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)-\frac{\sigma'\tau}{\sigma}\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right).$$

When the value of P is substituted, this right-hand side also vanishes identically; consequently

$$\sum_i \gamma_i w_i = 0,$$

and the third condition is satisfied.

Thus the three conditions, which are the analytical expression of the perpendicularity of the quadrinormal to the tangent, the binormal, and the trinormal, respectively, are satisfied.

48. Two sets of relations, cognate with some of the preceding relations, may be noted at this stage: their utilisation will come later.

(i) We have had the relations (§§ 31, 38)

$$\sum Y \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) = v_r, \quad \sum Y' \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) = -\sum_\mu g_{\mu r} v_\mu.$$

Let the equation

$$Y'' + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) Y = -\frac{1}{\Omega} \sum_\mu \sum_p a_{\mu p} w_p \frac{\partial y}{\partial x_\mu}$$

be multiplied throughout by $\frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right)$, and let the products on both sides be summed through the plenary space; then, using the property

$$\sum \frac{\partial y}{\partial x_\mu} \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) = \sum_\theta g_{\theta r} A_{\theta\mu},$$

we have

$$\begin{aligned}\sum Y'' \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) v_r &= -\frac{1}{\Omega} \sum_\mu \sum_p \sum_\theta a_{\mu p} A_{\theta\mu} g_{\theta r} w_p \\ &= -\sum_p g_{pr} w_p,\end{aligned}$$

by the customary value of $\sum_\mu a_{\mu p} A_{\theta u}$ as equal to 0 or to Ω, according as θ and p are different, or are the same; that is,

$$\sum Y'' \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) = -\sum_p g_{pr} w_p - \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) v_r.$$

Proceeding similarly from the equation

$$Y''' + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) Y' - 3\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right) Y = -\frac{1}{\Omega} \sum_\mu \sum_p a_{\mu p} t_p \frac{\partial y}{\partial x_\mu},$$

we find

$$\sum Y''' \frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) = -\sum_p g_{pr} t_p + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) \sum_p g_{pr} v_p - 3\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right) v_r.$$

These relations are analogous to the relations

$$\sum Y \frac{\partial y}{\partial x_r} = 0, \quad \sum Y' \frac{\partial y}{\partial x_r} = -v_r,$$

$$\sum Y'' \frac{\partial y}{\partial x_r} = -w_r,$$

$$\sum Y''' \frac{\partial y}{\partial x_r} = -t_r + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) v_r;$$

and the two sets of relations can be combined, to lead to additional results.

(ii) When the equation

$$Y''+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)Y=-\frac{1}{\Omega}\sum_\mu\sum_p a_{\mu p}w_p\frac{\partial y}{\partial x_\mu}$$

is differentiated along the geodesic, we have

$$\begin{aligned}Y'''+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)Y'-2\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)Y\\ =\frac{1}{\Omega}\sum_\mu\sum_p\sum_q\sum_\theta w_p\frac{\partial y}{\partial x_\mu}[a_{\theta p}\{q\theta,\mu\}+a_{\theta\mu}\{q\theta,p\}]x_q',\\ -\frac{1}{\Omega}\sum_\mu\sum_p\left[a_{\mu p}\left(t_p+\sum_\theta g_{\theta p}w_\theta\right)\frac{\partial y}{\partial x_\mu}\right]-\frac{1}{\Omega}\sum_\mu\sum_p a_{\mu p}w_p\frac{d}{ds}\left(\frac{\partial y}{\partial x_\mu}\right).\end{aligned}$$

On the right-hand side, the coefficient of $\frac{1}{\Omega}a_{\alpha\beta}w_\gamma\frac{\partial y}{\partial x_\alpha}$, in the double summation with regard to μ and p,

$$\begin{aligned}&=-g_{\gamma\beta}\text{ from the second term}\\ &\qquad+\sum_q\{q\beta,\gamma\}x_q',\text{ from the first term}\\ &=-g_{\gamma\beta}+g_{\gamma\beta}=0,\end{aligned}$$

for all values of α, γ, β. Also, the remaining coefficient of $\frac{1}{\Omega}a_{\mu p}w_p$

$$\begin{aligned}&=-\frac{d}{ds}\left(\frac{\partial y}{\partial x_\mu}\right)+\sum_q\sum_\epsilon\frac{\partial y}{\partial x_\epsilon}\{q\mu,\epsilon\}x_q'\\ &=-\sum x_q'\left[\frac{\partial^2 y}{\partial x_\mu\,\partial x_q}-\sum_\epsilon\frac{\partial y}{\partial x_\epsilon}\{q\mu,\epsilon\}\right]=-\sum_q\eta_{\mu q}x_q';\end{aligned}$$

and therefore the right-hand side

$$=-\frac{1}{\Omega}\sum_\mu\sum_p a_{\mu p}t_p\frac{\partial y}{\partial x_\mu}-\frac{1}{\Omega}\sum_\mu\sum_p\sum_q a_{\mu p}w_p\eta_{\mu q}x_q'.$$

But we have the equation

$$Y'''+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)Y'-3\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)Y=-\frac{1}{\Omega}\sum_\mu\sum_p a_{\mu p}t_p\frac{\partial y}{\partial x_\mu};$$

and therefore

$$\frac{1}{\Omega}\sum_\mu\sum_p\sum_q a_{\mu p}w_p\eta_{\mu q}x_q'=-\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)Y,$$

a relation, in form analogous to that of the relation

$$\frac{1}{\Omega}\sum_i\sum_j\sum_k a_{ij}v_j\eta_{jk}x_k'=\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)Y$$

of §§ 38, 40.

As an immediate verification, on multiplying by Y, and adding throughout the plenary space, there results the known relation

$$\frac{1}{\Omega}\sum_\mu\sum_p a_{\mu p}w_p v_\mu=-\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right).$$

49. Proceeding from the typical Frenet equation

$$Y' = \frac{l_3}{\sigma} - \frac{y'}{\rho},$$

after differentiation and substitution, we find successively

$$Y'' + Y\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) = \frac{l_4}{\sigma\tau} + l_3 \frac{d}{ds}\left(\frac{1}{\sigma}\right) - y' \frac{d}{ds}\left(\frac{1}{\rho}\right),$$

$$Y''' + Y'\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) - 3Y\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right)$$
$$= \frac{l_5}{\sigma\tau\kappa} + l_4 \sigma \frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right) + l_3\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma\tau^2}\right\} - y' \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right),$$

the last of which accords with the formulæ of § 45; and thus

$$\frac{l_5}{\sigma\tau\kappa} + l_4 \sigma \frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right) + l_3\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma\tau^2}\right\} - y' \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) = -\frac{1}{\Omega}\sum_r \sum_i a_{ri} t_i \frac{\partial y}{\partial x_r}.$$

Let this equation be squared as to both sides, and let the sums of the squares be taken through the dimension-range: then, owing to the properties

$$\sum l_4 l_5 = 0, \quad \sum l_3 l_5 = 0, \quad \sum y' l_5 = 0, \quad \sum l_4 l_3 = 0, \quad \sum l_4 y' = 0, \quad \sum l_3 y' = 0,$$

we have

$$\frac{1}{\sigma^2\tau^2\kappa^2} + \frac{1}{\sigma^4\tau^4}(\sigma\tau' + 2\sigma'\tau)^2 + \frac{1}{\sigma^2\tau^4} - \frac{2}{\sigma\tau^2}\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) + \left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)\right\}^2 + \left\{\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)\right\}^2$$
$$= \frac{1}{\Omega^2}\sum_r \sum_i \sum_p \sum_j a_{ri} t_i a_{pj} t_j A_{rp}$$
$$= \frac{1}{\Omega}\sum_r \sum_i a_{ri} t_i t_r,$$

which accordingly gives an expression for the coil of the amplitudinal geodesic in terms of magnitudes connected with the amplitude, the quantities ρ, σ, τ, having already been obtained in terms of such magnitudes.

Ex. It is easy to verify the relations

$$\sum l_3 Y' = \frac{1}{\sigma}; \quad \sum l_3 Y'' = \frac{d}{ds}\left(\frac{1}{\sigma}\right), \quad \sum l_4 Y'' = \frac{1}{\sigma\tau};$$

$$\sum l_3 Y''' = \frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} + \frac{1}{\tau^2}\right), \quad \sum l_4 Y''' = \sigma \frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right), \quad \sum l_5 Y''' = \frac{1}{\sigma\tau\kappa},$$

which are derivable from the Frenet equations modified (§ 8) for the geodesic.

Some concomitants in the system for the amplitude.

50. At this stage, it is convenient to use the expressions for the various combinations of curvatures that have been found, in order to obtain the geometrical interpretations of some types of covariants arising out of the system of homogeneous forms, which represent the analytical values of

$$\frac{1}{\rho}, \quad \frac{d}{ds}\left(\frac{1}{\rho}\right), \quad \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right),$$

together with the permanent form U which is equal to unity. For this purpose, we associate, with the foregoing expression for $\sum_r \sum_i a_{ri} t_i t_r$, the expressions

$$\sum_r \sum_i a_{ri} u_r t_i = \Omega \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right),$$

$$\sum_r \sum_i a_{ri} v_r t_i = \Omega \left[-\frac{1}{\sigma^2 \tau^2} + \frac{1}{\rho} \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) + \frac{1}{\sigma} \frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) \right],$$

$$\sum_r \sum_i a_{ri} w_r t_i = \Omega \left[\frac{1}{\sigma\tau} \frac{d}{ds}\left(\frac{1}{\sigma\tau}\right) - \frac{\rho'}{\rho^2} \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) - \frac{\sigma'}{\sigma^2} \frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) \right],$$

obtained in § 46; and we recall the six similar expressions collected at the end of § 40. The covariants, of the indicated types, are analogous in formation to the covariant

$$\begin{vmatrix} \sum\sum a_{ri} u_r u_i, & \sum\sum a_{ri} u_r v_i \\ \sum\sum a_{ri} v_r u_i, & \sum\sum a_{ri} v_r v_i \end{vmatrix}$$

considered in § 35, the geometrical value of this determinant being

$$= \Omega^2 \begin{vmatrix} 1, & \dfrac{1}{\rho} \\ \dfrac{1}{\rho}, & \dfrac{1}{\rho^2} + \dfrac{1}{\sigma^2} \end{vmatrix} = \frac{\Omega^2}{\sigma^2}.$$

It can be represented in each of the equivalent forms

$$\sum_i \sum_j \sum_k \sum_l \begin{vmatrix} a_{ik}, & a_{il} \\ a_{jk}, & a_{jl} \end{vmatrix} \begin{vmatrix} u_i, & u_j \\ v_i, & v_j \end{vmatrix} \begin{vmatrix} u_k, & u_l \\ v_k, & v_l \end{vmatrix},$$

$$\Omega \begin{vmatrix} A_{11}, & A_{12}, & \dots, & A_{1n}, & u_1, & v_1 \\ A_{21}, & A_{22}, & \dots, & A_{2n}, & u_2, & v_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{n1}, & A_{n2}, & \dots, & A_{nn}, & u_n, & v_n \\ u_1, & u_2, & \dots, & u_n, & 0, & 0 \\ v_1, & v_2, & \dots, & v_n, & 0, & 0 \end{vmatrix}.$$

(i). We have

$$\begin{vmatrix} \sum\limits_r \sum\limits_p a_{rp}u_r u_p, & \sum\limits_r \sum\limits_p a_{rp}u_r v_p \\ \sum\limits_r \sum\limits_p a_{rp}u_r w_p, & \sum\limits_r \sum\limits_p a_{rp}v_r w_p \end{vmatrix} = \Omega^2 \begin{vmatrix} 1, & \dfrac{1}{\rho} \\ -\dfrac{\rho'}{\rho^2}, & -\dfrac{\rho'}{\rho^3} - \dfrac{\sigma'}{\sigma^3} \end{vmatrix} = -\Omega^2 \frac{\sigma'}{\sigma^3};$$

and the left-hand side can be represented in each of the equivalent forms

$$\sum_i \sum_j \sum_k \sum_l \begin{vmatrix} a_{ik}, & a_{il} \\ a_{jk}, & a_{jl} \end{vmatrix} \begin{vmatrix} u_i, & u_j \\ v_i, & v_j \end{vmatrix} \begin{vmatrix} u_k, & u_l \\ w_k, & w_l \end{vmatrix},$$

$$\Omega \begin{vmatrix} A_{11}, & A_{12}, & \dots, & A_{1n}, & u_1, & v_1 \\ A_{21}, & A_{22}, & \dots, & A_{2n}, & u_2, & v_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{n1}, & A_{n2}, & \dots, & A_{nn}, & u_n, & v_n \\ u_1, & u_2, & \dots, & u_n, & 0, & 0 \\ w_1, & w_2, & \dots, & w_n, & 0, & 0 \end{vmatrix}.$$

Similarly, we have

$$\begin{vmatrix} \sum\limits_r \sum\limits_p a_{rp}u_r w_p, & \sum\limits_r \sum\limits_p a_{rp}u_r t_p \\ \sum\limits_r \sum\limits_p a_{rp}v_r w_p, & \sum\limits_r \sum\limits_p a_{rp}v_r t_p \end{vmatrix} = \Omega^2 \left\{ \frac{\rho'}{\rho^2\sigma^2\tau^2} + \frac{\sigma'}{\sigma^3}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) - \frac{\rho'}{\rho^2\sigma}\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) \right\},$$

and the left-hand side can be expressed in each of the equivalent forms

$$\sum_i \sum_j \sum_k \sum_l \begin{vmatrix} a_{ik}, & a_{il} \\ a_{jk}, & a_{jl} \end{vmatrix} \begin{vmatrix} u_i, & u_j \\ v_i, & v_j \end{vmatrix} \begin{vmatrix} w_k, & w_l \\ t_k, & t_l \end{vmatrix},$$

$$\Omega \begin{vmatrix} A_{11}, & A_{12}, & \dots, & A_{1n}, & u_1, & v_1 \\ A_{21}, & A_{22}, & \dots, & A_{2n}, & u_2, & v_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{n1}, & A_{n2}, & \dots, & A_{nn}, & u_n, & v_n \\ w_1, & w_2, & \dots, & w_n, & 0, & 0 \\ t_1, & t_2, & \dots, & t_n, & 0, & 0 \end{vmatrix}.$$

Likewise for all the analogous combinations, which can be constructed from determinants of two rows.

(ii). We can construct, in the same fashion, other covariants by means of determinants of three rows. Thus the determinant

$$\begin{vmatrix} \sum\limits_r \sum\limits_p a_{rp}u_r u_p, & \sum\limits_r \sum\limits_p a_{rp}u_r v_p, & \sum\limits_r \sum\limits_p a_{rp}u_r w_p \\ \sum\limits_r \sum\limits_p a_{rp}u_r v_p, & \sum\limits_r \sum\limits_p a_{rp}v_r v_p, & \sum\limits_r \sum\limits_p a_{rp}v_r w_p \\ \sum\limits_r \sum\limits_p a_{rp}u_r w_p, & \sum\limits_r \sum\limits_p a_{rp}v_r w_p, & \sum\limits_r \sum\limits_p a_{rp}w_r w_p \end{vmatrix},$$

in geometrical significance, has the value

$$\Omega^3 \begin{vmatrix} 1, & \dfrac{1}{\rho}, & -\dfrac{\rho'}{\rho^2} \\ \dfrac{1}{\rho}, & \dfrac{1}{\rho^2}+\dfrac{1}{\sigma^2}, & -\left(\dfrac{\rho'}{\rho^3}+\dfrac{\sigma'}{\sigma^3}\right) \\ -\dfrac{\rho'}{\rho^2}, & -\left(\dfrac{\rho'}{\rho^3}+\dfrac{\sigma'}{\sigma^3}\right), & \dfrac{1}{\sigma^2\tau^2}+\dfrac{\rho'^2}{\rho^4}+\dfrac{\sigma'^2}{\sigma^4} \end{vmatrix}, \quad =\Omega^3\frac{1}{\sigma^4\tau^2};$$

and it can be represented in the form

$$\sum_\alpha\sum_\beta\sum_\gamma\sum_i\sum_j\sum_k \begin{vmatrix} a_{\alpha i}, & a_{\alpha j}, & a_{\alpha k} \\ a_{\beta i}, & a_{\beta j}, & a_{\beta k} \\ a_{\gamma i}, & a_{\gamma j}, & a_{\gamma k} \end{vmatrix} \begin{vmatrix} u_\alpha, & u_\beta, & u_\gamma \\ v_\alpha, & v_\beta, & v_\gamma \\ w_\alpha, & w_\beta, & w_\gamma \end{vmatrix} \begin{vmatrix} u_i, & u_j, & u_k \\ v_i, & v_j, & v_k \\ w_i, & w_j, & w_k \end{vmatrix}$$

as also in the equivalent form

$$\Omega^2 \begin{vmatrix} A_{11}, & A_{12}, \ldots, & A_{1n}, & u_1, & v_1, & w_1 \\ A_{21}, & A_{22}, \ldots, & A_{2n}, & u_2, & v_2, & w_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n1}, & A_{n2}, \ldots, & A_{nn}, & u_n, & v_n, & w_n \\ u_1, & u_2, \ldots, & u_n, & 0, & 0, & 0 \\ v_1, & v_2, \ldots, & v_n, & 0, & 0, & 0 \\ w_1, & w_2, \ldots, & w_n, & 0, & 0, & 0 \end{vmatrix}.$$

Likewise for analogous combinations, by determinants of three rows, involving three of the four sets of quantities u_r, v_r, w_r, t_r.

(iii). The results enable us to construct similar covariants of four rows. Thus we have the concomitant

$$\begin{vmatrix} \sum_r\sum_p a_{rp}u_ru_p, & \sum_r\sum_p a_{rp}u_rv_p, & \sum_r\sum_p a_{rp}u_rw_p, & \sum_r\sum_p a_{rp}u_rt_p \\ \sum_r\sum_p a_{rp}u_rv_p, & \sum_r\sum_p a_{rp}v_rv_p, & \sum_r\sum_p a_{rp}v_rw_p, & \sum_r\sum_p a_{rp}v_rt_p \\ \sum_r\sum_p a_{rp}u_rw_p, & \sum_r\sum_p a_{rp}v_rw_p, & \sum_r\sum_p a_{rp}w_rw_p, & \sum_r\sum_p a_{rp}w_rt_p \\ \sum_r\sum_p a_{rp}u_rt_p, & \sum_r\sum_p a_{rp}v_rt_p, & \sum_r\sum_p a_{rp}w_rt_p, & \sum_r\sum_p a_{rp}t_rt_p \end{vmatrix}.$$

When we substitute the known geometrical values for the double sums in the various terms, and then subtract from the second, third, and fourth rows respectively, the product of the first row by

$$\frac{1}{\rho}, \quad \frac{d}{ds}\left(\frac{1}{\rho}\right), \quad \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right),$$

the determinant can be expressed in the form

$$\frac{\Omega^4}{\sigma^2}\begin{vmatrix} 1, & \dfrac{d}{ds}\left(\dfrac{1}{\sigma}\right), & -\dfrac{1}{\sigma\tau^2}+\dfrac{d^2}{ds^2}\left(\dfrac{1}{\rho}\right) \\ \dfrac{d}{ds}\left(\dfrac{1}{\sigma}\right), & \dfrac{1}{\sigma^2\tau^2}+\left\{\dfrac{d}{ds}\left(\dfrac{1}{\sigma}\right)\right\}^2, & \Upsilon \\ -\dfrac{1}{\sigma\tau^2}+\dfrac{d^2}{ds^2}\left(\dfrac{1}{\rho}\right), & \Upsilon, & \Phi \end{vmatrix},$$

where

$$\Upsilon=-\frac{\sigma\tau'+\sigma'\tau}{\sigma^3\tau^3}+\frac{d}{ds}\left(\frac{1}{\sigma}\right)\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right),$$

$$\Phi=\frac{1}{\sigma^2\tau^2\kappa^2}+\frac{1}{\sigma^2\tau^4}+\frac{(\sigma\tau'+2\sigma'\tau)^2}{\sigma^4\tau^4}-\frac{2}{\sigma\tau^2}\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)+\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)\right\}^2.$$

When the last determinant is evaluated, a final value of the original determinant is found to be

$$\Omega^4\frac{1}{\sigma^6\tau^4\kappa^2}.$$

The covariant can be represented in the form

$$\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta\sum_i\sum_j\sum_k\sum_l \begin{vmatrix} a_{\alpha i}, & a_{\alpha j}, & a_{\alpha k}, & a_{\alpha l} \\ a_{\beta i}, & a_{\beta j}, & a_{\beta k}, & a_{\beta l} \\ a_{\gamma i}, & a_{\gamma j}, & a_{\gamma k}, & a_{\gamma l} \\ a_{\delta i}, & a_{\delta j}, & a_{\delta k}, & a_{\delta l} \end{vmatrix}\begin{vmatrix} u_\alpha, & u_\beta, & u_\gamma, & u_\delta \\ v_\alpha, & v_\beta, & v_\gamma, & v_\delta \\ w_\alpha, & w_\beta, & w_\gamma, & w_\delta \\ t_\alpha, & t_\beta, & t_\gamma, & t_\delta \end{vmatrix}\begin{vmatrix} u_i, & u_j, & u_k, & u_l \\ v_i, & v_j, & v_k, & v_l \\ w_i, & w_j, & w_k, & w_l \\ t_i, & t_j, & t_k, & t_l \end{vmatrix},$$

and also in the form

$$\Omega^3\begin{vmatrix} A_{11}, & A_{12}, & \dots, & A_{1n}, & u_1, & v_1, & w_1, & t_1 \\ A_{21}, & A_{22}, & \dots, & A_{2n}, & u_2, & v_2, & w_2, & t_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{n1}, & A_{n2}, & \dots, & A_{nn}, & u_n, & v_n, & w_n, & t_n \\ u_1, & u_2, & \dots, & u_n, & 0, & 0, & 0, & 0 \\ v_1, & v_2, & \dots, & v_n, & 0, & 0, & 0, & 0 \\ w_1, & w_2, & \dots, & w_n, & 0, & 0, & 0, & 0 \\ t_1, & t_2, & \dots, & t_n, & 0, & 0, & 0, & 0 \end{vmatrix}.$$

(iv). As corollaries from the foregoing results, their special forms in three special instances may be noted.

When the amplitude is a surface, $n=2$; and we then have

$$\frac{\Omega^2}{\sigma^2}=\begin{vmatrix} A, & H, & u_1, & v_1 \\ H, & B, & u_2, & v_2 \\ u_1, & u_2, & 0, & 0 \\ v_1, & v_2, & 0, & 0 \end{vmatrix}=\Omega(u_2v_1-v_2u_1)^2,$$

with the modified notation $A, H, B, = A_{11}, A_{12}, A_{22}$, indicated (p. 31) for a surface: that is,

$$\frac{\Omega}{\sigma^2} = (u_2 v_1 - v_2 u_1)^2,$$

in accordance with the expression for the torsion in § 106.

When the amplitude is a region, $n=3$; and then

$$\frac{\Omega^3}{\sigma^4 \tau^2} = \Omega^2 \begin{vmatrix} A, & H, & G, & u_1, & v_1, & w_1 \\ H, & B, & F, & u_2, & v_2, & w_2 \\ G, & F, & C, & u_3, & v_3, & w_3 \\ u_1, & u_2, & u_3, & 0, & 0, & 0 \\ v_1, & v_2, & v_3, & 0, & 0, & 0 \\ w_1, & w_2, & w_3, & 0, & 0, & 0 \end{vmatrix} = \Omega^2 \begin{vmatrix} u_1, & v_1, & w_1 \\ u_2, & v_2, & w_2 \\ u_3, & v_3, & w_3 \end{vmatrix}^2,$$

with the modified notation indicated (p. 32) for a region: that is,

$$\frac{\Omega}{\sigma^4 \tau^2} = \begin{vmatrix} u_1, & v_1, & w_1 \\ u_2, & v_2, & w_2 \\ u_3, & v_3, & w_3 \end{vmatrix}^2,$$

in accordance with the result in § 179.

When the amplitude is a domain, $n=4$; and we similarly find

$$\frac{\Omega}{\sigma^6 \tau^4 \kappa^2} = \begin{vmatrix} u_1, & v_1, & w_1, & t_1 \\ u_2, & v_2, & w_2, & t_2 \\ u_3, & v_3, & w_3, & t_3 \\ u_4, & v_4, & w_4, & t_4 \end{vmatrix}^2,$$

in accordance with the result in § 288.

CHAPTER IV

Geodesic Properties relative to the Tangent Homaloid

Principal lines of a geodesic within the tangent homaloid of the amplitude.

51. Proceeding in a manner similar to that adopted for the determination of the respective directions of the binormal, the trinormal, and the quartinormal, we can obtain the principal lines of a geodesic in successive rank, so long as they are found to obey the same type of law in their analytical formation. In each such instance, a beginning has been made, through space-positions in the $(n+1)$-fold homaloid

$$\left\| \bar{y}-y, \quad \frac{\partial y}{\partial x_1}, \ \ldots, \ \frac{\partial y}{\partial x_n}, \quad Y \right\| = 0,$$

which contains the n-fold tangent homaloid of the amplitude and the prime normal of any geodesic, and so contains the osculating plane of the geodesic at the central point O. The tangent at a second point of the geodesic was proved (§ 33) to lie within the $(n+1)$-fold homaloid which, because it contains the osculating plane at O and the tangent at a second point, contains the osculating flat of the geodesic at O. This osculating flat contains the binormal; the typical direction-cosines of the binormal are therefore linearly expressible in terms of the quantities

$$\frac{\partial y}{\partial x_1}, \ \ldots\ldots, \ \frac{\partial y}{\partial x_n}, \ Y;$$

and, because the binormal is at right angles to the prime normal, the constituent Y does not enter into the expression of those typical direction-cosines; that is, a typical direction-cosine of the binormal has the form

$$l_3 = \sum a_i \frac{\partial y}{\partial x_i}.$$

Next, it was proved (§ 39) that the tangent at a third consecutive point of the geodesic lies within the $(n+1)$-fold homaloid which accordingly, because it contains the osculating flat at O and a consecutive tangent not in that flat, contains the osculating block of the geodesic at O. This osculating block contains the trinormal; the typical direction-cosines of the trinormal are therefore linearly expressible in terms of the quantities

$$\frac{\partial y}{\partial x_1}, \ \ldots\ldots, \ \frac{\partial y}{\partial x_n}, \ Y;$$

and because the trinormal is at right angles to the prime normal, the constituent Y does not enter into the expression of these typical direction-cosines; that is, a typical direction-cosine of the trinormal has the form

$$l_4 = \sum_i \beta_i \frac{\partial y}{\partial x_i}.$$

And so on, in succession. The proved inclusion, within the n-fold tangent homaloid of the amplitude, of the principal line of rank k in the orthogonal frame of the geodesic (where, for example, $k=2$ for the binormal, $k=3$ for the trinormal, $k=4$ for the quartinormal) secures that the tangent to the geodesic at a k-th successive point lies within the $(n+1)$-fold homaloid H. This homaloid, already containing the osculating homaloid of the geodesic of k dimensions and now containing also an additional tangent not lying in that homaloid, therefore contains the osculating homaloid of the geodesic of $(k+1)$ dimensions: that is, it contains the principal line of rank $k+1$ in the orthogonal frame of the geodesic. Thus the typical direction-cosines of the new principal line are expressible linearly in terms of the quantities

$$\frac{\partial y}{\partial x_1}, \;\ldots\ldots, \;\frac{\partial y}{\partial x_n}, \;Y;$$

and because this principal line is at right angles to the prime normal, the constituent Y does not enter into the expression of these direction-cosines, which therefore have the form

$$l_{k+1} = \sum_i \theta_i \frac{\partial y}{\partial x_i}.$$

Consequently the principal line of rank $k+1$ in the orthogonal frame of the geodesic lies within the n-fold tangent homaloid of the amplitude.

But a limit to the number of principal lines, which can lie within this n-fold homaloid, is imposed by its dimensionality; and the corresponding limit upon the expression of successive typical direction-cosines is imposed by the number of parametric magnitudes available. This limit is easily discerned, in both regards.

The n-fold tangent homaloid is analytically determinate, so far as concerns includible directions, by means of n leading lines: any set of n such lines can be substituted for an initially given set: in particular, any set of n such lines, orthogonal to one another, can be selected. It has been proved that the tangent to the curve lies within the homaloid. The prime normal is orthogonal to the homaloid, as being at right angles to every direction in it because of the relation

$$\sum Y \frac{\partial y}{\partial x_r} = 0, \quad (r=1, \ldots, n),$$

and therefore the prime normal is external to the homaloid. The binormal, being of rank two, lies within it: likewise, the trinormal, being of rank three; and so on in succession, up to (but not beyond) the principal line of rank n. Thus there

can be an aggregate, of n principal lines of the orthogonal frame of an amplitudinal geodesic, lying within the n-fold tangent homaloid of the amplitude at the point O.

Next, consider the analytical possibilities and limitations. Let an integer N, where N is greater than n but otherwise is unrestricted, denote the number of dimensions in the plenary homaloidal space of the amplitude. Then, for the specification of any point in that plenary space, N coordinates y are necessary and sufficient; and, for the specification of any point in the amplitude, each of the N coordinates y is a function of n parameters denoted by $x_1, \ldots, x_n$, the exclusive limitations on the forms of these functions being those stated in § 10. Hence, for the purpose of expressing directions lying within the n-fold tangent homaloid of the amplitude, there are Nn disposable quantities of the type

$$\frac{\partial y_m}{\partial x_i},$$

for $m=1, \ldots, N$, and $i=1, \ldots, n$. When any spatial direction lies within the n-fold homaloid, its typical direction-cosine is bound to be of the form

$$l=\kappa_1 \frac{\partial y}{\partial x_1}+\kappa_2 \frac{\partial y}{\partial x_2}+\ldots+\kappa_n \frac{\partial y}{\partial x_n},$$

where the quantities $\kappa_1, \ldots, \kappa_n$, are magnitudes not explicitly involving any of the quantities $\frac{\partial y}{\partial x}$, and occur in the expression of each cosine of the whole set for the postulated direction. Now there are N such cosines for a direction, linearly independent of one another; consequently, when the N direction-cosines of a line are assigned, N of the Nn disposable quantities $\frac{\partial y}{\partial x}$ are required. Consequently when n such lines, constituting a leading set for any n-fold homaloid are assigned, so that their expressions are free from homogeneous linear relations, the total number of quantities $\frac{\partial y}{\partial x}$ thus required for their expression is Nn, the precise number of quantities that are available. The exclusive limitations in § 10, already cited, allow the Nn magnitudes adequate freedom for the expression of the n directions; the absence of homogeneous linear relations among the directions of any n principal lines in the orthogonal frame of an amplitudinal geodesic precludes the possibility of their availability for the expression of more than n such directions.

Two inferences follow, as immediate consequences. The first is: the $(n+1)$-fold homaloid H, represented by the equations

$$\left\| \bar{y}-y, \ \frac{\partial y}{\partial x_1}, \ \ldots, \ \frac{\partial y}{\partial x_n}, \ Y \right\|=0,$$

contains the first $n+1$ principal lines in the orthogonal frame of an amplitudinal

geodesic, and therefore is the osculating $(n+1)$-fold homaloid of that geodesic. The second is: the n-fold homaloid, which is represented by the equations

$$\left\| \bar{y}-y, \frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n} \right\| = 0,$$

and which therefore is tangent homaloid of the amplitude, contains the tangent and the normals of rank 2, rank 3, ..., rank n, of every geodesic in the amplitude through the point O.

As the directions of the tangent, the binormal, the trinormal, and all the succeeding principal lines as far as the $(n+1)$th principal line inclusive, of every amplitudinal geodesic through the point O, lie within the n-fold homaloid which touches the amplitude at O, these n directions will be called *gremial*; and the curvatures, associated with the gremial directions of rank later than the tangent in the Frenet tableau, will be called the gremial curvatures. The n gremial directions of any geodesic, combined with its prime normal, can be taken as a complete set of guiding lines for the $(n+1)$-fold homaloid represented by the equations

$$\left\| \bar{y}-y, \frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n}, Y \right\|,$$

this homaloid being, in fact, the $(n+1)$-fold osculating homaloid of the amplitudinal geodesic. When convenient for brevity, the remaining principal directions in the orthogonal frame of the geodesic may be described as non-gremial.

52. It has appeared that, because the gremial lines lie within the n-fold tangent homaloid so that their direction-cosines are expressible in terms of the parameters of leading lines in that homaloid, the typical direction-cosines of the gremial lines are of the form

$$l_i = \sum_r c_{ir} \frac{\partial y}{\partial x_r},$$

for $i=1, 3, 4, \ldots, n+1$. To determine the various coefficients c_{ir}, the equation which provides the value of l_i has been made to conform to the Frenet scheme. In this method, successive arc-derivatives of the circular curvature of the amplitudinal geodesic are required at the successive increases of i for each stage; thus for l_4, the quantity $\frac{d}{ds}\left(\frac{1}{\rho}\right)$ is required; and for l_5, the quantity $\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)$ is required. Incidentally in the process, the perpendicularity of each principal line to all the principal lines of earlier rank led to the requirement of linear conditions, to be satisfied by the coefficients c_{ir}; and having forms of the type

$$\sum_r c_{ir} u_r = 0, \quad \sum_r c_{ir} v_r = 0, \quad \sum_r c_{ir} w_r = 0,$$

there being $i-2$ such conditions for the coefficients in l, where $3 \leqslant i \leqslant n+1$. These conditions, however, have been used for the verification of the values of c_{ir}

only after the values have been determined: the conditions have not been used to determine such values.

The foregoing n equations express the typical quantities y', $l_3, \ldots, l_{n+1}$, linearly in terms of the n typical quantities $\frac{\partial y}{\partial x_r}$, for $r=1, \ldots, n$. These typical direction-cosines, belonging to a set of n directions which are orthogonal among themselves, are certainly free from any homogeneous linear relation with coefficients independent of the actual quantities l in the system. Consequently the set of n equations can be resolved, so as to express the set of quantities $\frac{\partial y}{\partial x_r}$, for $r=1, \ldots, n$, in terms of the corresponding typical direction-cosines l; and the resolved equations are

$$\frac{\partial y}{\partial x_i}=\sum_t C_{ti} l_t, \quad (i=1, \ldots, n),$$

the summation being for the values $t=1, 3, 4, \ldots, n+1$. To determine the coefficients C_{ti}, on the assumption that the coefficients c_{ir} are known, we have, on multiplying by l_t and adding through the space-dimensionality,

$$\begin{aligned} C_{ti} &= \sum l_t \frac{\partial y}{\partial x_i} = \sum \left(\sum_r c_{tr} \frac{\partial y}{\partial x_r}\right) \frac{\partial y}{\partial x_i} \\ &= \sum_r A_{ir} c_{tr}, \end{aligned}$$

for the values $i=1, \ldots, n$, and $t=1, 3, \ldots, n+1$. In particular, we have

$$\begin{aligned} C_{1i} &= \sum_r A_{ir} c_{ir} \\ &= \sum_r A_{ir} x_r' = u_i ; \end{aligned}$$

also, by § 34,

$$C_{3i} = \sum l_3 \frac{\partial y}{\partial x_i} = \sigma \left(\frac{1}{\rho} u_i - v_i\right);$$

by § 39,

$$C_{4i} = \sum l_4 \frac{\partial y}{\partial x_i} = \left\{\tau \frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\} u_i - \sigma' \tau v_i - \sigma \tau w_i ;$$

and by § 45,

$$\begin{aligned} C_{5i} &= \sum l_5 \frac{\partial y}{\partial x_i} \\ &= \left[\frac{\sigma}{\rho\tau} + \frac{d}{ds}\left\{\tau \frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}\right] u_i - \left\{\frac{\sigma}{\tau} + \frac{d}{ds}(\sigma'\tau)\right\} v_i \\ &\qquad - \left\{\frac{1}{\sigma} \frac{d}{ds}(\sigma^2 \tau)\right\} w_i - \sigma \tau t_i . \end{aligned}$$

But the set of gremial directions consists only of the aggregate $l_1 (=y')$, l_3, $l_4, \ldots, l_{n+1}$, in the Frenet system. There still remain the non-gremial directions $l_2 (=Y)$, $l_{n+2}, \ldots, l_N$, in that system, the integer N denoting (as before) the order

of dimensionality of the plenary space: and each of these non-gremial directions $l_{n+2}, \ldots, l_N$, is orthogonal to the $(n+1)$-fold osculating homaloid H of the amplitudinal geodesic as represented by the equations

$$\left\| \bar{y}-y, \frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n}, Y \right\| = 0.$$

Now every direction within that homaloid has its typical direction-cosine given by an expression

$$\xi Y + \xi_1 \frac{\partial y}{\partial x_1} + \ldots\ldots + \xi_n \frac{\partial y}{\partial x_n},$$

where $\xi, \xi_1, \ldots, \xi_n$, are parameters appropriate to the direction. Hence every spatial direction, represented typically by $\bar{y}-y$ when drawn through O, is orthogonal to that $(n+1)$-fold homaloid if, and only if, the relation

$$\sum \left\{ (\bar{y}-y) \left(\xi Y + \xi_1 \frac{\partial y}{\partial x_1} + \ldots\ldots + \xi_n \frac{\partial y}{\partial x_n} \right) \right\} = 0$$

is satisfied for all possible values of the parameters $\xi, \xi_1, \ldots, \xi_n$: that is, if all the $n+1$ relations

$$\sum \{(\bar{y}-y) Y\} = 0,$$

$$\sum \left\{ (\bar{y}-y) \frac{\partial y}{\partial x_r} \right\} = 0, \quad (r=1, \ldots, n),$$

are satisfied. Accordingly, this set of relations constitutes the set of $n+1$ equations representing the $(N-n-1)$-fold homaloid orthogonal to the $(n+1)$-fold osculating homaloid of the geodesic. This orthogonal homaloid contains the $N-n-1$ non-gremial directions $l_{n+2}, \ldots, l_N$; each of these is orthogonal to the $(n+1)$-fold osculating homaloid of the geodesic; and we therefore have the relations

$$\sum l_p Y = 0,$$

$$\sum l_p \frac{\partial y}{\partial x_r} = 0, \quad (r=1, \ldots, n),$$

for all the values $p=n+2, n+3, \ldots, N$.

53. We now are in a position to obtain an expression of an entirely different character for the magnitude η_{ij}, already (§ 21) defined by the equation

$$\eta_{ij} = \frac{\partial^2 y}{\partial x_i \partial x_j} - \sum_k \{ij, k\} \frac{\partial y}{\partial x_k}, \quad (i, j, =1, \ldots, n).$$

As was there remarked, all such quantities satisfy the relations

$$\sum \eta_{ij} \frac{\partial y}{\partial x_r} = 0,$$

for all the values $r=1, \ldots, n$, in virtue of the relations (§ 21)

$$[ij, r] - \sum_k \{ij, k\} A_{rk} = 0.$$

Consequently, because each member of the gremial set $l_1, l_3, \ldots, l_{n+1}$, is expressible as a linear combination of the quantities $\frac{\partial y}{\partial x}$ in a form

$$l_k = \sum_r c_{kr} \frac{\partial y}{\partial x_r},$$

the relations

$$\sum \eta_{ij} l_k = 0$$

are satisfied for all the values $i, j, =1, 2, \ldots, n$, and $k=1, 3, 4, \ldots, n+1$.

Now η_{ij} does not itself denote an intrinsic magnitude of the amplitude or of a geodesic in that amplitude; and therefore it is not of an invariantive character. But, still being a magnitude in the system of the amplitudinal configuration, it is expressible by reference to any system of axes in the plenary homaloidal space of the amplitude; and therefore, when we take the complete orthogonal frame of an amplitudinal geodesic as providing such a system of axes for the plenary space, there will be some relation of the type

$$\eta_{ij} = \sum_\mu P_{ij}^{(\mu)} l_\mu,$$

where the μ-summation is for the range $1, 2, \ldots, N$, while the coefficients $P_{ij}^{(\mu)}$ have to be determined. Owing to the conditions

$$\sum l_p^2 = 1, \quad \sum l_p l_q = 0,$$

where p and q are different, and $p, q, =1, \ldots, N$, we have, for each value of p,

$$\sum l_p \eta_{ij} = P_{ij}^{(p)},$$

the summation on the left-hand side extending through the plenary space. Now when $p=1, 3, 4, \ldots, n+1$, the left-hand side vanishes; and therefore

$$P_{ij}^{(p)} = 0,$$

for $p=1, 3, 4, \ldots, n+1$. Also, when $p=2$ so that l_2 becomes Y, we have

$$P_{ij}^{(2)} = \sum Y \eta_{ij} = L_{ij},$$

where L_{ij} denotes the typical secondary magnitude (§ 30) of the amplitude. Hence we have

$$\eta_{ij} = Y L_{ij} + \sum_{n+2}^{N} l_\mu P_{ij}^{(\mu)},$$

for all the combinations $i, j, =1, \ldots, n$, the coefficients $P_{ij}^{(\mu)}$, for $\mu = n+2$, $n+3, \ldots, N$, being still undetermined. Consequently

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = Y L_{ij} + \sum_r \left[\{ij, r\} \frac{\partial y}{\partial x_r} \right] + \sum_{n+2}^{N} l_\mu P_{ij}^{(\mu)},$$

for all these combinations of i and j. These relations constitute a set of equations,

partly differential in form, satisfied by the space-coordinates of any point in the amplitude.

Before passing to further investigations, it is worth noting one use of this result in connection with the relation

$$\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) Y=\frac{1}{\Omega} \sum_i \sum_j \sum_k a_{ij} v_i x_k' \eta_{jk}$$

of § 40, and with the relation

$$-\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right) Y=\frac{1}{\Omega} \sum_i \sum_j \sum_k a_{ij} w_i x_k' \eta_{jk}$$

of § 48. When the foregoing values of the quantities η_{jk} are substituted in the first of these relations, we find

$$\begin{aligned}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) Y &=\frac{1}{\Omega} \sum_i \sum_j \sum_k a_{ij} v_i x_k' \left\{YL_{jk}+\sum_{n+2}^{N} l_\mu P_{jk}^{(\mu)}\right\} \\ &=\frac{1}{\Omega} Y \sum_i \sum_j a_{ij} v_i v_j+\frac{1}{\Omega} \sum_{n+2}^{N} l_\mu \left[\sum_i \sum_j \sum_k a_{ij} v_i x_k' P_{jk}^{(\mu)}\right].\end{aligned}$$

Now we have had the relation

$$\frac{1}{\rho^2}+\frac{1}{\sigma^2}=\frac{1}{\Omega} \sum_i \sum_j a_{ij} v_i v_j ;$$

and as the lines represented by the typical direction-cosines $l_{n+2}, \ldots, l_N$, are orthogonal to one another, so that no linear relation can exist among these direction-cosines, we have

$$\sum_i \sum_j \sum_k a_{ij} v_i x_k' P_{jk}^{(\mu)}=0,$$

for all the values $\mu=n+2, \ldots, N$.

Proceeding similarly from the second of the cited results, and using the established relation

$$-\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)=\frac{1}{\Omega} \sum_i \sum_j a_{ij} w_i v_j,$$

we are led to the relations

$$\sum_i \sum_j \sum_k a_{ij} w_i x_k' P_{jk}^{(\mu)}=0,$$

again for all the values $\mu=n+2, \ldots, N$.

There are other relations of like character, the number of them depending on the dimensionality of the amplitude.

*Curves * of circular curvature.*

54. Curves of circular curvature are of fundamental organic importance on surfaces existing in triple homaloidal space; and it will appear that they are of

* The phrase "curves of curvature", in preference to the phrase "lines of curvature" customarily associated with surfaces in triple homaloidal space, is used because the word "line" has been reserved (p. 1) to denote a straight line.

similar organic importance in any n-fold amplitude which is primary *, that is, which exists in a plenary homaloidal space of $n+1$ dimensions. But such curves of curvature neither possess the same kind of organic importance nor are characterised by a similar analytical simplicity of expression, when the amplitude is not primary to its plenary homaloidal space.

The construction of curves of circular curvature in any configuration emerges from one or other of two properties, distinct in their geometrical bearing, the same in their analytical expression. By the one property, an amplitudinal curve of curvature is such that the prime normals of amplitudinal geodesic tangents at successive points of the curve intersect one another. By the other property, an amplitudinal curve of curvature has a direction such that the circular curvature of its amplitudinal geodesic tangent is a maximum or a minimum among the circular curvatures of all the amplitudinal geodesics which can be drawn through the point. We shall consider these implied definitions in turn, and shall prove that they lead to the same curves.

The property, that the prime normals of consecutive geodesic tangents meet one another, can be expressed as follows. Let y denote (as usual) the typical space-coordinate at a point O of such a curve, and let Y denote the corresponding typical direction-cosine of the prime normal of the tangential geodesic at O; then a point, at a distance D along the normal measured from O, has coordinates typified by

$$\bar{y}-y=YD.$$

We take this point to be the intersection of this prime normal with the prime normal of the geodesic tangent at a consecutive point of the curve typified by a coordinate $y+dy$; at that consecutive point, the prime normal of the new geodesic tangent has a typical direction-cosine $Y+dY$, and the distance from the consecutive point along the new prime normal up to the supposed point of intersection can be represented by $D+dD$. Then we have relations

$$\bar{y}-(y+dy)=(Y+dY)(D+dD),$$

as many in number as the number of point-coordinates. As the quantity typified by $\bar{y}$ is the same in both sets of equations, we have the set of conditions

$$-dy=D\,.\,dY+Y\,.\,dD+dY\,.\,dD.$$

Now for any set of direction-cosines, we have $\sum Y^2=1$; and therefore

$$\sum Y\,dY=0.$$

Also the prime normal of an amplitudinal geodesic is at right angles to every direction in the n-fold tangent homaloid of the region: so that, as $dy=y'ds$, where ds denotes the elementary arc of the curve, we have

$$\sum Y\,dy=(\sum Yy')\,ds=0.$$

* As an example, we may cite regions in a plenary quadruple space: see *G.F.D.*, vol. ii, chap. xvi.

Hence multiplying the foregoing typical condition by Y, and adding for all the space-dimensions, we have

$$dD=0.$$

Thus

$$-dy=D\,.\,dY,$$

that is,

$$-y'=DY'=D\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right),$$

there being one such relation for each space-coordinate.

We first multiply this relation by y', and then add all the products for the space-dimensions; and we find

$$D=\rho,$$

or the two normals intersect at the centre of curvature of the first, a geometrical result to be expected.

We then have

$$l_3\frac{\rho}{\sigma}=0,$$

or, as not all the quantities l_3 can vanish because $\sum l_3^2=1$, and as ρ does not vanish, we have

$$\frac{1}{\sigma}=0:$$

that is, the torsion of the geodesic tangent vanishes.

The property that the circular curvature of the geodesic tangent touching a curve of curvature through O is a maximum or a minimum among the circular curvatures of geodesics drawn in all possible directions through O, can be expressed as follows. We have seen that the quantities denoted by v_i are such (p. 70) that

$$v_i=\sum_r L_{ri}x_r',\quad 2v_i=\frac{\partial}{\partial x_i'}\left(\frac{1}{\rho}\right),$$

for all the values of i, the secondary magnitudes L_{ri} themselves involving the direction-variables $x_1', \ldots, x_n'$, of the tangent to the curve. Now by § 30, the circular curvature of the geodesic in this direction can be expressed as a function of these direction-variables—the non-rational character of its form for a general amplitude having no bearing on the present issue; and these variables x' are subject to the relation

$$\sum_i\sum_j A_{ij}x_i'x_j'=1.$$

The symbol u_i has been defined (p. 71) by the equation

$$u_i=\sum_r A_{ri}x_r'.$$

The critical conditions that, among the values of $1/\rho$ arising from all the directions subject to the sole condition $\sum_i\sum_j A_{ij}x_i'x_j'=1$, there shall be a maximum or a minimum for the curve of curvature, are

$$\frac{\partial}{\partial x_i'}\left(\frac{1}{\rho}\right)=\lambda u_i,$$

where λ is a multiplier left undetermined by the critical conditions: or the conditions are

$$2v_i = \lambda u_i,$$

for $i=1, \ldots, n$. Let these equations be multiplied by x_i' for the successive values of i, and the results be added: then, because

$$\sum_i v_i x_i' = \frac{1}{\rho}, \quad \sum u_i x_i' = 1,$$

we have

$$\frac{2}{\rho} = \lambda,$$

or the critical conditions are

$$\frac{u_i}{\rho} - v_i = 0,$$

for $i=1, \ldots, n$.

But in connection with the binormal, we have proved (§ 35) that

$$\frac{1}{\sigma} \sum l_3 \frac{\partial y}{\partial x_i} = \frac{u_i}{\rho} - v_i,$$

for all the specified values of i; and therefore, in the present instance, we have

$$\frac{1}{\sigma} \sum l_3 \frac{\partial y}{\partial x_i} = 0.$$

It has been shewn that the binormal lies in the n-fold tangent homaloid of the amplitude, so that it is not at right angles to every direction in that homaloid: the quantities $\sum l_3 \frac{\partial y}{\partial x_i}$ therefore cannot vanish simultaneously; and therefore we must have

$$\frac{1}{\sigma} = 0,$$

or the torsion of the geodesic must vanish.

Thus the two definitions lead to this same property of a vanishing torsion for the geodesic tangent to a curve of circular curvature: and the analytical equations, which determine the direction of such a curve at a point, are

$$\frac{u_i}{\rho} - v_i = 0, \quad (i=1, \ldots, n).$$

One other fact must be noted. In the foregoing investigation, it was proved that

$$dD = 0, \quad D = \rho;$$

and therefore we have

$$\rho' = 0.$$

But this property is not continuous along any geodesic: it is attached solely to the point of the geodesic where the geodesic touches a line of curvature of the region. In character, it is distinct from the property that the radius of circular

curvature of a geodesic tangent to a line of curvature is either a maximum or a minimum among the radii of circular curvature of all amplitudinal geodesics which can be drawn through the point.

Thus, in connection with the curves of circular curvature at a point O in an amplitude, two questions emerge. One of these requires the equation determining the corresponding radii, which may be called the principal radii, of circular curvature. The other of them requires the equations determining the direction-variables of the curves of circular curvature: and these can be taken in the form

$$\frac{v_1}{u_1}=\frac{v_2}{u_2}=\ldots=\frac{v_n}{u_n},$$

or in the equivalent form

$$\frac{1}{u_1}\frac{\partial}{\partial x_1'}\left(\frac{1}{\rho^2}\right)=\frac{1}{u_2}\frac{\partial}{\partial x_2'}\left(\frac{1}{\rho^2}\right)=\ldots=\frac{1}{u_n}\frac{\partial}{\partial x_n'}\left(\frac{1}{\rho^2}\right),$$

where we substitute for $1/\rho^2$ the quartic homogeneous function of $x_1', \ldots, x_n'$, obtained in § 30.

These equations lead to a simple determination of the curves of circular curvature, when the amplitude is primary, that is, when the plenary homaloidal space is of $n+1$ dimensions. They will be discussed later for the primary amplitudes. For all other n-fold amplitudes, their simplicity is more formal than actual. Thus, in the quantity v_i, which is equal to $\sum_r L_{ri}x_r'$, the quantities L_{ri} themselves implicitly involve the direction-variables x'; and so the equation

$$\left|\frac{1}{\rho}A_{ij}-L_{ij}\right|=0,$$

which undoubtedly is satisfied, is not essentially free from the direction-variables: it does not determine the maximum and minimum values of ρ.

The n equations, which determine the curves of circular curvature at any point of an amplitude, can be stated in a different analytical form. As we have

$$v_i=\tfrac{1}{2}\frac{\partial}{\partial x_i'}\left(\frac{1}{\rho}\right),\quad u_i=\tfrac{1}{2}\frac{\partial}{\partial x_i'}\left(\sum_i\sum_j A_{ij}x_i'x_j'\right),$$

while

$$\frac{1}{\rho^2}=\sum_i\sum_j\sum_k\sum_l Z_{ijkl}x_i'x_j'x_k'x_l',\quad \sum_i\sum_j A_{ij}x_i'x_j'=1,$$

the equations express the property that the discriminant of the quaternary form

$$\sum_i\sum_j\sum_k\sum_l Z_{ijkl}x_i'x_j'x_k'x_l'-\frac{1}{\rho^2}\left(\sum_i\sum_j A_{ij}x_i'x_j'\right)^2$$

shall vanish. But, in the absence of any body of algebraical results appertaining to quaternary forms of n variables, such a statement is mainly descriptive: it will, however, prove useful in the discussion of surfaces existing freely in multiple space.

Locus of centres of circular curvature of concurrent geodesics.

55. In connection with the curves of circular curvature, which provide the maximum and the minimum radii of circular curvature among all the amplitudinal geodesics concurrent in O, a brief consideration may be given here to the locus (if any) of the centres of circular curvature. Fuller details are considered later when the amplitude is of specific (and not general) dimensions, whether the dimensionality of the plenary homaloidal space is, or is not, specific. Here, we shall consider only one or two instances, in which the non-gremial range (§ 51) is restricted.

When the amplitude is primary, its plenary space is of $n+1$ dimensions. For such an amplitude, some properties of which are considered hereafter (Chapter VII), no discussion of the locus of the centre of circular curvature of amplitudinal geodesics is necessary; for the locus merely consists of portions of the prime normal which is common to all the geodesics.

Accordingly, we assume that the number of dimensions of the plenary space is $n+m$, where $m>1$. The direction-cosines and the magnitude of the radius of circular curvature are given by the set of equations typified by

$$\frac{Y}{\rho}=\sum_i \sum_j \eta_{ij} x_i' x_j'.$$

Let the typical space-coordinate of the centre of circular curvature be denoted by y_0, so that we have

$$y_0-y=Y\rho,$$

and therefore

$$\frac{y_0-y}{\rho^2}=\frac{Y}{\rho}=\sum_i \sum_j \eta_{ij} x_i' x_j',$$

typical of $n+m$ equations.

Now for all indices $i, j, k, =1, \ldots, n$, we have

$$\sum \eta_{ij}\frac{\partial y}{\partial x_k}=0,$$

the summation extending throughout the plenary space; and therefore, for each of the n values of k, we have

$$\sum\left\{(y_0-y)\frac{\partial y}{\partial x_k}\right\}=0,$$

so that the locus lies in the m-fold homaloid which is orthogonal to the amplitude. But it is not to be inferred that the locus in question has some m-fold range, its equations being the eliminant resulting from the removal of the direction-variables from the earlier equations; for it can happen that, especially with a specific non-gremial range for the plenary space, even the number of equations can be restricted.

Thus consider the specific instance when the plenary space is of $n+2$ dimensions, so that the foregoing homaloid orthogonal to the amplitude is a plane. A plane is determinate, as to orientation, by any two guiding lines; and the direction-cosines of every line in the plane are expressible linearly in terms of those of the two lines. Every line with the equations

$$\frac{\bar{y}_1-y_1}{(\eta_1)_{ij}}=\frac{\bar{y}_2-y_2}{(\eta_2)_{ij}}=\dots,$$

for all the values of i and j, lies in the plane because of the relation

$$\sum(\eta_\mu)_{ij}\frac{\partial y_\mu}{\partial x_k}=0;$$

and therefore every set of quantities typified by η_{ij} can be expressed in terms of two sets only. If the two sets, typified by η_{11} and η_{12}, be chosen as the sets of reference, there is the aggregate of relations

$$\| \ \eta_{ij}, \ \ \eta_{11}, \ \ \eta_{12} \ \|=0,$$

for all combinations i, j, $=1, \dots, n$, taken independently of one another. Let

$$\mathbf{a}=\sum\eta_{11}{}^2, \quad \mathbf{h}=\sum\eta_{11}\eta_{12}, \quad \mathbf{b}=\sum\eta_{12}{}^2, \quad (\mathbf{ab})^{\frac{1}{2}}\cos\omega=\mathbf{h},$$

so that $\mathbf{a}^{-\frac{1}{2}}\eta_{11}$, $\mathbf{b}^{-\frac{1}{2}}\eta_{12}$, are typical direction-cosines of two guiding lines in the orthogonal plane, while ω denotes the inclination of those lines.

We require two coordinate axes in the orthogonal plane. At the cost of symmetry but with no loss of generality, we choose these two guiding lines as such axes. Denoting by z and t the coordinates of a centre of circular curvature, referred to them, we have

$$z+t\cos\omega=\mathbf{a}^{-\frac{1}{2}}\sum\{\eta_{11}(y_0-y)\}, \quad z\cos\omega+t=\mathbf{b}^{-\frac{1}{2}}\sum\{\eta_{12}(y_0-y)\};$$

and there is the relation

$$z^2+2zt\cos\omega+t^2=\rho^2.$$

Because of the aggregate of relations $\|\,\eta_{ij},\,\eta_{11},\,\eta_{12}\,\|=0$, there are equations

$$\eta_{ij}=\alpha_{ij}\eta_{11}+\beta_{ij}\eta_{12},$$

where the quantities α_{ij}, β_{ij}, being determined by the relations

$$\mathbf{a}\alpha_{ij}+\mathbf{h}\beta_{ij}=\sum\eta_{11}\eta_{ij}, \quad \mathbf{h}\alpha_{ij}+\mathbf{b}\beta_{ij}=\sum\eta_{12}\eta_{ij},$$

are independent of the direction-variables $x_1', \dots, x_n'$. If then we take quantities P and Q according to the definitions

$$P=x_1'^2+\alpha_{22}x_2'^2+2\alpha_{13}x_1'x_3'+\dots+2\alpha_{ij}x_i'x_j'+\dots+\alpha_{nn}x_n'^2,$$
$$Q=2x_1'x_2'+\beta_{22}x_2'^2+2\beta_{13}x_1'x_3'+\dots+2\beta_{ij}x_i'x_j'+\dots+\beta_{nn}x_n'^2,$$

the space-coordinates of the centre are given by

$$\frac{y_0-y}{\rho^2}=\eta_{11}P+\eta_{12}Q,$$

and the plane-coordinates of the centre are given by

$$\frac{\mathbf{a}^{\frac{1}{2}}}{\rho^2}(z+t\cos\omega)=\mathbf{a}P+\mathbf{h}Q,$$

$$\frac{\mathbf{b}^{\frac{1}{2}}}{\rho^2}(z\cos\omega+t)=\mathbf{h}P+\mathbf{b}Q.$$

Also

$$\frac{1}{\rho^2}=\sum\left(\frac{Y}{\rho}\right)^2=\sum(\eta_{11}P+\eta_{12}Q)^2=\mathbf{a}P^2+2\mathbf{h}PQ+\mathbf{b}Q^2\,;$$

and it is easy to verify the relation

$$\rho^2=z^2+2zt\cos\omega+t^2.$$

Thus there are two equations, which involve the coordinates of a point on the centre-locus in the plane and which involve also the n direction-variables $x_1', \dots, x_n'$, the last being subject to the single permanent relation

$$\sum_i\sum_j A_{ij}x_i'x_j'=1.$$

Consequently there are only three equations from which to eliminate n quantities. No eliminant free from the n variables, that is, no equation valid for all directions, is possible, if n is greater than 2. If, however, limitations were imposed on the directions of geodesics, implying a geometrical selection of geodesics and thereby imposing algebraical relations among the direction-variables, these imposed relations would provide additional equations which might render an elimination possible for the system.

It is needless to dwell on the instance when $n=2$; such a value implies that the configuration (in the case of the plenary space considered) is a surface existing freely in quadruple space. The centre-locus is known * to be a lemniscate curve lying in the orthogonal plane of the surface.

As the alternative (it is not difficult to see that, for the n-fold configuration in a plenary space of $n+2$ dimensions, there is only one alternative), consider the aggregate of geodesics the originating directions of which lie in an arbitrarily selected superficial orientation within the n-fold configuration. To specify such an orientation, take two arbitrary directions indicated by $p_1', p_2', \dots, p_n'$; $q_1', q_2', \dots, q_n'$; then the direction-variables of any geodesic originating in this superficial orientation are such that the relations

$$\left\|\begin{matrix} x_1', & x_2', & \dots, & x_n' \\ p_1', & p_2', & \dots, & p_n' \\ q_1', & q_2', & \dots, & q_n' \end{matrix}\right\|=0$$

are satisfied. Hence

$$x_k'=\lambda p_k'+\mu q_k', \quad (k=1, \dots, n),$$

* *G.F.D.*, vol. i, § 242.

where the range of variation is provided by the parameters λ and μ, the quantities p' and q' being constants for the range; and, as there are equations

$$\sum_i \sum_j A_{ij} x_i' x_j' = 1,$$

$$\sum_i \sum_j A_{ij} p_i' p_j' = 1, \quad \sum_i \sum_j A_{ij} (p_i' q_j' + p_j' q_i') = \cos \epsilon, \quad \sum_i \sum_j A_{ij} q_i' q_j' = 1,$$

where ϵ denotes the inclination of the directions p' and q', we have

$$\lambda^2 + 2\lambda\mu \cos \epsilon + \mu^2 = 1,$$

as a permanent condition to be satisfied by λ, μ.

Thus

$$\frac{Y}{\rho} = \sum_i \sum_j \eta_{ij} x_i' x_j'$$

$$= \lambda^2 \left(\sum_i \sum_j \eta_{ij} p_i' p_j' \right) + \lambda\mu \left\{ \sum_i \sum_j \eta_{ij} (p_i' q_j' + p_j' q_i') \right\} + \mu^2 \left\{ \sum_i \sum_j \eta_{ij} q_i' q_j' \right\};$$

and thus the equations determining the coordinates z and t in the orthogonal plane become

$$\bar{z} = \mathrm{a}^{\frac{1}{2}} \frac{z + t \cos \omega}{z^2 + 2zt \cos \omega + t^2} = \lambda^2 A_1 + 2\lambda\mu H_1 + \mu^2 B_1,$$

$$\bar{t} = \mathrm{b}^{\frac{1}{2}} \frac{z \cos \omega + t}{z^2 + 2zt \cos \omega + t^2} = \lambda^2 A_2 + 2\lambda\mu H_2 + \mu^2 B_2,$$

where

$$A_1 = \sum_i \sum_j \{ \sum (\eta_{11} \eta_{ij}) p_i' p_j' \}, \quad A_2 = \sum_i \sum_j \{ \sum (\eta_{12} \eta_{ij}) p_i' p_j' \},$$

with like values for H_1, B_1, H_2, B_2, all these quantities being independent of λ and μ. Also there is the permanent condition affecting λ and μ. The eliminant is

$$4 \begin{vmatrix} A_1 - \bar{z}, & H_1 - \bar{z} \cos \epsilon \\ A_2 - \bar{t}, & H_2 - \bar{t} \cos \epsilon \end{vmatrix} \begin{vmatrix} H_1 - \bar{z} \cos \epsilon, & B_1 - \bar{z} \\ H_2 - \bar{t} \cos \epsilon, & B_2 - \bar{t} \end{vmatrix} = \begin{vmatrix} A_1 - \bar{z}, & B_1 - \bar{z} \\ A_2 - \bar{t}, & B_2 - \bar{t} \end{vmatrix}^2,$$

a lemniscate curve lying in the orthogonal plane.

Hence the locus of the centre of circular curvature of geodesics, originating in a superficial orientation in any n-fold amplitude existing freely in a plenary space of $n+2$ dimensions, is a lemniscate curve lying in the plane orthogonal to the amplitude.

56. Next, consider the case when the n-fold configuration lies in a plenary homaloidal space of $n+3$ dimensions. The orthogonal homaloid then is of three dimensions, that is, it is a flat; and therefore any centre-locus of geodesics in the configuration must lie in the flat.

Now a flat contains not more than three reciprocally independent directions. In the present instance, as always, all the directions typified by η_{ij} lie in the orthogonal homaloid and consequently are directions in a flat; hence the direction-cosines of any one such direction are expressible by means of three selected direc-

tions. Accordingly, choosing the directions typified by η_{11}, η_{12}, η_{22}, as leading lines for the flat, we have relations

$$\eta_{ij} = a_{ij}\eta_{11} + \beta_{ij}\eta_{12} + \gamma_{ij}\eta_{22},$$

typical of the direction-cosines of any direction; and the magnitudes a_{ij}, β_{ij}, γ_{ij}, are non-directional magnitudes of position defined by the equations

$$\begin{aligned} \textstyle\sum \eta_{11}\eta_{ij} &= \mathbf{a}a_{ij} + \mathbf{h}\beta_{ij} + \mathbf{g}\gamma_{ij}, \\ \textstyle\sum \eta_{12}\eta_{ij} &= \mathbf{h}a_{ij} + \mathbf{b}\beta_{ij} + \mathbf{f}\gamma_{ij}, \\ \textstyle\sum \eta_{22}\eta_{ij} &= \mathbf{g}a_{ij} + \mathbf{f}\beta_{ij} + \mathbf{c}\gamma_{ij}, \end{aligned}$$

where, as usual,

$$\mathbf{a} = \textstyle\sum \eta_{11}^2, \qquad \mathbf{b} = \sum \eta_{12}^2, \qquad \mathbf{c} = \sum \eta_{22}^2,$$
$$\mathbf{f} = \textstyle\sum \eta_{12}\eta_{22}, \qquad \mathbf{g} = \sum \eta_{11}\eta_{22}, \qquad \mathbf{h} = \sum \eta_{11}\eta_{12}.$$

Also we write

$$\mathbf{f} = (\mathbf{bc})^{\frac{1}{2}} \cos a, \quad \mathbf{g} = (\mathbf{ca})^{\frac{1}{2}} \cos \beta, \quad \mathbf{h} = (\mathbf{ab})^{\frac{1}{2}} \cos \gamma.$$

Let the three selected lines be taken as coordinate axes of reference in the flat; and let z, t, u, denote the coordinates of the centre of circular curvature of the amplitudinal geodesic in the direction x_1', ..., x_n', so that

$$\begin{aligned} \mathbf{a}^{\frac{1}{2}}(z + t\cos\gamma + u\cos\beta) &= \textstyle\sum \{\eta_{11}(y_0 - y)\}, \\ \mathbf{b}^{\frac{1}{2}}(z\cos\gamma + t + u\cos a) &= \textstyle\sum \{\eta_{12}(y_0 - y)\}, \\ \mathbf{c}^{\frac{1}{2}}(z\cos\beta + t\cos a + u) &= \textstyle\sum \{\eta_{22}(y_0 - y)\}, \end{aligned}$$

while

$$\rho^2 = z^2 + t^2 + u^2 + 2tu\cos a + 2uz\cos\beta + 2zt\cos\gamma.$$

Using the relations which express the quantities η_{ij} in terms of η_{11}, η_{12}, η_{22}, we have

$$\frac{Y}{\rho} = \eta_{11}P + \eta_{12}Q + \eta_{22}R,$$

where P, Q, R, are homogeneous quadratic functions of x_1', ..., x_n', while

$$\frac{1}{\rho^2} = (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, \mathbf{h})(P, Q, R)^2.$$

(It is easy to verify that the two values of ρ^2 are in accord.) Thus we have

$$\bar{z} = \frac{\mathbf{a}^{\frac{1}{2}}}{\rho^2}(z + t\cos\gamma + u\cos\beta) = \mathbf{a}P + \mathbf{h}Q + \mathbf{g}R,$$

$$\bar{t} = \frac{\mathbf{b}^{\frac{1}{2}}}{\rho^2}(z\cos\gamma + t + u\cos a) = \mathbf{h}P + \mathbf{b}Q + \mathbf{f}R,$$

$$\bar{u} = \frac{\mathbf{c}^{\frac{1}{2}}}{\rho^2}(z\cos\beta + t\cos a + u) = \mathbf{g}P + \mathbf{f}Q + \mathbf{c}R,$$

the n quantities x_1', ..., x_n', being subject to the permanent relation

$$\sum_i \sum_j A_{ij}x_i'x_j' = 1.$$

Consequently we have four equations from which, at this stage, it is necessary to eliminate the n magnitudes x'. Such elimination is possible only if, either $n=3$, or there are other conditions limiting the range of independence of the direction-variables x'. When $n=3$, the amplitude becomes a region; the subject will be discussed in the investigations concerning regions (Chap. XXI). In the alternative possibility, there must be conditions which shall leave not more than three independent magnitudes to be eliminated; and therefore the n direction-variables x' must be rendered equivalent either to two parameters or to three parameters.

When the direction of the geodesic originates in a superficial orientation, made definite by the assignment of two directions, with sets of direction-variables $p_1', \ldots, p_n'$, and $q_1', \ldots, q_n'$, there are two parameters κ, λ, and the direction-variables of any geodesic, originating in the superficial orientation, are given by

$$x_i' = \kappa p_i' + \lambda q_i',$$

for $i=1, \ldots, n$. The permanent relation becomes of the form

$$\kappa^2 + 2\kappa\lambda \cos\epsilon + \lambda^2 = 1.$$

When the direction of the geodesic originates in a volumetric orientation, made definite by the assignment of three non-complanar directions with sets of direction-variables $p_1, \ldots, p_n'$; $q_1, \ldots, q_n'$; $r_1', \ldots, r_n'$, there are three parameters κ, λ, μ, and the direction-variables of any geodesic, originating in the volumetric orientation, are given by

$$x_i' = \kappa p_i' + \lambda q_i' + \mu r_i',$$

for $i=1, \ldots, n$. The permanent relation becomes of the form

$$\kappa^2 + \lambda^2 + \mu^2 + 2\lambda\mu\cos\epsilon_1 + 2\mu\kappa\cos\epsilon_2 + 2\kappa\lambda\cos\epsilon_3 = 1,$$

where ϵ_1, ϵ_2, ϵ_3, denote the inclinations of pairs of the three leading lines assigned for the orientation.

We consider the alternatives in succession.

I. When

$$x_i' = \kappa p_i' + \lambda q_i',$$

we have

$$\begin{aligned} \frac{Y}{\rho} &= \sum_i \sum_j \eta_{ij} x_i' x_j' \\ &= \kappa^2 \left(\sum_i \sum_j \eta_{ij} p_i' p_j' \right) + \kappa\lambda \left\{ \sum_i \sum_j \eta_{ij} (p_i' q_j' + p_j' q_i') \right\} + \lambda^2 \left(\sum_i \sum_j \eta_{ij} q_i' q_j' \right), \end{aligned}$$

and therefore

$$\begin{aligned} \bar{z} &= \kappa^2 A_1 + 2\kappa\lambda H_1 + \lambda^2 B_1, \\ \bar{t} &= \kappa^2 A_{12} + 2\kappa\lambda H_{12} + \lambda^2 B_{12}, \\ \bar{u} &= \kappa^2 A_2 + 2\kappa\lambda H_2 + \lambda^2 B_2, \end{aligned}$$

with an obvious significance for the symbols A, H, B; and there is the permanent relation

$$1 = \kappa^2 + 2\kappa\lambda\cos\epsilon + \lambda^2.$$

Thus there are four equations involving the two eliminable quantities κ and λ; the eliminant, therefore, consists of two equations. One of these obviously is

$$\begin{vmatrix} \bar{z}, & A_1, & H_1, & B_1 \\ \bar{t}, & A_{12}, & H_{12}, & B_{12} \\ \bar{u}, & A_2, & H_2, & B_2 \\ 1, & 1, & \cos\epsilon, & 1 \end{vmatrix} = 0;$$

when account is taken of the values of $\bar{z}$, $\bar{t}$, $\bar{u}$, the equation represents a sphere lying in the flat of reference—that is, the flat orthogonal to the amplitude. Again, if Δ denote the determinant

$$\begin{vmatrix} A_1, & H_1, & B_1 \\ A_{12}, & H_{12}, & B_{12} \\ A_2, & H_2, & B_2 \end{vmatrix},$$

and if c_μ represent the minor of C_μ in Δ (for $C=A$, H, B, and for $\mu=1$, 12, 2), we have

$$\kappa^2\Delta = a_1\bar{z} + a_{12}\bar{t} + a_2\bar{u},$$
$$2\kappa\lambda\,\Delta = h_1\bar{z} + h_{12}\bar{t} + h_2\bar{u},$$
$$\lambda^2\Delta = b_1\bar{z} + b_{12}\bar{t} + b_2\bar{u};$$

and therefore

$$4(a_1\bar{z} + a_{12}\bar{t} + a_2\bar{u})(b_1\bar{z} + b_{12}\bar{t} + b_2\bar{u}) = (h_1\bar{z} + h_{12}\bar{t} + h_2\bar{u})^2,$$

another eliminant equation. When the values of $\bar{z}$, $\bar{t}$, $\bar{u}$, are inserted, the denominators ρ^2 disappear; and the equation represents a quadric cone in the flat with its vertex at the origin.

Hence the locus of the centres of circular curvature of concurrent geodesics, originating in a superficial orientation, is the skew-quartic intersection of a sphere and a cone in a flat, the vertex of the cone being on the surface of the sphere.

II. When

$$x_i' = \kappa p_i' + \lambda q_i' + \mu r_i',$$

we have

$$\begin{aligned} \frac{Y}{\rho} &= \sum_i \sum_j \eta_{ij} x_i' x_j' \\ &= \kappa^2 \sum_i \sum_j \eta_{ij} p_i' p_j' + \lambda\mu \sum_i \sum_j \eta_{ij}(q_i' r_j' + q_j' r_i') \\ &\quad + \lambda^2 \sum_i \sum_j \eta_{ij} q_i' q_j' + \mu\kappa \sum_i \sum_j \eta_{ij}(r_i' p_j' + r_j' p_i') \\ &\quad + \mu^2 \sum_i \sum_j \eta_{ij} r_i' r_j' + \mu\lambda \sum_i \sum_j \eta_{ij}(p_i' q_j' + p_j' q_i'); \end{aligned}$$

and therefore the equations for the flat-coordinates of the centre of curvature of an amplitudinal geodesic become

$$\bar{z} = (A_1, B_1, C_1, F_1, G_1, H_1 \between \kappa, \lambda, \mu)^2,$$
$$\bar{t} = (A_{12}, B_{12}, C_{12}, F_{12}, G_{12}, H_{12} \between \kappa, \lambda, \mu)^2,$$
$$\bar{u} = (A_2, B_2, C_2, F_2, G_2, H_2 \between \kappa, \lambda, \mu)^2,$$

with an obvious significance for the symbols A, B, C, F, G, H; and there is also the permanent relation

$$1=(1,\ 1,\ 1,\ \cos\epsilon_1,\ \cos\epsilon_2,\ \cos\epsilon_3)(\kappa,\ \lambda,\ \mu)^2.$$

Thus there are four equations involving the three eliminable magnitudes κ, λ, μ; the eliminant consists of a single equation in the coordinates z, t, u, and therefore the locus is a surface in the flat orthogonal to the amplitude. Later (§ 257) the same elimination has to be performed for the case $n=3$, that is, for a region in sextuple space, all the algebraic forms being the same; and therefore, anticipating the later algebraic result, we infer that the surface is of degree eight and has, at the initial point O, a (real or imaginary) conical point of order four.

Curves of spherical curvature.

57. Analogous to amplitudinal curves of circular curvature, which have been proved to be associated with centres of circular curvature of amplitudinal geodesic tangents of the curves, there exist curves of spherical curvature. These occur in the following manner.

For brevity, we may call the line, which passes through the centre of circular curvature of any curve and is drawn parallel to the binormal of the curve, the binormal axis; and we know, from the geometry of the curve, that the centre of spherical curvature lies along this binormal axis, at a distance $\sigma\rho'$ from the centre of circular curvature. In general, consecutive normals of a skew curve do not meet: also the prime normals of consecutive geodesic tangents to a skew curve in an amplitude do not meet, the exception to this general property being provided by the curves * of circular curvature. Similarly, the binormal axes belonging to consecutive points of a curve do not meet, in general: so also, in general, the binormal axes of consecutive geodesic tangents to a skew curve in an amplitude do not meet. The succeeding investigation shews that, here also, there are exceptions to this general property.

Accordingly, we consider the amplitudinal geodesic tangent at any point of a curve: along its prime normal, towards its centre of curvature, we measure a distance a; and through the point thus obtained, we draw a line parallel to the binormal and measure a distance β along this line to a point S. Then the typical coordinate of the point S thus obtained is given by

$$\bar{y}-y=aY+\beta l_3.$$

The laws, determining the organic magnitudes a and β, are at our disposal: it is a question whether they can be assigned so that, when an amplitudinal geodesic tangent is drawn at a consecutive point of the curve and when a corresponding

* Such curves are not always skew: thus we have families of surfaces in homaloidal triple space which have plane curves of curvature.

construction (with the duly modified values of α and β) is made in connection with this consecutive geodesic tangent, the same point can be attained: in other words, are there curves such that the binormal axes of consecutive geodesic tangents intersect? The consecutive point on the curve we denote by $y+dy$, that is, $y+y'ds$; the consecutive values of α and β will be denoted by $\alpha+\alpha'ds$, and $\beta+\beta'ds$, respectively, the values of α' and β' being unformulated beforehand: the consecutive value of Y is $Y+Y'ds$, that is,

$$Y+\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)ds\,;$$

and the consecutive value of l_3 is $l_3+l_3'ds$, that is,

$$l_3+\left(\frac{l_4}{\tau}-\frac{Y}{\sigma}\right)ds.$$

If then the two binormal axes meet, and if the foregoing value $\bar{y}$ is the typical coordinate of their intersection, we have

$$\bar{y}-(y+y'ds)=(\alpha+\alpha'ds)\left\{Y+\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)ds\right\}+(\beta+\beta'ds)\left\{l_3+\left(\frac{l_4}{\tau}-\frac{Y}{\sigma}\right)ds\right\},$$

where $\bar{y}$ is the same as before: hence, as ds tends to zero, in the limit we have, from the two equations together, the relation

$$-y'=\alpha'Y+\alpha\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)+\beta'l_3+\beta\left(\frac{l_4}{\tau}-\frac{Y}{\sigma}\right),$$

typical for each space-coordinate y. Hence, owing to the conditions attaching to the direction-cosines of the principal lines in the orthogonal frame of the geodesic tangent, we have the set of relations

$$-1=-\frac{\alpha}{\rho},\quad \alpha'-\frac{\beta}{\sigma}=0,\quad \frac{\alpha}{\sigma}+\beta'=0,\quad \frac{\beta}{\tau}=0.$$

From the first two of these, we have

$$\alpha=\rho,\quad \beta=\sigma\alpha'=\sigma\rho':$$

that is, the point of intersection (or the aforesaid point S) is the centre of spherical curvature of the geodesic tangent at O.

The third equation now gives

$$\sigma\rho''+\sigma'\rho'+\frac{\rho}{\sigma}=0.$$

But, always,

$$R^2=\rho^2+\sigma^2\rho'^2,$$

so that

$$RR'=\rho'(\rho+\sigma^2\rho''+\sigma\sigma'\rho')\,;$$

and therefore the value of R' belonging to an amplitudinal geodesic, at a point on

a curve of spherical curvature where it touches the curve, is zero. This property is analogous to the property that the value of ρ' belonging to an amplitudinal geodesic at a point on a curve of circular curvature where it touches the curve, is zero.

Finally, the fourth equation gives $\beta/\tau=0$: or, as β does not vanish, we must have

$$\frac{1}{\tau}=0.$$

Hence the tilt of an amplitudinal geodesic, touching a curve of spherical curvature, vanishes at the point of contact with the curve.

Moreover, this last property leads to equations satisfied by the direction-variables of a curve of spherical curvature. In § 39, it was proved that the direction-cosines of the trinormal of any geodesic in the amplitude are such that

$$\frac{1}{\tau}\sum l_4\frac{\partial y}{\partial x_r}=\left\{\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}u_r-\sigma' v_r-\sigma w_r,$$

for all the values $r=1, \ldots, n$. Now the direction of this trinormal lies within the n-fold tangent homaloid of the amplitude, so that the quantities $\sum l_4\frac{\partial y}{\partial x_r}$ do not all vanish, and at non-singularities of the amplitude no one of them can be infinite; hence the vanishing of the tilt of a geodesic at a point, as characteristic of the direction of that particular geodesic, requires the equations

$$\left\{\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}u_r-\sigma' v_r-\sigma w_r=0,$$

for $r=1, \ldots, n$. These, in effect, are only $n-1$ independent equations; for when we multiply the equation of rank r in the succession by x_r', add the products, and use the relations

$$\sum_r u_r x_r'=1,\quad \sum_r v_r x_r'=\frac{1}{\rho},\quad \sum_r w_r x_r'=\frac{d}{ds}\left(\frac{1}{\rho}\right),$$

we obtain a mere identity.

The equations for the direction-variables of the curves of spherical curvature passing through a point O of the amplitude can be expressed in a somewhat simpler form. If we denote by V the form $\sum_i\sum_j L_{ij}x_i'x_j'$ which is the value of $\frac{1}{\rho}$, and by W the form $\sum_i\sum_j\sum_k e_{ijk}x_i'x_j'x_k'$ which is the value of $\frac{d}{ds}\left(\frac{1}{\rho}\right)$, the foregoing equations become

$$\sigma(Wu_i-w_i)+\sigma'(Vu_i-v_i)=0,$$

or, even more simply,

$$\frac{w_1-Wu_1}{v_1-Vu_1}=\frac{w_2-Wu_2}{v_2-Vu_2}=\ldots=\frac{w_n-Wu_n}{v_n-Vu_n}=-\frac{\sigma'}{\sigma}.$$

The equalities of the first n fractions constitute only $n-2$ equations, because

$$\sum(w_i - Wu_i)x_i' = 0, \quad \sum(v_i - Vu_i)x_i' = 0;$$

for a full complement of $n-1$ equations to determine the ratios of $x_1', \ldots, x_n'$, the explicit value of $\frac{\sigma'}{\sigma}$ must be retained, in any of the forms already established (§ 35).

Curves of globular curvature.

58. Likewise there are amplitudinal curves of curvature of orbicular rank higher than spherical in the succession of orbicular curvatures (§ 9). After the preceding instances, it will suffice to add the initial results affecting curves of globular curvature, which prove to be associated with the centre of globular curvature in a manner analogous to the connection of curves of circular curvature with the centre of circular curvature of an amplitudinal geodesic, and of curves of spherical curvature with its centre of spherical curvature.

The trinormal axis of a curve may be defined (similarly to the binormal axis of § 57) as a line through the centre of spherical curvature of a curve parallel to the trinormal of the curve: and we know, from the geometry of a curve, that the centre of globular curvature lies along the trinormal axis, at a distance *

$$\tau \frac{RR'}{\sigma\rho'}$$

from the centre of spherical curvature, R denoting the radius of spherical curvature. Just as prime normals of consecutive geodesic tangents to a curve usually do not meet (the exception occurring when the curve is a curve of circular curvature of the amplitude), and as their binormal axes usually do not meet (the exception occurring when the curve is a curve of spherical curvature of the amplitude), so also their trinormal axes usually do not meet (it being assumed that the plenary homaloidal space is of more than four dimensions). But here also there are curves in the amplitude such that the trinormal axes of their successive amplitudinal geodesic tangents do intersect.

Accordingly, along the prime normal of the amplitudinal geodesic tangent of a curve we measure a distance α; from the point so obtained, we measure a distance β along a direction parallel to the binormal of the geodesic; and from this point so obtained, we measure a distance γ along a direction parallel to the trinormal of the geodesic. The typical coordinate of this final point is given by the equation

$$\bar{y} - y = \alpha Y + \beta l_3 + \gamma l_4.$$

For the present purpose, we have to determine α, β, γ, as to their law of existence, so that, if possible, the same final point can be attained when a similar construc-

* *G.F.D.*, vol. i, p. 246.

tion (with the duly modified values of α, β, γ) is made in connection with the amplitudinal geodesic at a consecutive point of the specified curve. The possibility will be realised if the quantities α, β, γ, are such that

$$-\frac{dy}{ds}-\frac{d\alpha}{ds}Y-\frac{d\beta}{ds}l_3-\frac{d\gamma}{ds}l_4$$
$$=\alpha\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)+\beta\left(\frac{l_4}{\tau}-\frac{Y}{\sigma}\right)+\gamma\left(\frac{l_5}{\kappa}-\frac{l_3}{\tau}\right),$$

a relation to hold for each of the set of quantities associated with y, the typical space-coordinate of the point on the curve. Necessary and sufficient conditions are provided by the equations

$$1=\frac{\alpha}{\rho},\quad \frac{d\alpha}{ds}=\frac{\beta}{\sigma},\quad \frac{d\beta}{ds}=-\frac{\alpha}{\sigma}+\frac{\gamma}{\tau},\quad -\frac{d\gamma}{ds}=\frac{\beta}{\tau},$$

and

$$\frac{\gamma}{\kappa}=0.$$

From the first of these relations, we have

$$\alpha=\rho\,;$$

and from the second, we then have

$$\beta=\sigma\rho'.$$

The third equation now gives

$$\frac{\gamma}{\tau}=\frac{\rho}{\sigma}+\sigma\rho''+\sigma'\rho'\,:$$

or, because $R^2=\rho^2+\sigma^2\rho'^2$, so that

$$RR'=(\rho+\sigma^2\rho''+\sigma'\rho')\rho',$$

we have

$$\gamma=\tau\frac{RR'}{\sigma\rho'}.$$

It therefore appears, from these values for α, β, γ, respectively, that the point attained in the construction is the centre of globular curvature of the amplitudinal geodesic tangent of the curve.

The fourth equation now becomes

$$\frac{d}{ds}\left(\tau\frac{RR'}{\sigma\rho'}\right)+\frac{\sigma\rho'}{\tau}=0.$$

But the radius of globular curvature, denoted by Γ, is given * by

$$\Gamma^2=R^2+\left(\tau\frac{RR'}{\sigma\rho'}\right)^2$$

* *G.F.D.*, vol. i, p. 246.

always, so that

$$\Gamma\Gamma' = RR' + \tau\frac{RR'}{\sigma\rho'}\frac{d}{ds}\left(\tau\frac{RR'}{\sigma\rho'}\right),$$

always ; hence, in the present case,

$$\Gamma' = 0\,;$$

that is, at a point of contact between an amplitudinal geodesic and the curve (which will be called a curve of globular curvature of the amplitude), the globular curvature of the geodesic itself is either a maximum or a minimum along the geodesic.

Finally, as the quantity γ is not zero, the fifth equation gives the relation

$$\frac{1}{\kappa} = 0\,;$$

that is, at a point on a curve of globular curvature, the coil of the amplitudinal geodesic tangent vanishes.

The last property makes it possible to frame equations for the ratios of the direction-variables $x_1', \ldots, x_n'$, these being rendered actually determinate by the permanent equation $\sum_i \sum_j A_{ij}x_i'x_j' = 1$. In § 45, it was proved that the parameters γ, in the expression of the direction-cosines of the quartinormal in terms of the parameters of the tangent homaloid, are given by equations which give the quantities

$$\frac{1}{\kappa}\sum_\mu A_{\mu r}\gamma_\mu, \quad (r = 1, \ldots, n),$$

in terms of known magnitudes. These parameters γ are finite ; and therefore, at any direction on an amplitudinal geodesic where the coil vanishes, each such expression must vanish. Hence we have

$$Pu_r - \left\{\frac{\sigma}{\tau} + \frac{d}{ds}(\sigma'\tau)\right\}v_r - Sw_r - \sigma\tau t_r = 0,$$

for the values $r = 1, \ldots, n$, where

$$P = \frac{\sigma}{\rho\tau} + \frac{d}{ds}\left\{\tau\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}, \quad S = \frac{1}{\sigma}\frac{d}{ds}(\sigma^2\tau).$$

Now

$$\tau\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) = \sigma'\tau V + \sigma\tau W,$$

so that

$$\frac{d}{ds}\left\{\tau\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\} = \left\{\frac{d}{ds}(\sigma'\tau)\right\}V + \left\{\frac{1}{\sigma}\frac{d}{ds}(\sigma^2\tau)\right\}W + \sigma\tau T.$$

Consequently, the foregoing equation can be taken in the form

$$\left\{\frac{\sigma}{\tau} + \frac{d}{ds}(\sigma'\tau)\right\}(Vu_r - v_r) + \left\{\frac{1}{\sigma}\frac{d}{ds}(\sigma^2\tau)\right\}(Wu_r - w_r) + \sigma\tau(Tu_r - t_r) = 0\,;$$

and this relation is valid for $r=1, \ldots, n$. But in the aggregate of these relations there are only $n-1$ members, linearly independent of one another, because the equations

$$\sum_r x_r'(Vu_r - v_r)=0, \quad \sum_r x_r'(Wu_r - w_r)=0, \quad \sum_r x_r'(Tu_r - t_r)=0,$$

are satisfied identically.

Manifestly, the relations

$$\begin{vmatrix} u_i, & u_j, & u_k, & u_l \\ v_i, & v_j, & v_k, & v_l \\ w_i, & w_j, & w_k, & w_l \\ t_i, & t_j, & t_k, & t_l \end{vmatrix} = 0$$

are satisfied for all the combinations $i, j, k, l, =1, \ldots, n$, for a direction on an amplitudinal geodesic at which the coil vanishes: the result is in accordance with the expression (p. 118) obtained for the magnitude $\Omega^4/(\sigma^6\tau^4\kappa^2)$. But this set of relations contains only $n-4$ linearly independent members.

Similarly, the relations

$$\begin{vmatrix} Vu_i - v_i, & Wu_i - w_i, & Tu_i - t_i \\ Vu_j - v_j, & Wu_j - w_j, & Tu_j - t_j \\ Vu_k - v_k, & Wu_k - w_k, & Tu_k - t_k \end{vmatrix} = 0,$$

or, what is the equivalent,

$$V\begin{vmatrix} u_i, & w_i, & t_i \\ u_j, & w_j, & t_j \\ u_k, & w_k, & t_k \end{vmatrix} - W\begin{vmatrix} u_i, & v_i, & t_i \\ u_j, & v_j, & t_j \\ u_k, & v_k, & t_k \end{vmatrix} + T\begin{vmatrix} u_i, & v_i, & w_i \\ u_j, & v_j, & w_j \\ u_k, & v_k, & w_k \end{vmatrix} - \begin{vmatrix} v_i, & w_i, & t_i \\ v_j, & w_j, & t_j \\ v_k, & w_k, & t_k \end{vmatrix} = 0,$$

are satisfied for all the combinations $i, j, k, =1, \ldots, n$, for such a direction on an amplitudinal geodesic; but this set of relations contains only $n-3$ linearly independent members.

In order to have the complement of equations, adequate to determine the ratios of the direction-variables x', we retain the first set of $n-1$ equations.

CHAPTER V

Riemann's Measure of Curvature of an Amplitude

Small geodesic triangles.

59. The preceding investigations have been concerned with the values of various magnitudes at any arbitrary point in the amplitude. Occasionally, the immediate vicinity of such a point has been considered, almost entirely for the purpose of proceeding to limiting results which ensue when the vicinity is reduced so as to be negligible. In the present chapter, we shall discuss a matter that is concerned with a non-evanescent range in the amplitude. Lengths measured along geodesics will be taken into account; but, owing to the (implicitly assumed) complete generality of the amplitude, we shall proceed by approximations in powers of small quantities. The aim of the investigation is the construction of the Riemann measure of curvature of an amplitude in a superficial or plane orientation and the derivation of a geometrical significance of the measure. As will appear from brief historical notes hereafter * inserted, Riemann himself left no recoverable interpretation of the measure; and subsequent investigators have usually had recourse to geodesic surfaces, in order to initiate the calculations. But it will be seen that the investigation can be completed without the assistance of such surfaces: and, indeed, their consideration is rather a sequel, than an introduction, to the construction of the measure as obtained in what follows.

Let O be any point in the amplitude, OPX and OQY two amplitudinal geodesics drawn in any assigned directions at O, the march of OPX and of OQY being definite because (§ 19) of the assignment of directions at O. Let a small length OP, denoted by x, be taken along the arc of the geodesic OX: and a small length OQ, denoted by y, be taken along the arc of the geodesic OY. Let the points P and Q be joined by an amplitudinal geodesic PQ; as the arc PQ is small, there can be no question of the Jacobian conjugates † of P and of Q along the geodesic, so that PQ is possible and unique. Thus there is a triangle OPQ, with geodesic sides; the internal angles OPQ and OQP will be denoted by P and Q respectively.

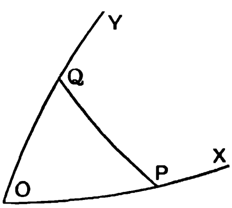

Fig. 1.

The direction-variables at O of the geodesic OPX will be denoted by

* See, in particular, §§ 74, 108.

† In the customary sense associated with the Jacobi test for the maximum or the minimum of an integral; see *e.g.* my *Calculus of Variations*, § 173.

$p_1', p_2', \dots, p_n'$, and those at O of the geodesic OQY will be denoted by $q_1', q_2', \dots, q_n'$; differentiations along OPX and along OQY will be denoted by ds_1 and by ds_2 respectively. Thus, at P, the parametric variable corresponding to p_r is equal to

$$p_r + xp_r' + \tfrac{1}{2}x^2p_r'' + \tfrac{1}{6}x^3p_r''' + \dots;$$

it will be found sufficient, when dealing with parameters, to neglect powers and products of the small quantities x and y that are of net order higher than three. Similarly, at P, for the geodesic OPX in the direction PX, when the direction-variable corresponding to p_r' is denoted by P_r', for $r=1, \dots, n$, we have

$$P_r' = p_r' + xp_r'' + \tfrac{1}{2}x^2p_r''' + \dots;$$

it will be found sufficient, when dealing with direction-variables, to neglect small quantities of net order higher than two.

But at P, we shall require the direction-variables of the geodesic PQ in the direction PQ; these will be denoted by $z_1', z_2', \dots, z_n'$. As they will be obtained by approximations, we shall take

$$z_r' = t_r' + T_r + \Theta_r + \dots,$$

where t_r' is finite, T_r is of the first order of small quantities, Θ_r is of the second order, and higher orders are neglected. Also, at P, we shall require magnitudes such as z_r'' connected with the geodesic PQ; and therefore

$$z_r'' = -\sum_i \sum_j \{ij, r\}_P z_i'z_j',$$

where the symbol $\{ij, r\}_P$ denotes the value at P (and not at O) of the implied magnitude. Again, at P, we shall require the values of the primary magnitudes which, at O, are denoted by A_{ij}; for, there, we shall have the permanent relation

$$\sum_i \sum_j (A_{ij})_P z_i'z_j' = 1,$$

and, there, we shall have the equation

$$-\cos P = \cos XPQ = \sum_i \sum_j (A_{ij})_P z_i'P_j'.$$

In each instance we use approximations. Now

$$(A_{ij})_P = A_{ij} + x\frac{dA_{ij}}{ds_1} + \tfrac{1}{2}x^2\frac{d^2A_{ij}}{ds_1^2} + \dots,$$

there being no necessity to retain higher powers of x; and thus we shall require combinations

$$\sum_i \sum_j p_i'q_j'\frac{dA_{ij}}{ds_k}, \quad \sum_i \sum_j p_i'q_j'\frac{d^2A_{ij}}{ds_k^2}.$$

Accordingly, we proceed to construct these combinations. We shall also evaluate

z_r''; the value (as will be seen) is required only up to the first order of small quantities inclusive.

60. As regards the quantity z_r'' at P for the direction PQ along the geodesic PQ, the value of which is required up to the first order of small quantities, we have

$$\{ij, r\}_P = \{ij, r\} + x \sum_k p_k' \frac{\partial}{\partial x_k} \{ij, r\},$$

$$z_i' z_j' = t_i' t_j' + t_i' T_j + t_j' T_i,$$

quantities of higher orders being negligible for this approximation. Also (§ 23)

$$\frac{\partial}{\partial x_k} \{ij, r\} = \{ijk, r\} + \sum_a [\{ki, a\}\{ja, r\} + \{kj, a\}\{ia, r\}]$$
$$- \frac{1}{3\Omega} \sum_\beta a_{r\beta} [(\beta i, jk) + (\beta j, ik)];$$

and therefore, up to the first order inclusive in the approximation,

$$\begin{aligned} z_r'' &= - \sum_i \sum_j \{ij, r\}_P z_i' z_j' \\ &= - \sum_i \sum_j \{ij, r\} t_i' t_j' - \sum_i \sum_j (t_i' T_j + t_j' T_i) \{ij, r\} \\ &\qquad - x \sum_k \sum_i \sum_j t_i' t_j' p_k' \frac{\partial}{\partial x_k} \{ij, r\} \\ &= t_r'' - \sum_i \sum_j (t_i' T_j + t_j' T_i) \{ij, r\} \\ &\qquad - x \sum_i \sum_j \sum_k \{ijk, r\} t_i' t_j' p_k' \\ &\qquad - x \sum_a \sum_i \sum_j \sum_k [\{ki, a\}\{ja, r\} + \{kj, a\}\{ia, r\}] t_i' t_j' p_k' \\ &\qquad - \frac{x}{3\Omega} \sum_\beta \sum_i \sum_j a_{r\beta} [(\beta i, jk) + (\beta j, ik)] t_i' t_j' p_k', \end{aligned}$$

which is the required value, so far as it can be developed at this stage. Later (p. 154), the quantities T_i and T_j will be found; anticipating them, so that the fully explicit value of z_r'' may be known, we take

$$T_l = - x \sum_\lambda \sum_\mu \{\lambda\mu, l\} p_\lambda' t_\mu'.$$

Thus

$$\begin{aligned} - \sum_i \sum_j t_i' T_j \{ij, r\} &= x \sum_i \sum_j \sum_\lambda \sum_\mu \{\lambda\mu, l\} \{ij, r\} t_i' t_\mu' p_\lambda' \\ &= x \sum_a \sum_i \sum_j \sum_k \{kj, a\} \{ia, r\} t_i' t_j' p_k', \end{aligned}$$

and

$$- \sum_i \sum_j t_j' T_i \{ij, r\} = x \sum_a \sum_i \sum_j \sum_k \{ki, a\} \{ja, r\} t_i' t_j' p_k';$$

and therefore we can take, as the value of z_r'',

$$z_r''=t_r''-x\sum_i\sum_j\sum_k\{ijk,r\}t_i't_j'p_k'$$
$$-\frac{x}{3\Omega}\sum_\beta\sum_i\sum_j a_{r\beta}[(\beta i,jk)+(\beta j,ik)]t_i't_j'p_k'.$$

Next, in connection with the desired combinations involving the first arc-derivatives and the second arc-derivatives of the primary magnitudes A_{ij}, we shall require certain combinations connected with the Christoffel symbols $\{hk, l\}$. We define them by the relations

$$g_{\mu i}{}^{(t)}=\sum\{im,\mu\}t_m',$$

$$g_\mu{}^{(ty)}=g_\mu{}^{(yt)}=\sum_i g_{\mu i}{}^{(t)}y_i'=\sum_m g_{\mu m}{}^{(y)}t_m'=\sum_i\sum_m\{im,\mu\}y_m't_i'=\sum_i\sum_m\{im,\mu\}y_i't_m';$$

and we shall also use the extended definition of u_μ in the form

$$u_\mu{}^{(p)}=\sum_m A_{\mu m}p_m',$$

for all the values μ, m, $=1, \ldots, n$. Manifestly

$$g_\mu{}^{(yy)}=\sum_i\sum_m\{im,\mu\}y_i'y_m'=-y_\mu'',$$

for any set of direction-variables * $y_i', \ldots, y_n'$.

Further, the direction-variables $p_1', \ldots, p_n'$, are associated with an arc-differentiation ds_1 along the geodesic OPX, and the direction-variables $q_1', \ldots, q_n'$ with an arc-differentiation ds_2 along the geodesic OQY; we shall associate $y_1', \ldots, y_n'$ with any (unspecified) arc-differentiation ds_k along an amplitudinal geodesic through O. In particular, if $ds_k=ds_1$, then $y'=p'$; and if $ds_k=ds_2$, then $y'=q'$.

We have (§ 12)

$$\frac{\partial A_{ij}}{\partial x_l}=\sum_\mu[A_{i\mu}\{lj,\mu\}+A_{j\mu}\{li,\mu\}],$$

and therefore

$$\frac{dA_{ij}}{ds_k}=\sum_l\frac{\partial A_{ij}}{\partial x_l}\frac{dx_l}{ds_k}$$
$$=\sum_l\frac{\partial A_{ij}}{\partial x_l}y_l'$$
$$=\sum_\mu\{A_{i\mu}g_{\mu j}{}^{(y)}+A_{j\mu}g_{\mu i}{}^{(y)}\}.$$

Consequently

$$\sum_i p_i'\frac{dA_{ij}}{ds_k}=\sum_\mu\{u_\mu{}^{(p)}g_{\mu j}{}^{(y)}+A_{j\mu}g_\mu{}^{(yp)}\},$$

$$\sum_j q_j'\frac{dA_{ij}}{ds_k}=\sum_\mu\{A_{i\mu}g_\mu{}^{(yq)}+u_\mu{}^{(q)}g_{\mu i}{}^{(y)}\},$$

* These symbols y, for temporary use only, are of course distinct from the space coordinates.

and

$$\sum_i \sum_j p_i' q_j' \frac{dA_{ij}}{ds_k} = \sum_\mu \{u_\mu^{(p)} g_\mu^{(vq)} + u_\mu^{(q)} g_\mu^{(vp)}\};$$

and a particular instance of the last of these results is

$$\sum_i \sum_j t_i' t_j' \frac{dA_{ij}}{ds_1} = 2 \sum_\mu u_\mu^{(t)} g_\mu^{(vt)}.$$

Next, we have

$$\frac{\partial^2 A_{ij}}{\partial x_l \partial x_m} = \frac{\partial}{\partial x_m} \sum_\mu [A_{i\mu}\{lj, \mu\} + A_{j\mu}\{li, \mu\}]$$

$$= \sum_\mu \sum_\lambda \{lj, \mu\}[A_{\lambda\mu}\{mi, \lambda\} + A_{\lambda i}\{m\mu, \lambda\}]$$

$$+ \sum_\mu \sum_\lambda \{li, \mu\}[A_{\lambda\mu}\{mj, \lambda\} + A_{\lambda j}\{m\mu, \lambda\}]$$

$$+ \sum_\mu A_{i\mu}\{lmj, \mu\} + \sum_\lambda \sum_\mu A_{i\mu}[\{lm, \lambda\}\{j\lambda, \mu\} + \{mj, \lambda\}\{l\lambda, \mu\}]$$

$$- \frac{1}{3\Omega} \sum_\theta \sum_\mu A_{i\mu} a_{\mu\theta}[(\theta l, jm) + (\theta j, lm)]$$

$$+ \sum_\mu A_{j\mu}\{lmi, \mu\} + \sum_\lambda \sum_\mu A_{j\mu}[\{lm, \lambda\}\{i\lambda, \mu\} + \{mi, \lambda\}\{l\lambda, \mu\}]$$

$$- \frac{1}{3\Omega} \sum_\theta \sum_\mu A_{j\mu} a_{\mu\theta}[(\theta l, im) + (\theta i, lm)].$$

The aggregate of all the terms containing the Riemann four-index symbols

$$= -\tfrac{1}{3}[(il, jm) + (jl, im)].$$

Hence

$$\frac{d^2 A_{ij}}{ds_k^2} = \sum_l \frac{\partial A_{ij}}{\partial x_l} \frac{d^2 x_l}{ds_k^2} + \sum_l \sum_m \frac{\partial^2 A_{ij}}{\partial x_l \partial x_m} \frac{dx_l}{ds_k} \frac{dx_m}{ds_k}$$

$$= -\sum_l \sum_e \sum_f \frac{\partial A_{ij}}{\partial x_l} [\{ef, l\} y_e' y_f'] + \sum_l \sum_m \frac{\partial^2 A_{ij}}{\partial x_l \partial x_m} y_l' y_m'$$

$$= -\tfrac{1}{3}[(il, jm) + (jl, im)] y_l' y_m'$$

$$+ \sum_\mu [A_{i\mu}\{lmj, \mu\} + A_{j\mu}\{lmi, \mu\}] y_l' y_m'$$

$$+ 2 \sum_\lambda \sum_\mu \{A_{\lambda\mu} g_{\lambda i}^{(v)} g_{\mu j}^{(v)}\}$$

$$+ 2 \sum_\lambda \sum_\mu g_{\lambda\mu}^{(v)} \{A_{i\lambda} g_{\mu j}^{(v)} + A_{j\lambda} g_{\mu i}^{(v)}\}.$$

Denoting by $x_1', \dots, x_n'$, and $z_1', \dots, z_n'$, two sets of direction-variables at O which later will be made to be $t_1', \dots, t_n'$, simultaneously for one application, and to be

p_1', ... , p_n', and t_1', ... , t_n', respectively for a second application, we find, when x' and z' denote different variables,

$$\begin{aligned}\sum_i \sum_j x_i' z_j' \frac{d^2 A_{ij}}{ds_k^2} &= -\tfrac{2}{3}\sum_\alpha \sum_\beta \sum_\gamma \sum_\delta (\alpha\beta, \gamma\delta)[(s_{\alpha\beta} t_{\gamma\delta} + s_{\gamma\delta} t_{\alpha\beta})] \\ &\quad + \sum_\mu u_\mu^{(x)} [\sum_i \sum_j \sum_k \{ijk, \mu\} y_i' y_j' z_k'] \\ &\quad + \sum_\mu u_\mu^{(z)} [\sum_i \sum_j \sum_k \{ijk, \mu\} y_i' y_j' x_k'] \\ &\quad + 2\sum_\lambda \sum_\mu \{A_{\lambda\mu} g_\lambda^{(xy)} g_\mu^{(zy)}\} \\ &\quad + 2\sum_\lambda \sum_\mu g_{\lambda\mu}^{(y)} \{u_\lambda^{(x)} g_\mu^{(yz)} + u_\lambda^{(z)} g_\mu^{(yx)}\},\end{aligned}$$

where

$$s_{\alpha\beta} = x_\alpha' y_\beta' - x_\beta' y_\alpha', \quad t_{\gamma\delta} = z_\gamma' y_\delta' - z_\delta' y_\gamma' ;$$

but when x' and z' denote the same variables, the first term in

$$\sum_i \sum_j x_i' x_j' \frac{d^2 A_{ij}}{ds_k^2}$$

is equal to

$$-\tfrac{2}{3}\sum_\alpha \sum_\beta \sum_\gamma \sum_\delta (\alpha\beta, \gamma\delta) s_{\alpha\beta} s_{\gamma\delta},$$

the remaining terms changing solely by making z' the same as x'.

Moreover, if the variables x' should belong to ds_k, then $s_{\alpha\beta}$ vanishes for all values of α and β; and all the terms, involving the four-index symbols, disappear. Those terms also disappear from the former expression if the variables z' should belong to ds_k, because the quantities $t_{\gamma\delta}$ (for all values of γ and δ) then vanish.

Magnitude and direction of the third side of a geodesic triangle.

61. In obtaining the unknown elements of the geodesic triangle OPQ, namely the length of the small arc PQ, the direction-variables of the geodesic PQ in the direction PQ at P leading to the value of the angle OPQ, and the direction-variables of the same geodesic QP in the direction QP at Q leading to the value of the angle OQP, we denote the length of the arc PQ by the quantity

$$\bar{w} = w + W_2 + W,$$

where w is of the first order, W_2 denotes the aggregate of terms of the second order (it will be found that W_2 is zero), and W denotes the aggregate of terms of the third order, terms of higher order being negligible for the present aim.

The position Q can be attained from O in two different diagrammatic paths: (i), by proceeding directly along the geodesic OQ: and (ii), by proceeding along the geodesic OX up to P, and thence along the geodesic PQ up to Q. When the

changes in the parametric variables are traced, the values at Q must be the same, by the one path as by the other. Consider the amplitudinal parameter denoted by x_r at O. At Q, after the geodesic path OQ has been described, its value is

$$x_r + yq_r' + \tfrac{1}{2}y^2 q_r'' + \tfrac{1}{6}y^3 q_r''',$$

up to the third order of small quantities. At the same point Q, after the combined path OP, PQ has been described, its value is

$$x_r + xp_r' + \tfrac{1}{2}x^2 p_r'' + \tfrac{1}{6}x^3 p_r''' + \bar{w} z_r + \tfrac{1}{2}\bar{w}^2 z_r'' + \tfrac{1}{6}\bar{w}^3 z_r''',$$

accurately up to the third order, though it will appear that some portions of the last two terms in the expression are negligible. The equality of these two values gives the relation

$$\begin{aligned}\bar{w} z_r' &+ \tfrac{1}{2}\bar{w}^2 z_r'' + \tfrac{1}{6}\bar{w}^3 z_r''' \\ &= yq_r' - xp_r' + \tfrac{1}{2}(y^2 q_r'' - x^2 p_r'') + \tfrac{1}{6}(y^3 q_r''' - x^3 p_r'''),\end{aligned}$$

accurate (or, so far as concerns parts of the left-hand side, to be made accurate) up to the third order of small quantities. And there is one such relation for each of the n values of r, connected with the aggregate of parameters.

Moreover, for the determination of the direction-variables z_r' at P in the direction PQ along the geodesic PQ, there is the permanent arc-relation for any direction at P, being

$$\sum_i \sum_j (A_{ij})_P z_i' z_j' = 1.$$

We proceed to the specified approximations, taking successive orders in the small quantities; and we begin with the first two orders of approximation in the parametric relations.

In the foregoing typical relation, terms of the first order can arise only out of $\bar{w} z_r'$ which is equal to

$$(w + W_2 + W)(t_r' + T_r + \Theta_r);$$

and therefore, in order that terms of the first order may balance, we must have

$$wt_r' = yq_r' - xp_r'.$$

As regards terms of the second order, as $\bar{w}^2$ is equal to w^2 up to the second order, we need take only the finite part of z_r'' in the term $\tfrac{1}{2}\bar{w}^2 z_r''$; thus the terms of the second order contributed by this term

$$\begin{aligned}&= \tfrac{1}{2}w^2 t_r'' \\ &= -\tfrac{1}{2}\sum_i \sum_j \{ij, r\}(yq_i' - xp_i')(yq_j' - xp_j') \\ &= \tfrac{1}{2}y^2 q_r'' + \tfrac{1}{2}x^2 p_r'' + xy \sum_i \sum_j [\{ij, r\}(p_i' q_j' + p_j' q_i')].\end{aligned}$$

As $\bar{w}^3$ is of the third order at least, no second-order terms are contributed by $\tfrac{1}{6}\bar{w}^3 z_r'''$. The terms of the second order contributed from $\bar{w} z_r'$

$$= W_2 t_r' + wT_r.$$

Hence, in order that terms of the second order in the relation may balance, there is a residuary condition

$$W_2 t_r' + wT_r = -x^2 p_r'' - xy \sum_i \sum_j [\{ij, r\}(p_i' q_j' + p_j' q_i')].$$

On the right-hand side, the coefficient of $-x\{ij, r\}$

$$\begin{aligned} &= y(p_i' q_j' + p_j' q_i') - 2xp_i' p_j' \\ &= w(p_i t_j' + p_j' t_i'); \end{aligned}$$

and therefore the residuary condition is

$$W_2 t_r' + wT_r = -xw \sum_i \sum_j [\{ij, r\}(p_i' t_j' + p_j t_i')],$$

the summation on the right-hand side being for i and j as a completed combination. But that right-hand side, with the notation of p. 148, can be taken in the form

$$-xw \sum_l g_{rl}^{(p)} t_l';$$

and thus a form, final at this stage, for the condition is

$$W_2 t_r' + w\left[T_r + x\sum_l g_{rl}^{(p)} t_l'\right] = 0,$$

that is,

$$W_2 t_r' + w[T_r + xg_r^{(pt)}] = 0,$$

holding for $r = 1, \ldots, n$.

Before proceeding to the third-order terms, it is desirable to consider the residuary conditions (if any) arising from the first two approximations in connection with the permanent relation

$$\sum_i \sum_j (A_{ij})_P z_i' z_j' = 1$$

at the point P. We have

$$(A_{ij})_P = A_{ij} + x\frac{dA_{ij}}{ds_1} + \tfrac{1}{2}x^2 \frac{d^2 A_{ij}}{ds_1^2} + \ldots,$$

and

$$z_i' z_j' = (t_i' + T_i + \Theta_i)(t_j' + T_j + \Theta_j),$$

all the portions being accurate up to the stated terms. Thus the residuary condition from the finite terms is

$$\sum_i \sum_j A_{ij} t_i' t_j' = 1,$$

that is, the quantities $t_1', \ldots, t_n'$, are the direction-variables of a direction through O. (It could be regarded as " parallel " to the direction PQ at P.)

The terms, of the first order of small quantities in the permanent relation, must balance ; and this requirement leaves the residuary condition

$$x \sum_i \sum_j t_i' t_j' \frac{dA_{ij}}{ds_1} + \sum_i \sum_j \{A_{ij}(t_i' T_j + t_j' T_i)\} = 0$$

When the values of the coefficients of $t_i't_j'$ are inserted in the first summation, and when the second summation is re-arranged, this condition becomes

$$2x\Big[\sum_\mu u_\mu{}^{(t)} g_\mu{}^{(pt)}\Big] + 2\Big[\sum_i u_i{}^{(t)} T_i\Big] = 0,$$

that is,

$$\sum_\mu [u_\mu{}^{(t)}\{T_\mu + xg_\mu{}^{(pt)}\}] = 0.$$

Some simplifying inferences can be made at once from these residuary conditions.

The relations

$$wt_r' = yq_r' - xp_r', \quad \sum_i \sum_j A_{ij} t_i' t_j' = 1,$$

the former holding for $r=1, \ldots, n$, shew that a plane triangle OP_0Q_0 can be constructed, such that the lengths of the linear sides OP_0, OQ_0, P_0Q_0, are x, y, w, and that the angles are $P_0OQ_0=\epsilon$, $OP_0Q_0=P_0$, $OQ_0P_0=Q_0$, there being relations

$$w \sin P_0 = y \sin \epsilon, \quad w \cos P_0 = x - y \cos \epsilon,$$
$$w \sin Q_0 = x \sin \epsilon, \quad w \sin P_0 = y - x \cos \epsilon,$$
$$x^2 - 2xy \cos \epsilon + y^2 = w^2,$$
$$\epsilon + P_0 + Q_0 = \pi.$$

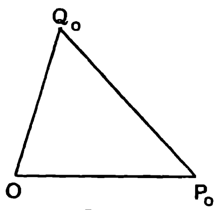

FIG. 2.

Moreover, P_0 is the finite part in the measure of the geodesic angle OPQ, and Q_0 is the finite part in the measure of the geodesic angle OQP, so that $P-P_0$, $Q-Q_0$, are small quantities. They will be found to be of the second order.

In the next place, let the second-order residuary relation

$$W_2 t_r' + w[T_r + xg_r{}^{(pt)}] = 0$$

be multiplied by $u_r{}^{(t)}$, and let the products for all the values of r be summed. Then, as

$$\sum \{u_r{}^{(t)} t_r'\} = 1,$$

and as there is a residuary condition

$$\sum_\mu [u_\mu{}^{(t)}\{T_\mu + xg_\mu{}^{(pt)}\}] = 0$$

from the arc-relation, we have

$$W_2 = 0,$$

that is, the second-order part of the length of the geodesic arc $\bar{w}$ vanishes, and we have

$$\bar{w} = w + W,$$

where W is of the third order of small quantities.

Further, because $W_2=0$, we have

$$T_r + xg_r{}^{(pt)} = 0,$$

for all the values of r, where

$$g_r^{(pt)} = \sum_i \sum_j \{ij, r\} p_i' t_j' ;$$

and thus the direction-variable z_r' at P for the geodesic arc PQ in the direction PQ is given by

$$z_r' = t_r' - x g_r^{(pt)} + \Theta_r, \quad r = 1, \dots, n,$$

where Θ_r is the second-order portion of z_r'.

Accordingly, we now have to determine the third-order magnitude W and the second-order quantities $\Theta_1, \dots, \Theta_n$. For this purpose, we have to consider the third-order terms in the n relations (p. 151) which arise out of the values of the parameters, and the second-order terms which arise out of the permanent arc-relation (p. 152) at P.

62. For the third-order terms in the n parametric relations, we note that, as $\bar{w} = w + W$, where W is of the third order, we can take

$$\bar{w}^2 = w^2, \quad \bar{w}^3 = w^3,$$

accurately up to the third order inclusive. Hence, in the term $\frac{1}{6}\bar{w}^3 z_r'''$, we shall require only the finite parts (not involving small quantities) in z''', so that its contribution is

$$\tfrac{1}{6} w^3 t_r'''.$$

Also, in the term $\frac{1}{2}\bar{w}^2 z_r''$, we shall require only the first-order terms in z_r''; and therefore its contribution

$$= -\tfrac{1}{2} w^2 x \sum_i \sum_j \sum_k \{ijk, r\} t_i' t_j' p_k'$$
$$- \frac{w^2 x}{6\Omega} \sum_\beta \sum_i \sum_j \sum_k a_{r\beta} [(\beta i, jk) + (\beta j, ik)] t_i' t_j' p_k'.$$

The total contribution of the third order from the term $\bar{w} z_r'$

$$= w\Theta_r + W t_r'.$$

Thus the residuary condition from the third-order terms in the typical parametric relation is

$$w\Theta_r + W t_r' + \tfrac{1}{6} w^3 t_r''' + \tfrac{1}{6} x^3 p_r''' - \tfrac{1}{6} y^3 q_r''' - \tfrac{1}{2} w^2 x \sum_i \sum_j \sum_k \{ijk, r\} t_i' t_j' p_k'$$
$$= \frac{w^2 x}{6\Omega} \sum_\beta \sum_i \sum_j a_{r\beta} [(\beta i, jk) + (\beta j, ik)] t_i' t_j' p_k'.$$

Now

$$\begin{aligned} w^3 t_r''' &= - \sum \{ijk, r\} w^3 t_i' t_j' t_k' \\ &= - \sum \{ijk, r\} (y q_i' - x p_i')(y q_j' - x p_j')(y q_k' - x p_k') \\ &= y^3 q_r''' - x^3 p_r''' + 3xy \sum \{ijk, r\} p_k' q_j' (y q_i' - x p_i') \\ &= y^3 q_r''' - x^3 p_r''' + 3xyw \sum_i \sum_j \sum_k \{ijk, r\} p_k' q_j' t_i'. \end{aligned}$$

Also we have

$$xyw\Big[\sum_i\sum_j\sum_k\{ijk,r\}p_k'q_j't_i'\Big]-xw^2\Big[\sum_i\sum_j\sum_k\{ijk,r\}t_i't_j'p_k'\Big]$$
$$=xw\sum_i\sum_j\sum_k\{ijk,r\}t_i'p_k'(yq_j'-wt_j')$$
$$=x^2w\sum_i\sum_j\sum_k\{ijk,r\}t_i'p_j'p_k'$$
$$=x^2w\sum_i\sum_j\sum_k\{ijk,r\}p_i'p_j't_k'\,;$$

and therefore the residuary condition becomes

$$w\Big[\Theta_r+\tfrac{1}{2}x^2\sum_i\sum_j\sum_k\{ijk,r\}p_i'p_j't_k'\Big]+Wt_r'$$
$$=\frac{w^2x}{6\Omega}\sum_\beta\sum_i\sum_j\sum_k a_{r\beta}[(\beta i,jk)+(\beta j,ik)]t_i't_j'p_k',$$

holding for $r=1,\ \dots,\ n$.

For the second-order terms which arise out of the permanent arc-relation at P, we have, up to this order,

$$z_1'^2=t_1'^2-2xt_1'g_1^{(pt)}+x^2\{g_1^{(pt)}\}^2+2t_1'\Theta_1,$$

and so for other combinations of the direction-variables z' at P for the geodesic PQ; also, to the same order,

$$(A_{ij})_P=A_{ij}+x\frac{dA_{ij}}{ds_1}+\tfrac{1}{2}x^2\frac{d^2A_{ij}}{ds_1^2},$$

for all values of i and j. Hence the residuary condition, arising from these second-order terms, can be taken in the form

$$2\sum_i\sum_j A_{ij}t_i'\Theta_j+x^2\sum_i\sum_j[A_{ij}g_i^{(pt)}g_j^{(pt)}]$$
$$-2x^2\sum_i\sum_j\frac{dA_{ij}}{ds_1}t_i'g_j^{(pt)}+\tfrac{1}{2}x^2\sum_i\sum_j t_i't_j'\frac{d^2A_{ij}}{ds_1^2}=0.$$

The first term is equal to

$$2\sum u_r^{(t)}\Theta_r.$$

As regards the third term, we have, from the result in § 60,

$$\sum_j t_j'\frac{dA_{ij}}{ds_1}=\sum_\mu\{u_\mu^{(t)}g_{\mu i}^{(p)}+A_{i\mu}g_\mu^{(pt)}\},$$

because the direction-variables p' are connected with the arc ds_1; and therefore the whole third term

$$=-2x^2\sum_i\sum_\mu g_i^{(pt)}\{u_\mu^{(t)}g_{\mu i}^{(p)}+A_{i\mu}g_\mu^{(pt)}\}$$
$$=-2x^2\sum_i\sum_\mu[A_{i\mu}g_i^{(pt)}g_\mu^{(pt)}]-2x^2\sum_i\sum_\mu u_\mu^{(t)}g_{\mu i}^{(p)}g_i^{(pt)}.$$

As regards the fourth term, we use the result on p. 150, making $x'=t'$, $z'=t'$, and

$y'=p'$, and note that, because x' and z' are made the same, the second result must be used; thus

$$\sum_i \sum_j t_i' t_j' \frac{d^2 A_{ij}}{ds_1^2} = -\tfrac{2}{3} \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta (\alpha\beta, \gamma\delta) s_{\alpha\beta} s_{\gamma\delta}$$
$$+2 \sum_\mu u_\mu^{(t)} \Big[\sum_i \sum_j \sum_k \{ijk, \mu\} p_i' p_j' t_k' \Big]$$
$$+2 \sum_\lambda \sum_\mu A_{\lambda\mu} g_\lambda^{(pt)} g_\mu^{(pt)}$$
$$+4 \sum_\lambda \sum_\mu u_\lambda^{(t)} g_{\lambda\mu}^{(p)} g_\mu^{(pt)}.$$

When these values are inserted, and when cancelling terms are removed, the residuary condition from the second-order terms in the arc-relation at P can be expressed in the form

$$\sum_r u_r^{(t)} \Big[2\Theta_r + x^2 \sum_i \sum_j \sum_k \{ijk, r\} p_i' p_j' t_k' \Big] = \tfrac{1}{3} x^2 \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta (\alpha\beta, \gamma\delta) s_{\alpha\beta} s_{\gamma\delta},$$

where, for all values of λ, μ,

$$s_{\lambda\mu} = t_\lambda' p_\mu' - t_\mu' p_\lambda'.$$

This single residuary condition must now be combined with the n residuary conditions from the parametric relations. Let the latter be multiplied by $u_1^{(t)}$, $u_2^{(t)}$, ..., $u_n^{(t)}$, and let the results be added; then, on using the single residuary condition and also the property

$$\sum_r u_r^{(t)} t_r' = 1,$$

we have

$$W + \tfrac{1}{6} w x^2 \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta (\alpha\beta, \gamma\delta)\, s_{\alpha\beta} s_{\gamma\delta}$$
$$= \frac{w^2 x}{6\Omega} \sum_r \sum_\beta \sum_i \sum_j \sum_k u_r^{(t)} a_{r\beta} [(\beta i, jk) + (\beta j, ik)]\, t_i' t_j' p_k'.$$

But

$$\sum_r u_r^{(t)} a_{r\beta} = \Omega t_\beta',$$

so that the right-hand side

$$= \tfrac{1}{6} w^2 x \sum_\beta \sum_i \sum_j \sum_k [(\beta i, jk) + (\beta j, ik)]\, t_\beta' t_i' t_j' p_k,$$

and therefore, owing to the relations

$$(\beta i, jk) = -(i\beta, jk), \quad (\beta j, ik) = -(j\beta, ik),$$

the right-hand side vanishes. Consequently

$$W = -\tfrac{1}{6} w x^2 \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta (\alpha\beta, \gamma\delta)\, s_{\alpha\beta} s_{\gamma\delta}.$$

Now we have

$$w s_{\lambda\mu} = \begin{vmatrix} w t_\lambda', & p_\lambda' \\ w t_\mu', & p_\mu' \end{vmatrix} = -y(p_\lambda' q_\mu' - p_\mu' q_\lambda'),$$

for all values of λ, μ; and therefore

$$W = -\tfrac{1}{6}\frac{x^2y^2}{w}\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta(\alpha\beta,\gamma\delta)(p_\alpha'q_\beta' - p_\beta'q_\alpha')(p_\gamma'q_\delta' - p_\delta'q_\gamma').$$

Also, as the angle XOY between the directions $p_1', \ldots, p_n'$, and $q_1', \ldots, q_n'$, is denoted by ϵ, we have

$$\begin{aligned}\sin^2\epsilon &= \Big(\sum_i\sum_j A_{ij}p_i'p_j'\Big)\Big(\sum_i\sum_j A_{ij}q_i'q_j'\Big) - \Big(\sum_i\sum_j A_{ij}p_i'q_j'\Big)^2 \\ &= \sum_\alpha\sum_\beta\sum_\gamma\sum_\delta(A_{\alpha\gamma}A_{\beta\delta} - A_{\alpha\delta}A_{\beta\gamma})(p_\alpha'q_\beta' - p_\beta'q_\alpha')(p_\gamma'q_\delta' - p_\delta'q_\gamma').\end{aligned}$$

Riemann postulated *, as his measure of curvature of the n-fold amplitude in the superficial orientation at O, defined by the two directions OPX and OQY at O, the quantity

$$\frac{\displaystyle\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta(\alpha\beta,\gamma\delta)(p_\alpha'q_\beta' - p_\beta'q_\alpha')(p_\gamma'q_\delta' - p_\delta'q_\gamma')}{\displaystyle\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta(A_{\alpha\gamma}A_{\beta\delta} - A_{\alpha\delta}A_{\beta\gamma})(p_\alpha'q_\beta' - p_\beta'q_\alpha')(p_\gamma'q_\delta' - p_\delta'q_\gamma')}.$$

Denoting this Riemann measure by K, we have

$$\begin{aligned}W &= -\tfrac{1}{6}\frac{Kx^2y^2}{w}\sin^2\epsilon \\ &= -\tfrac{1}{6}K\frac{x^2y^2\sin^2\epsilon}{(x^2 - 2xy\cos\epsilon + y^2)^{\frac{1}{2}}};\end{aligned}$$

and the length of the small geodesic PQ, accurately up to the third order of small quantities inclusive, is

$$(x^2 - 2xy\cos\epsilon + y^2)^{\frac{1}{2}} - \tfrac{1}{6}K\frac{x^2y^2\sin^2\epsilon}{(x^2 - 2xy\cos\epsilon + y^2)^{\frac{1}{2}}}.$$

Also, the value of Θ_r now is given by

$$\begin{aligned}&\Theta_r + \tfrac{1}{2}x^2\Big[\sum_i\sum_j\sum_k\{ijk, r\}\,p_i'p_j't_k'\Big] \\ &= \tfrac{1}{6}K\frac{x^2y^2\sin^2\epsilon}{x^2 - 2xy\cos\epsilon + y^2}t_r' + \tfrac{1}{6}\frac{wx}{\Omega}\sum_\beta\sum_i\sum_j\sum_k a_{r\beta}[(\beta i, jk) + (\beta j, ik)]\,t_i't_j'p_k';\end{aligned}$$

and the value of the direction-variable z_r' of the geodesic PQ at P in the direction PQ is

$$z_r' = t_r' - x\Big[\sum_i\sum_j\{ij, r\}\,p_i't_j'\Big] + \Theta_r.$$

63. The direction-variables, which may be denoted by $\zeta_1', \ldots, \zeta_n'$, of the geodesic PQ at the point Q in the direction QP, can be derived from the values of $z_1', \ldots, z_n'$, by an interchange of the points P and Q, with a simultaneous interchange of the sets $p_1', \ldots, p_n'$, and $q_1', \ldots, q_n'$, and of the magnitudes x and y; and,

* References will be found later; see p. 192.

as the direction QP is opposite in sense to the direction PQ along the geodesic, the signs of $t_1', \dots, t_r'$, must be changed. The values of w and W are unaffected; and we have

$$\zeta_r' = -t_r' + y \sum_i \sum_j \{ij, r\} q_i' t_j' + \Psi_r,$$

where

$$\Psi_r - \tfrac{1}{2} y^2 \Big[\sum_i \sum_j \sum_k \{ijk, r\} q_i' q_j' t_k' \Big]$$
$$= -\tfrac{1}{6} K \frac{x^2 y^2 \sin^2 \epsilon}{x^2 - 2xy \cos \epsilon + y^2} t_r' + \tfrac{1}{6} \frac{wy}{\Omega} \sum_\beta \sum_i \sum_j \sum_k a_{r\beta} [(\beta i, jk) + (\beta j, ik)] t_i' t_j' q_k'.$$

Angles of the geodesic triangle.

64. To complete the elements of the geodesic triangle OPQ, we need the internal angles P and Q, the known angle O being denoted by ϵ, connected with the orientation at O. The typical direction-variable at P for the geodesic OPX in the direction PX

$$= p_i' + x p_i'' + \tfrac{1}{2} x^2 p_i''',$$

and a typical direction-variable at P for the geodesic PQ in the direction PQ

$$= t_j' - x g_j^{(pt)} + \Theta_j,$$

the value of Θ_j being known up to the second order of small quantities. Hence

$$-\cos P = \cos XPQ$$
$$= \sum_i \sum_j [(A_{ij})_P (p_i' + x p_i'' + \tfrac{1}{2} x^2 p_i''') \{t_j' - x g_j^{(pt)} + \Theta_j\}],$$

and

$$(A_{ij})_P = A_{ij} + x \frac{dA_{ij}}{ds_1} + \tfrac{1}{2} x^2 \frac{d^2 A_{ij}}{ds_1^2}$$

up to the second order of small quantities inclusive; and our approximation is to be taken to this order on the right-hand side of the expression for $-\cos P$.

The finite terms on the right-hand side

$$= \sum_i \sum_j A_{ij} p_i' t_j' = -\cos P_0.$$

Thus P and P_0 differ by small quantities, whatever be their order: retaining only the most significant part, we have

$$-\cos P = -\cos P_0 + (P - P_0) \sin P_0.$$

The aggregate of terms of the first order in the expression for $-\cos P$

$$= x \sum_i \sum_j \Big[\frac{dA_{ij}}{ds_1} p_i' t_j' \Big] + x \Big[\sum_i \sum_j A_{ij} t_j' p_i'' \Big] - x \Big[\sum_i \sum_j A_{ij} p_i' g_j^{(pt)} \Big].$$

Here, the term with the first summation

$$= x \sum_\mu [u_\mu^{(p)} g_\mu^{(pt)} + u_\mu^{(t)} g_\mu^{(pp)}];$$

as $p_i'' = -g_i^{(pp)}$, the second term

$$= -x \sum_i u_i^{(t)} g_i^{(pp)};$$

and the third term

$$= -x \sum_j u_j^{(p)} g_j^{(pt)}.$$

Accordingly, the aggregate of terms of the first order in the expression on the right-hand side for $-\cos P$ vanishes. We therefore have

$$(P-P_0)\sin P_0 = \text{aggregate of terms of the second order,}$$

this order being the limit of approximation sought.

The aggregate of these second-order terms

$$= \sum_i \sum_j (A_{ij} p_i' \Theta_j) + \tfrac{1}{2}x^2 \sum_i \sum_j \left(p_i' t_j' \frac{d^2 A_{ij}}{ds_1^2}\right) + \tfrac{1}{2}x^2 \sum_i \sum_j (A_{ij} t_j' p_i''')$$
$$+ x^2 \sum_i \sum_j \left(\frac{dA_{ij}}{ds_1} p_i'' t_j'\right) - x^2 \sum_i \sum_j \left\{\frac{dA_{ij}}{ds_1} p_i' g_j^{(pt)}\right\} - x^2 \sum_i \sum_j \{A_{ij} p_i'' g_j^{(pt)}\}.$$

In this aggregate, the first term

$$= \sum_j u_j^{(p)} \Theta_j.$$

For the second term, the expression on p. 150 is used for the cited instance when the variables x' (here p') belong to the element ds_k (here ds_1); and the second term

$$= \tfrac{1}{2}x^2 \sum_\mu u_\mu^{(p)} \left[\sum_i \sum_j \sum_k \{ijk, \mu\} p_i' p_j' t_k'\right]$$
$$+ \tfrac{1}{2}x^2 \sum_\mu u_\mu^{(t)} \left[\sum_i \sum_j \sum_k \{ijk, \mu\} p_i' p_j' p_k'\right]$$
$$+ x^2 \sum_i \sum_j \{A_{ij} g_i^{(pp)} g_j^{(pt)}\}$$
$$+ x^2 \sum_\lambda \sum_\mu g_{\lambda\mu}^{(p)} \{u_\lambda^{(p)} g_\mu^{(pt)} + u_\lambda^{(t)} g_\mu^{(pp)}\}.$$

The third term

$$= -\tfrac{1}{2}x^2 \sum_\mu \left[u_\mu^{(t)} \sum_i \sum_j \sum_k \{ijk, \mu\} p_i' p_j' p_k'\right].$$

The fourth term

$$= -x^2 \sum_i \sum_\lambda g_i^{(pp)} \{u_\lambda^{(t)} g_{\lambda i}^{(p)} + A_{i\lambda} g_\lambda^{(pt)}\}.$$

The fifth term

$$= -x^2 \sum_j \sum_\lambda g_j^{(pt)} \{\sum u_\lambda^{(p)} g_{\lambda j}^{(p)} + A_{j\lambda} g_\lambda^{(pp)}\}.$$

The sixth term

$$= x^2 \sum_i \sum_j \{A_{ij} g_i^{(pp)} g_j^{(pt)}\}.$$

Let the cancelling terms in this aggregate be removed; then the equation for $P - P_0$ becomes

$$(P-P_0)\sin P_0 = \sum_\mu u_\mu^{(p)} [\Theta_\mu + \tfrac{1}{2}x^2 \sum_i \sum_j \sum_k \{ijk, \mu\} p_i' p_j' t_k'].$$

The values of the quantities, of which $u_\mu{}^{(p)}$ are the coefficients in the summation, have been obtained in § 62. The sum of the parts involving K

$$=\tfrac{1}{6}K\frac{x^2y^2\sin^2\epsilon}{w^2}\left[\sum_\mu u_\mu{}^{(p)}t_\mu{}'\right];$$

but

$$\sum u_\mu{}^{(p)}t_\mu{}'=\sum_\lambda\sum_\mu A_{\lambda\mu}p_\lambda{}'t_\mu{}'=-\cos P_0,$$

while

$$x\sin\epsilon=w\sin Q_0,\quad y\sin\epsilon=w\sin P_0;$$

and therefore the aggregate of these parts

$$=-\tfrac{1}{6}Kxy\sin P_0\sin Q_0\cos P_0.$$

The sum of the parts involving the four-index symbols in the summation, when the values for Θ_μ have been substituted,

$$=\tfrac{1}{6}\frac{wx}{\Omega}\sum_\mu\sum_\beta\sum_i\sum_j\sum_k u_\mu{}^{(p)}a_{\mu\beta}\left[(\beta i, jk)+(\beta j, ik\right]t_i{}'t_j{}'p_k{}',$$

or, as

$$\sum_\mu u_\mu{}^{(p)}a_{\mu\beta}=\Omega p_\beta{}',$$

this sum

$$=\tfrac{1}{6}wx\sum_i\sum_j\sum_k\sum_l\left[(li, jk)+(lj, ik)\right]t_i{}'t_j{}'p_k{}'p_l{}'.$$

But

$$\sum_i\sum_j\sum_k\sum_l\{(li, jk)t_i{}'t_j{}'p_k{}'p_l{}'\}=\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta\left[(\alpha\beta, \gamma\delta)s_{\alpha\beta}s_{\gamma\delta}\right],$$

$$\sum_i\sum_j\sum_k\sum_l\{(lj, ik)t_i{}'t_j{}'p_k{}'p_l{}'\}=\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta\left[(\alpha\beta, \gamma\delta)s_{\alpha\beta}s_{\gamma\delta}\right],$$

where

$$s_{lm}=p_l{}'t_m{}'-p_m{}'t_l{}',$$

for all values. Also

$$ws_{lm}=y(p_l{}'q_m{}'-p_m{}'q_l{}');$$

and therefore the sum

$$=\tfrac{1}{3}\frac{xy^2}{w}\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta\left[(\alpha\beta, \gamma\delta)(p_\alpha{}'q_\beta{}'-p_\beta{}'q_\alpha{}')(p_\gamma{}'q_\delta{}'-p_\delta{}'q_\gamma{}')\right]$$

$$=\tfrac{1}{3}\frac{xy^2}{w}K\sin^2\epsilon$$

$$=\tfrac{1}{3}xy\,K\sin\epsilon\sin P_0.$$

Thus the equation for $P-P_0$ is

$$(P-P_0)\sin P_0=-\tfrac{1}{6}K\,xy\sin P_0\sin Q_0\cos P_0+\tfrac{1}{3}xy\,K\sin\epsilon\sin P_0,$$

and therefore

$$P-P_0=\tfrac{1}{3}xy\,K(\sin\epsilon-\tfrac{1}{2}\sin Q_0\cos P_0),$$

which accordingly gives the approximation for the angle P up to the second order of small quantities.

Manifestly the corresponding analysis would lead to the same degree of approximation for the angle Q, with the result

$$Q - Q_0 = \tfrac{1}{3}xyK(\sin\epsilon - \tfrac{1}{2}\sin P_0 \cos Q_0).$$

Thus all the required elements of the geodesic triangle OPQ, initially determined by a small arc OP equal to x along a geodesic OPX and a small arc OQ equal to y along a geodesic OQY, are known, the length of the third side PQ being

$$(x^2 - 2xy\cos\epsilon + y^2)^{\frac{1}{2}}\left\{1 - \tfrac{1}{6}K\frac{x^2y^2\sin^2\epsilon}{x^2 - 2xy\cos\epsilon + y^2}\right\}$$

up to the third order of small quantities inclusive, and the other two angles at P and Q being given by the foregoing values up to the second order inclusive.

65. On the basis of these analytical results, it is possible to establish a geometrical significance for the analytical measure denoted by K.

From the values of P and Q, it follows that

$$P - P_0 + Q - Q_0 = \tfrac{1}{3}xyK\{2\sin\epsilon - \tfrac{1}{2}\sin(P_0 + Q_0)\}.$$

But (p. 153)

$$P_0 + Q_0 + \epsilon = \pi;$$

and therefore

$$P + Q + \epsilon - \pi = \tfrac{1}{2}xyK\sin\epsilon.$$

Now $P + Q + \epsilon$ is the sum of the three angles of the geodesic triangle, while π is the sum for a plane triangle; the magnitude on the left side of the equation may be called the *angular excess* of the geodesic triangle. Again, the quantity $\frac{1}{2}xy\sin\epsilon$ is the area of the triangle up to the second order of small quantities; so that, for the small geodesic triangle, the result can be stated

$$\text{area of geodesic triangle} = \frac{1}{K}\,(\text{angular excess of the triangle}).$$

When, on the surface of a sphere of radius R, a triangle is drawn having arcs of great circles for its sides (that is, having geodesic sides, so that there is a geodesic triangle), the area and the spherical excess are connected by the equation

$$\text{area of triangle} = R^2\,(\text{spherical excess of the triangle}).$$

On the analogy of the meaning of curvature for a circle as being equal to $1/r$ where r is the radius of the circle, we may call $1/R^2$ the sphericity of a sphere of radius R. We therefore are in a position to call K the *sphericity* of the amplitude at the point O in the superficial orientation defined by the two geodesics drawn through the point O. For a general amplitude, the quantity K varies not merely with the orientation at O but also with the point O itself; and therefore the sphericity is defined for a superficial orientation at a point. But there are amplitudes for which K is everywhere constant. If K vanishes for all orientations

everywhere, the amplitude is a homaloid. An orb, in a space of $n+1$ dimensions, represented by the equation in space-coordinates

$$\sum_{i=1}^{n+1} y_i^2 = c^2,$$

is a primary n-fold amplitude of constant sphericity $1/c^2$. Riemann himself propounded an amplitude, of constant sphericity $1/\kappa^2$ for all superficial orientations at any point, having its arc-element represented by the equation

$$ds^2 = \frac{dx_1^2 + dx_2^2 + \ldots + dx_n^2}{\left\{1 + \frac{1}{4\kappa^2}(x_1^2 + \ldots + x_n^2)\right\}^2};$$

but such an amplitude is only one of an unlimited set with the same character of constant sphericity for all superficial orientations at any point.

66. In connection with the preceding results, two matters may receive a brief mention: later, both of them will be considered in fuller detail for specific types of amplitudes, such as surfaces, regions, domains, the initial significance of the implied limitations being clearer for such types than for a general amplitude. One relates to geodesic surfaces, and demands consideration for amplitudes of more than two dimensions: such surfaces originated with Riemann *. The other relates to the theory of parallelism as associated with geodesics in any amplitude, the earliest investigations having originated with Levi-Civita *.

I. At the point O in the amplitude, two geodesics OPX and OQY are drawn in arbitrarily selected directions at O which, for brevity of description, may be called the linear directions OP and OQ. As these linear directions are distinct, they determine a plane QOP; and through the point O, any number of directions can be taken, all lying in the plane. Through O, in each such assumed direction, an amplitudinal geodesic can be drawn and it touches the plane; the aggregate of all these geodesics is defined to be a *geodesic surface* of the amplitude. Such a surface touches the plane at O, and it is said to be geodesic to the amplitude at O.

The point P on the amplitudinal geodesic OPX, and the point Q on the amplitudinal geodesic OQY, are points on this geodesic surface. But the amplitudinal geodesic joining P and Q does not lie in the surface that is geodesic at O to the amplitude †. There is a geodesic which lies in the surface and joins the two points P and Q; the direction-variables at P of the amplitudinal geodesic and those at P of the superficial geodesic are found to differ by small quantities of the second (and higher) orders even when the amplitude is only a region.

* Full references will be given later, during the detailed consideration of both subjects (Chaps. xvii, xxix, for geodesic surfaces; Chaps. x, xviii, xxxii, for parallels).

† Unless the amplitude is of constant sphericity—an exceptional and particular type that, throughout the argument, is implicitly excluded.

II. In the discussion of parallel geodesics, an essential property (arising as an inference from the definitions) is the equality of the angles made by two such geodesics where they meet a common basic geodesic. Thus if, in the figure on p. 145, a geodesic PT through P were declared parallel to OY (the parallelism being relative to OPX as a basic geodesic), the angle XPT would be equal to the angle XOY. When the amplitude is a surface, existing freely in a plenary homaloidal space and not regarded as geodesic to some containing amplitude of greater extent, the selection of a geodesic through P parallel to OY is definite and unique. But for all amplitudes more extensive than surfaces, there is an initial difficulty in selecting a single one, out of an unlimited number, of the directions through P satisfying this requirement of equal angles. Levi-Civita made a selection * by reference to the plenary homaloidal space: Severi made a selection † by reference to the surface which is geodesic to the amplitude at O; and the two selections give two sets of direction-variables which, agreeing up to the included first order of small quantities, differ ‡ in the second order of small quantities. But calculations shew that if, by either selection, a geodesic PT be drawn through P parallel to OY and a geodesic QR be drawn through Q parallel to OX, the geodesics PT and QR do not meet; and a geodesic parallelogram cannot thus be completed.

Other selections of parallel geodesics are possible. Thus, as the definition of a geodesic PT through P parallel to OY, the initiating direction at P might be made to lie in the plane orientation determined by the directions at P of the geodesics PX and PQ; but again there is a difficulty, for if a different point Q' on OY be taken, the direction at P of the amplitudinal geodesic PQ' does not lie in that plane orientation, for only the finite parts of its direction-variables give a line in that plane—a statement that can be established by simple calculations. Still another selection ** is possible: a geodesic PT can be taken so that, up to the first order inclusive, its direction-variables satisfy the requirement of equal angles TPX and YOX, and another geodesic QR can be drawn parallel to OX with the like requirement, leaving the terms of the second order to be determined so that PT and QR shall meet and thus form a geodesic parallelogram.

A discussion of the alternatives is reserved until the properties of regions are under consideration.

* §§ 119, 221, 377.

† §§ 222, 380.

‡ Again on an assumption that the amplitude is not of constant sphericity.

** See, in particular, §§ 232, 233.

CHAPTER VI

Sub-amplitudes in a General Amplitude

Parametric definition : amplitudinal normals.

67. Within an amplitude, configurations of various types may exist; they may be called sub-amplitudes, for brevity of general description. Each sub-amplitude necessarily is contained in the homaloidal space which is plenary to the amplitude ; and it will have its aggregate of properties, characteristic of itself, determined in relation to that space. It is, however, to be observed that the plenary homaloidal space of an included configuration is not necessarily as extensive as the plenary space of the including amplitude though, when less extensive, it will be contained in the amplitudinal space. Thus a small circle, on a sphere in triple space, has a plane for its plenary space, the plane lying in the triple space that contains the sphere ; but flexure of a small circle (sometimes called its geodesic curvature) of angular radius θ on a unit sphere is equal to $\cot\theta$, this flexure being extrinsic to the plane.

As a sub-amplitude exists within an amplitude, which stands to it as a non-homaloidal plenary space, it has a set of properties characteristic solely of its relations to that amplitude. The amplitude itself has properties, which must affect and may limit those relations ; and thus there will be a geometrical fusion of some properties of the amplitude, determined by its plenary homaloidal space, with some properties of the sub-amplitude relative solely to the amplitude. It is necessary to investigate these influences, to note the specially amplitudinal properties of a sub-amplitude, and to set out the relations between such properties and what may be called the spatial properties of the sub-amplitude and of the amplitude.

The simplest representation of sub-amplitudes, when they are completely contained in an amplitude of n dimensions, is effected by means of analytical relations, one or more than one, among the parameters characteristic of the amplitude. When m such independent relations are postulated, the sub-amplitude so defined is of $n-m$ dimensions.

Two lemmas may be proved at once. Let a sub-amplitude, of $n-1$ dimensions, be defined by a single parametric relation

$$\epsilon(x_1, x_2, \dots, x_n) = 0, \text{ or constant,}$$

so that every direction $x_1', \dots, x_n'$, in the sub-amplitude satisfies the equation

$$\epsilon_1 x_1' + \epsilon_2 x_2' + \dots + \epsilon_n x_n' = 0.$$

Consider an elementary arc dp within the amplitude, orthogonal to the sub-amplitude and therefore at right angles to every direction $x_1', x_2', \ldots, x_n'$; and let its direction-variables in the amplitude be denoted by

$$\frac{dx_1}{dp}, \frac{dx_2}{dp}, \ldots, \frac{dx_n}{dp};$$

then the condition

$$\sum_i \sum_j A_{ij} x_i' \frac{dx_j}{dp} = 0$$

must be satisfied for all values of $x_1', \ldots, x_n'$, which are subject to the single condition

$$\epsilon_1 x_1' + \epsilon_2 x_2' + \ldots + \epsilon_n x_n' = 0.$$

Consequently

$$\frac{1}{\epsilon_1} \sum_\mu A_{1\mu} \frac{dx_\mu}{dp} = \frac{1}{\epsilon_2} \sum_\mu A_{2\mu} \frac{dx_\mu}{dp} = \ldots = \frac{1}{\epsilon_n} \sum_\mu A_{n\mu} \frac{dx_\mu}{dp},$$

and therefore

$$\frac{1}{\sum_\mu a_{1\mu}\epsilon_\mu} \frac{dx_1}{dp} = \frac{1}{\sum_\mu a_{2\mu}\epsilon_\mu} \frac{dx_2}{dp} = \ldots = \frac{1}{\sum_\mu a_{n\mu}\epsilon_\mu} \frac{dx_n}{dp}.$$

Thus the amplitudinal normal to the $(n-1)$-fold subnormal is unique in direction. Let $1/T$ denote the common value of these fractions; and let $d\epsilon$ denote the increment of the magnitude $\epsilon(x_1, x_2, \ldots, x_n)$ for the variation dp along the normal to the sub-amplitude, so that $\frac{d\epsilon}{dp}$ may be regarded as the *normal enlargement* (or the *dilatation*) of the sub-amplitude, its value being

$$\begin{aligned} \frac{d\epsilon}{dp} &= \epsilon_1 \frac{dx_1}{dp} + \epsilon_2 \frac{dx_2}{dp} + \ldots + \epsilon_n \frac{dx_n}{dp} \\ &= \frac{1}{T} \sum_i \sum_j a_{ij}\epsilon_i\epsilon_j. \end{aligned}$$

Also we have

$$\begin{aligned} T^2 &= \sum_i \sum_j A_{ij} \left(T \frac{dx_i}{dp}\right)\left(T \frac{dx_j}{dp}\right) \\ &= \sum_i \sum_j A_{ij} \left(\sum_\lambda a_{i\lambda}\epsilon_\lambda\right)\left(\sum_\mu a_{j\mu}\epsilon_\mu\right) \\ &= \Omega \sum_i \sum_j a_{ij}\epsilon_i\epsilon_j. \end{aligned}$$

Hence we have

$$T = \Omega \frac{d\epsilon}{dp}, \quad \Omega \left(\frac{d\epsilon}{dp}\right)^2 = \sum_i \sum_j a_{ij}\epsilon_i\epsilon_j;$$

and therefore the unique amplitudinal direction, orthogonal to the $(n-1)$-fold sub-amplitude $\epsilon(x_1, \ldots, x_n) = 0$, is given by the equations

$$\Omega \frac{d\epsilon}{dp} \frac{dx_r}{dp} = \sum_\mu a_{r\mu}\epsilon_\mu,$$

for $r=1, \ldots, n$. We write

$$D_\epsilon = \sum_i \sum_j a_{ij}\epsilon_i\epsilon_j = \Omega\left(\frac{d\epsilon}{dp}\right)^2.$$

Next, let a sub-amplitude, of $n-2$ dimensions, be defined by a couple of parametric relations

$$\epsilon(x_1, \ldots, x_n)=0, \quad \omega(x_1, \ldots, x_n)=0.$$

The relation $\omega=0$, taken alone, defines a sub-amplitude of $n-1$ dimensions similar to the sub-amplitude defined by the relation $\epsilon=0$, taken alone. For this sub-amplitude $\omega=0$, let dq be an orthogonal amplitudinal elementary arc, and write

$$D_\omega = \sum_i \sum_j a_{ij}\omega_i\omega_j.$$

Then the normal enlargement $\dfrac{d\omega}{dq}$ is given by

$$\Omega\left(\frac{d\omega}{dq}\right)^2 = D_\omega\,;$$

and the direction-cosines of the amplitudinal normal arc dq are given by the equations

$$\Omega\frac{d\omega}{dq}\frac{dx_t}{dq} = \sum_\lambda a_{t\lambda}\omega_\lambda,$$

for $t=1, \ldots, n$.

We define the inclination of the two sub-amplitudes of $n-1$ dimensions, given by $\epsilon=0$ and by $\omega=0$ respectively, and lying within the plenary amplitude of n dimensions, to be equal to the angle between the directions of the elementary arcs dp and dq, drawn in the amplitude normal to the respective sub-amplitudes of dimensions $n-1$. When this angle is denoted by ζ, we have

$$\cos\zeta = \sum_i \sum_j A_{ij}\frac{dx_i}{dp}\frac{dx_j}{dq}\,;$$

and therefore

$$\begin{aligned}\Omega^2\frac{d\epsilon}{dp}\frac{d\omega}{dq}\cos\zeta &= \sum_i \sum_j A_{ij}\left(\sum_\lambda a_{i\lambda}\epsilon_\lambda\right)\left(\sum_\mu a_{j\mu}\omega_\mu\right)\\ &= \Omega\sum_i \sum_j a_{ij}\epsilon_\lambda\omega_\mu.\end{aligned}$$

We write

$$D_{\epsilon\omega} = \sum_i \sum_j a_{ij}\epsilon_i\omega_j,$$

a new concomitant intermediate between D_ϵ and D_ω; and now we have

$$\cos\zeta = \frac{D_{\epsilon\omega}}{D_\epsilon^{\frac{1}{2}}D_\omega^{\frac{1}{2}}},$$

giving the inclination of the two sub-amplitudes of dimensions $n-1$, while the interpretation of the concomitant $D_{\epsilon\omega}$ is

$$D_{\epsilon\omega} = \Omega\frac{d\epsilon}{dp}\frac{d\omega}{dq}\cos\zeta = (D_\epsilon D_\omega)^{\frac{1}{2}}\cos\zeta.$$

Equations of geodesics in a sub-amplitude.

68. An initial form of the characteristic intrinsic equations of geodesics belonging to a sub-amplitude of $n-m$ dimensions, which exists within the general amplitude and is defined by a set of m independent parametric equations

$$\theta_\mu(x_1, \ldots, x_n)=0, \quad (\mu=1, \ldots, m),$$

has already (§ 17) been obtained. We denote by $\bar{x}_i''$ the second arc-derivative of x_i along the geodesic in the sub-amplitude, reserving the former symbol x_i'' for the second arc-derivative of the same parameter along the amplitudinal geodesic. The intrinsic equations of geodesics in the sub-amplitude are

$$\sum_j A_{kj}\bar{x}_j'' + \sum_i \sum_j [ij, k] x_i' x_j' = \sum_{\mu=1}^m \lambda_\mu \frac{\partial \theta_\mu}{\partial x_k},$$

for $k=1, \ldots, n$; the magnitudes λ are multipliers left undetermined in the construction of the intrinsic equations.

We resolve these equations for $\bar{x}_1'', \ldots, \bar{x}_n''$; we multiply the foregoing typical equation by a_{kl}, the k being the same as for the typical equation; and we add for $k=1, \ldots, n$. Then

$$\bar{x}_l'' + \sum_i \sum_j \{ij, l\} x_i' x_j' = \frac{1}{\Omega} \sum_\mu \sum_k \lambda_\mu \left(a_{kl} \frac{\partial \theta_\mu}{\partial x_k} \right),$$

holding for each of the values $l=1, \ldots, n$. Multiply this typical equation for the value l by $\dfrac{\partial \theta_t}{\partial x_l}$, where t is any one of the numbers $1, \ldots, m$, in the set of parametric equations for the sub-amplitude; and add for all the values of l. Then

$$\sum_l \frac{\partial \theta_t}{\partial x_l} \bar{x}_l'' + \sum_l \sum_i \sum_j \{ij, l\} x_i' x_j' \frac{\partial \theta_t}{\partial x_l} = \frac{1}{\Omega} \sum_\mu \lambda_\mu \left(\sum_k \sum_l a_{kl} \frac{\partial \theta_\mu}{\partial x_k} \frac{\partial \theta_t}{\partial x_l} \right).$$

Now for any of the equations $\theta_t=0$, repeated differentiation along any arc in the direction $x_1', \ldots, x_n'$, gives

$$\sum_l \frac{\partial \theta_t}{\partial x_l} \frac{d^2 x_l}{ds^2} + \sum_i \sum_j \frac{\partial^2 \theta_t}{\partial x_i \partial x_j} x_i' x_j' = 0;$$

and therefore, for repeated differentiation along the geodesic,

$$\sum_i \sum_j \left[\frac{\partial^2 \theta_t}{\partial x_i \partial x_j} - \sum_l \frac{\partial \theta_t}{\partial x_l} \{ij, l\} \right] x_i' x_j' = -\frac{1}{\Omega} \sum_\mu \lambda_\mu \left(\sum_k \sum_l a_{kl} \frac{\partial \theta_\mu}{\partial x_k} \frac{\partial \theta_t}{\partial x_l} \right).$$

The concomitants on the right-hand side have occurred (§ 67) in the lemma dealing with the inclination of two sub-amplitudes of dimension $m-1$. For two such sub-amplitudes, given by

$$\theta_\alpha(x_1, \ldots, x_n)=0, \quad \theta_\beta(x_1, \ldots, x_n)=0,$$

we write

$$D_{\alpha\beta}=\sum_i\sum_j a_{ij}\frac{\partial\theta_\alpha}{\partial x_i}\frac{\partial\theta_\beta}{\partial x_j},\quad D_\alpha=\sum_i\sum_j a_{ij}\frac{\partial\theta_\alpha}{\partial x_i}\frac{\partial\theta_\alpha}{\partial x_j},\quad D_\beta=\sum_i\sum_j a_{ij}\frac{\partial\theta_\beta}{\partial x_i}\frac{\partial\theta_\beta}{\partial x_i};$$

and we denote their inclination (measured by the inclination of their amplitudinal normals) by $\zeta_{\alpha\beta}$, so that

$$(D_\alpha D_\beta)^{\frac{1}{2}}\cos\zeta_{\alpha\beta}=D_{\alpha\beta}.$$

Again, we write (for reasons set out later, in § 70)

$$-\frac{1}{\gamma_t}=\left(\frac{\Omega}{D_t}\right)^{\frac{1}{2}}\sum_i\sum_j\left[\frac{\partial^2\theta_t}{\partial x_i\partial x_j}-\sum_l\frac{\partial\theta_t}{\partial x_l}\{ij,\, l\}\right]x_i'x_j'.$$

When the various substitutions are used, the foregoing relation is expressible in the form

$$\frac{\Omega^{\frac{1}{2}}}{\gamma_t}=\sum_{\theta=1}^{m}\lambda_\theta D_\theta^{\frac{1}{2}}\cos\zeta_{t\theta}.$$

As the relation holds for $t=1, \ldots, m$, there are m such relations sufficient for the determination of the m multipliers λ_θ, in terms of these magnitudes γ_t.

The equations can be simplified in form. We write

$$\lambda_\alpha\left(\frac{D_\alpha}{\Omega}\right)^{\frac{1}{2}}=c_\alpha,$$

thus taking new quantities c instead of the multipliers λ. Again, the direction-cosines of the element dp_μ of amplitudinal arc, orthogonal to the sub-amplitude $\theta_\mu=0$ of $m-1$ dimensions, are given by

$$\Omega\frac{d\theta_\mu}{dp_\mu}\frac{dx_r}{dp_\mu}=\sum_k a_{kr}\frac{\partial\theta_\mu}{\partial x_k},$$

and

$$\Omega\left(\frac{d\theta_\mu}{dp_\mu}\right)^2=D_0,$$

so that

$$\frac{1}{\Omega}\sum_\mu\sum_k\lambda_\mu a_{kr}\frac{\partial\theta_\mu}{\partial x_k}=\sum_\mu\lambda_\mu\frac{d\theta_\mu}{dp_\mu}\frac{dx_r}{dp_\mu}=\sum_\mu c_\mu\frac{dx_r}{dp_\mu}.$$

Thus the intrinsic equations of the geodesics are

$$\bar{x}_l''+\sum_i\sum_j\{ij,\, l\}x_i'x_j'=\sum_{\mu=1}^{m}c_\mu\frac{dx_l}{dp_\mu},$$

while the coefficients c_μ on the right-hand sides are determinable by the equations

$$\sum_{t=1}^{m}c_t\cos\zeta_{tr}=\frac{1}{\gamma_r},$$

the former equations holding for $l=1, \ldots, n$, and the latter equations holding for $r=1, \ldots, m$.

Thus when $m=1$, there is only one coefficient c, and it is equal to $1/\gamma$; therefore the intrinsic equations of geodesics, in the sub-amplitude $\epsilon(x_1, \ldots, x_n)=0$ of $n-1$ dimensions, existing in the plenary amplitude, are

$$\bar{x}_l'' + \sum_i \sum_j \{ij, l\} x_i' x_j' = \frac{1}{\gamma_\epsilon} \frac{dx_l}{dp_\epsilon},$$

holding for $l=1, \ldots, n$; the value of γ_ϵ is given by

$$-\frac{1}{\gamma_\epsilon} = \left(\frac{\Omega}{D_\epsilon}\right)^{\frac{1}{2}} \sum_i \sum_j \left[\frac{\partial^2 \epsilon}{\partial x_i \partial x_j} - \sum \frac{\partial \epsilon}{\partial x_k} \{ij, k\}\right] x_i' x_j',$$

where

$$D_\epsilon = \sum_i \sum_j a_{ij} \frac{\partial \epsilon}{\partial x_i} \frac{\partial \epsilon}{\partial x_j} = \Omega \left(\frac{d\epsilon}{dp_\epsilon}\right)^2;$$

and the quantities $\dfrac{dx_l}{dp_\epsilon}$ are given by the equations

$$\sum_\mu a_{l\mu} \frac{\partial \epsilon}{\partial x_\mu} = \Omega \frac{d\epsilon}{dp_\mu} \frac{dx_l}{dp_\epsilon},$$

for $l=1, \ldots, n$.

Similarly, when $m=2$, and there is a sub-amplitude of $n-2$ dimensions given by the two parametric relations

$$\epsilon(x_1, \ldots, x_n)=0, \quad \omega(x_1, \ldots, x_n)=0,$$

there are two coefficients c_ϵ and c_ω given by the relations

$$\left.\begin{aligned} c_\epsilon + c_\omega \cos\zeta &= \frac{1}{\gamma_\epsilon} \\ c_\epsilon \cos\zeta + c_\omega &= \frac{1}{\gamma_\omega} \end{aligned}\right\},$$

where the value of γ_ϵ is as before: the value of γ_ω is given by

$$-\frac{1}{\gamma_\omega} = \left(\frac{\Omega}{D_\omega}\right)^{\frac{1}{2}} \sum_i \sum_j \left[\frac{\partial^2 \omega}{\partial x_i \partial x_j} - \sum \frac{\partial \omega}{\partial x_k} \{ij, k\}\right] x_i' x_j',$$

$$D_\omega = \sum_i \sum_j a_{ij} \frac{\partial \omega}{\partial x_i} \frac{\partial \omega}{\partial x_j} = \Omega \left(\frac{d\omega}{dp_\omega}\right)^2,$$

and also

$$\cos\zeta = \frac{D_{\epsilon\omega}}{(D_\epsilon D_\omega)^{\frac{1}{2}}},$$

$$D_{\epsilon\omega} = \sum_i \sum_j a_{ij} \frac{\partial \epsilon}{\partial x_i} \frac{\partial \omega}{\partial x_j}.$$

The intrinsic equations of the geodesic in this sub-amplitude of $n-2$ dimensions, defined by the two relations $\epsilon=0$ and $\omega=0$, are

$$\bar{x}_l'' + \sum_i \sum_j \{ij, l\} x_i' x_j' = c_\epsilon \frac{dx_l}{dp_\epsilon} + c_\omega \frac{dx_l}{dp_\omega},$$

for $l=1, \ldots, n$, while the values of $\dfrac{dx_l}{dp_\epsilon}$ and $\dfrac{dx_l}{dp_\omega}$ are given by

$$\Omega \frac{d\epsilon}{dp_\epsilon}\frac{dx_l}{dp_\epsilon}=\sum_\mu a_{l\mu}\frac{\partial\epsilon}{\partial x_\mu}, \quad \Omega \frac{d\omega}{dp_\omega}\frac{dx_l}{dp_\omega}=\sum_\mu a_{l\mu}\frac{\partial\omega}{\partial x_\mu}.$$

The values of c_ϵ and c_ω can be taken in the equivalent forms

$$\left.\begin{aligned} c_\epsilon &= \frac{1}{\sin^2\zeta}\left(-\frac{\cos\zeta}{\gamma_\omega}+\frac{1}{\gamma_\epsilon}\right) \\ c_\omega &= \frac{1}{\sin^2\zeta}\left(-\frac{\cos\zeta}{\gamma_\epsilon}+\frac{1}{\gamma_\omega}\right)\end{aligned}\right\}.$$

Before proceeding to determine the significance of the quantities γ_ϵ and γ_ω in this case, and all the quantities γ for the general sub-amplitude, we establish the due relation between the prime normal of the geodesic of the sub-amplitude and the tangent homaloid of the sub-amplitude. As the sub-amplitude exists in the plenary homaloidal space of the amplitude, being a configuration in that space, the relation indicated must agree with the general relation in § 20.

Tangent homaloid of the sub-amplitude: relation to geodesic prime normal.

69. The Cartesian equations, relative to the ultimate homaloidal plenary space, which represent the homaloid tangential to a sub-amplitude of $n-m$ dimensions existing in a given plenary amplitude of n dimensions, are constructed in the manner (§ 20) used for the tangent homaloid of the amplitude. A line

$$\frac{\bar{y}_1-y_1}{y_1'}=\frac{\bar{y}_2-y_2}{y_2'}=\ldots=\lambda,$$

which can be typified by the equation

$$\bar{y}-y=\lambda y',$$

touches the sub-amplitude, provided the direction-variables $x_1', \ldots, x_n'$, occurring in the quantities $y_1', y_2', \ldots$, in the form

$$y'=\frac{\partial y}{\partial x_1}x_1'+\frac{\partial y}{\partial x_2}x_2'+\ldots+\frac{\partial y}{\partial x_n}x_n',$$

satisfy the m equations

$$\sum_\mu \frac{\partial\theta_1}{\partial x_\mu}x_\mu'=0, \quad \sum_\mu \frac{\partial\theta_2}{\partial x_\mu}x_\mu'=0, \ldots, \quad \sum_\mu \frac{\partial\theta_m}{\partial x_\mu}x_\mu'=0.$$

Hence every point on such a line satisfies the equations

$$\bar{y}-y=a_1\frac{\partial y}{\partial x_1}+a_2\frac{\partial y}{\partial x_2}+\ldots+a_n\frac{\partial y}{\partial x_n},$$

where $a_1=\lambda x_1'$, $a_2=\lambda x_2'$, $\ldots$, $a_n=\lambda x_n'$, provided all the relations

$$a_1\frac{\partial\theta_i}{\partial x_1}+a_2\frac{\partial\theta_i}{\partial x_2}+\ldots+a_n\frac{\partial\theta_i}{\partial x_n}=0, \quad (i=1, \ldots, m),$$

are satisfied by the quantities a: that is, the homaloid of $n-m$ dimensions touching the sub-amplitude in question is represented through the equations typified by

$$\bar{y}-y=\sum_{\mu} a_{\mu}\frac{\partial y}{\partial x_{\mu}},$$

provided the parameters $a_1, \ldots, a_n$, satisfy the m relations

$$\sum_{\mu} a_{\mu}\frac{\partial \theta_i}{\partial x_{\mu}}=0, \quad (i=1, \ldots, m).$$

We shall require the conditions necessary to secure that a spatial direction, with spatial direction-cosines $Z_1, Z_2, \ldots$, shall be orthogonal to this tangent homaloid: that is, shall be at right angles to every direction in the homaloid. The requirement will be met, and will be secured, if, and only if, the relation

$$\sum_{j} Z_j\left(a_1\frac{\partial y_j}{\partial x_1}+a_2\frac{\partial y_j}{\partial x_2}+\ldots+a_n\frac{\partial y_j}{\partial x_n}\right)=0$$

be satisfied for all admissible values of the parameters a. These parameters must themselves obey the m conditions

$$\sum_{\mu} a_{\mu}\frac{\partial \theta_i}{\partial x_{\mu}}=0, \quad (i=1, \ldots, m),$$

associated with the parametric equations of the sub-amplitude. The necessary and sufficient conditions for this purpose are that the equations

$$\sum_{j} Z_j\frac{\partial y_j}{\partial x_t}=\kappa_1\frac{\partial \theta_1}{\partial x_t}+\kappa_2\frac{\partial \theta_2}{\partial x_t}+\ldots+\kappa_m\frac{\partial \theta_m}{\partial x_t},$$

for $t=1, \ldots, n$, be satisfied for general values of the set of quantities $\kappa_1, \kappa_2, \ldots, \kappa_m$.

Moreover, for any point on a line through the point $y_1, y_2, \ldots$, drawn in such a direction $Z_1, Z_2, \ldots$, the $n-m$ equations

$$\left\|\begin{matrix}\sum_{j}(\bar{y}_j-y_j)\frac{\partial y_j}{\partial x_t}, & \frac{\partial \theta_1}{\partial x_t}, & \ldots, & \frac{\partial \theta_m}{\partial x_t}\end{matrix}\right\|=0,$$

arising from the array for $t=1, \ldots, n$, are satisfied. Thus these equations represent a homaloid; it is orthogonal to the tangent homaloid of the sub-amplitude; and it is called the orthogonal homaloid of the sub-amplitude.

Let $\eta_1, \eta_2, \ldots$ be a point in the sub-amplitude contiguous to $y_1, y_2, \ldots$, and from it let a perpendicular be drawn to the foregoing tangent homaloid; let Π denote the length of this perpendicular, let $\overline{Y}_1, \overline{Y}_2, \ldots$ denote its direction-cosines, and let $\bar{y}_1, \bar{y}_2, \ldots$ be the space-coordinates of its foot in the homaloid, so that we have

$$\begin{aligned}\overline{Y}_j\Pi&=\eta_j-\bar{y}_j\\&=\eta_j-\left(y_j+\sum_{\mu} a_{\mu}\frac{\partial y_j}{\partial x_{\mu}}\right),\end{aligned}$$

for all the values $j=1, \ldots, N$, corresponding to the dimensions of the plenary homaloidal space of the whole amplitude. Because Π is the length of the perpendicular, the magnitude

$$\sum_j \left\{\eta_j - \left(y_j + \sum_\mu a_\mu \frac{\partial y_j}{\partial x_\mu}\right)\right\}^2$$

must be a minimum among all the values that can be acquired for values of the parameters $a_1, \ldots, a_n$, which satisfy the m relations

$$\sum_\mu a_\mu \frac{\partial \theta_i}{\partial x_\mu} = 0, \quad (i=1, \ldots, m).$$

The equations, critical for this minimum, are the set

$$\sum_j \left[\frac{\partial y_j}{\partial x_t}\left\{\eta_j - \left(y_j + \sum_\mu a_\mu \frac{\partial y_j}{\partial x_\mu}\right)\right\}\right] = \lambda_1 \frac{\partial \theta_1}{\partial x_t} + \lambda_2 \frac{\partial \theta_2}{\partial x_t} + \ldots + \lambda_m \frac{\partial \theta_m}{\partial x_t},$$

for all the values $t=1, 2, \ldots, n$, the quantities λ being multipliers that are left undetermined in the construction of the critical equations.

In the first place, these n equations may be written

$$\left(\sum_j \overline{Y}_j \frac{\partial y_i}{\partial x_t}\right) \Pi = \lambda_1 \frac{\partial \theta_1}{\partial t} + \lambda_2 \frac{\partial \theta_2}{\partial x_t} + \ldots + \lambda_m \frac{\partial \theta_m}{\partial x_t};$$

and it therefore follows that (as is to be expected) the perpendicular Π is orthogonal to the homaloid touching the sub-amplitude, that is, it is orthogonal to the sub-amplitude.

In the next place, the n equations may also be written in the different form

$$\sum_j \sum_\mu \left(a_\mu \frac{\partial y_j}{\partial x_t} \frac{\partial y_j}{\partial x_\mu}\right) + \sum_{k=1}^m \lambda_k \frac{\partial \theta_k}{\partial x_t} = \sum_j \frac{\partial y_j}{\partial x_t}(\eta_j - y_j),$$

that is,

$$\sum_\mu A_{t\mu} a_\mu + \sum_{k=1}^m \lambda_k \frac{\partial \theta_k}{\partial x_t} = \sum_j \frac{\partial y_j}{\partial x_t}(\eta_j - y_j),$$

for all the n values $t=1, \ldots, n$; and the right-hand sides must be evaluated. Now, for each of the quantities $\eta_j - y_j$, we have

$$\eta - y = y'\delta + \tfrac{1}{2} y_c'' \delta^2 + \ldots,$$

where δ denotes an arc-distance in the sub-amplitude between the points $\eta_1, \eta_2, \ldots$ and $y_1, y_2, \ldots$, in the direction $y_1', y_2', \ldots$; and the symbols y_c'' are typical of second differentiation along some curve, touching that direction but otherwise remaining unspecified as yet. Now

$$y' = \sum_\mu \frac{\partial y}{\partial x_\mu} x_\mu',$$

and therefore

$$\sum_j \frac{\partial y_j}{\partial x_t} y_j' = \sum_j \sum_\mu \frac{\partial y_j}{\partial x_t} \frac{\partial y_j}{\partial x_\mu} x_\mu' = \sum_\mu A_{t\mu} x_\mu'.$$

Again, we have

$$y_c'' = \sum_\mu \frac{\partial y}{\partial x_\mu}(x_\mu'')_c + \sum_i \sum_j \frac{\partial^2 y}{\partial x_i \partial x_j} x_i' x_j',$$

where $(x_\mu'')_c$ denotes the second variation along the unspecified curve touching the direction y_1', y_2', ... ; and therefore

$$\begin{aligned}\sum \frac{\partial y}{\partial x_t} y_c'' &= \sum_t \sum_\mu \frac{\partial y}{\partial x_t}\frac{\partial y}{\partial x_\mu}(x_\mu'')_c + \sum_t \sum_i \sum_j \frac{\partial y}{\partial x_t}\frac{\partial^2 y}{\partial x_i \partial x_j} x_i' x_j' \\ &= \sum_\mu A_{t\mu}(x_\mu'')_c + \sum_i \sum_j [ij, t] x_i' x_j' \\ &= \sum_\mu A_{t\mu}[(x_\mu'')_c + \sum_i \sum_j \{ij, \mu\} x_i' x_j'].\end{aligned}$$

When these relations are used, the foregoing typical equation can be transformed to

$$\sum_\mu A_{t\mu}[a_\mu - x_\mu'\delta - \tfrac{1}{2}\{(x_\mu'')_c + \sum_i \sum_j \{ij, \mu\} x_i' x_j'\}\delta^2] = -\sum_k \lambda_k \frac{\partial \theta_k}{\partial x_t},$$

powers of δ higher than the second being neglected, and the equation holding for $t = 1, \ldots, n$. Thus there are n equations to determine the n quantities $a_1, \ldots, a_n$; resolved for these quantities, they give

$$a_\mu - x_\mu'\delta - \tfrac{1}{2}[(x_\mu'')_c + \sum_i \sum_j \{ij, \mu\} x_i' x_j']\delta^2 = -\frac{1}{\Omega}\sum_t \sum_k a_{t\mu}\lambda_k \frac{\partial \theta_k}{\partial x_t},$$

for the n values of μ.

Let these quantities be multiplied by $\dfrac{\partial \theta_\beta}{\partial x_\mu}$, where β is any one of the integers $1, 2, \ldots, m$; and let the results be summed for the n values of μ. We have

$$\sum_\mu a_\mu \frac{\partial \theta_\beta}{\partial x_\mu} = 0,$$

for each value of β, because of the relations satisfied by the parameters; and

$$\sum_\mu x_\mu' \frac{\partial \theta_\beta}{\partial x_\mu} = 0,$$

because the direction $x_1', \ldots, x_n'$, lies in the sub-amplitude. Also

$$\sum_\mu \sum_t a_{t\mu} \frac{\partial \theta_k}{\partial x_t}\frac{\partial \theta_\beta}{\partial x_\mu} = D_{k\beta};$$

and thus the relation becomes

$$\tfrac{1}{2}\delta^2 \sum_\mu \frac{\partial \theta_\beta}{\partial x_\mu}[(x_\mu'')_c + \sum_i \sum_j \{ij, \mu\} x_i' x_j'] = \frac{1}{\Omega}\sum_k \lambda_k D_{k\beta}.$$

But second differentiation of the equation $\theta_\beta(x_1, \ldots, x_n) = 0$ gives an equation

$$\sum_\mu \frac{\partial \theta_\beta}{\partial x_\mu}(x_\mu'')_c + \sum_i \sum_j \frac{\partial^2 \theta_\beta}{\partial x_i \partial x_j} x_i' x_j' = 0,$$

and therefore the foregoing relation becomes

$$\tfrac{1}{2}\delta^2 \sum_i \sum_j \left[\frac{\partial^2\theta_\beta}{\partial x_i \partial x_j} - \sum_\mu \frac{\partial\theta_\beta}{\partial x_\mu}\{ij, \mu\}\right] x_i' x_j' = -\frac{1}{\Omega}\sum \lambda_k D_{k\beta}.$$

On the left-hand side, the coefficient of $\tfrac{1}{2}\delta^2$ has been denoted by the symbol

$$-\frac{1}{\gamma_\beta}\left(\frac{D_\beta}{\Omega}\right)^{\frac{1}{2}},$$

where the significance of γ_β has yet to be found; we also have

$$D_{k\beta} = (D_k)^{\frac{1}{2}}(D_\beta)^{\frac{1}{2}} \cos \zeta_{k\beta};$$

and we shall write

$$\lambda_k \left(\frac{D_k}{\Omega}\right)^{\frac{1}{2}} = \tfrac{1}{2}\delta^2 c_k,$$

for the values $1, \ldots, m$, of k. With these postulations, we have

$$\frac{1}{\gamma_\beta} = \sum_k c_k \cos \zeta_{k\beta},$$

for $\beta = 1, \ldots, m$, as the m relations for the determination of the m quantities c_k which take the place of the m quantities λ_k.

We now return to the consideration of the magnitude Π and its direction-cosines $\overline{Y}_1, \overline{Y}_2, \ldots$, as given by the typical equation

$$\begin{aligned} \overline{Y}\Pi &= \eta - y - \sum_\mu a_\mu \frac{\partial y}{\partial x_\mu} \\ &= y'\delta + \tfrac{1}{2}y_c''\delta^2 - \sum_\mu a_\mu \frac{\partial y}{\partial x_\mu}, \end{aligned}$$

accurately up to the second power of δ inclusive. Further, with the foregoing determination of the quantities a_μ, we have

$$\begin{aligned} \sum_\mu a_\mu \frac{\partial y}{\partial x_\mu} = \left(\sum_\mu \frac{\partial y}{\partial x_\mu} x_\mu'\right)\delta + \tfrac{1}{2}\delta^2 \sum_\mu \frac{\partial y}{\partial x_\mu}[(x_\mu'')_c + \sum_i \sum_j \{ij, \mu\} x_i' x_j'] \\ - \frac{1}{\Omega}\sum_t \sum_k \sum_\mu \lambda_k \frac{\partial y}{\partial x_\mu} a_{t\mu} \frac{\partial \theta_k}{\partial x_t}. \end{aligned}$$

The last term, being a triple summation,

$$= -\frac{1}{\Omega}\sum_k \sum_\mu \lambda_k \frac{\partial y}{\partial x_\mu}\left(\frac{d\theta_k}{dp_k}\frac{dx_\mu}{dp_k}\Omega\right),$$

after the expressions (p. 165) for an amplitudinal direction orthogonal to the sub-amplitude $\theta_\mu = 0$; and this expression, in its turn,

$$\begin{aligned} &= -\sum_k \lambda_k \frac{d\theta_k}{dp_k}\frac{dy}{dp_k} \\ &= -\sum_k \lambda_k \left(\frac{D_k}{\Omega}\right)^{\frac{1}{2}} \frac{dy}{dp_k} = -\tfrac{1}{2}\delta^2 \sum_k c_k \frac{dy}{dp_k}, \end{aligned}$$

when we use the value of $\dfrac{d\theta_k}{dp_k}$ from p. 166. Also

$$\sum_\mu \frac{\partial y}{\partial x_\mu} x_\mu' = y' ;$$

and

$$y_c'' - \sum_\mu \frac{\partial y}{\partial x_\mu} (x_\mu'')_c = \sum_i \sum_j \frac{\partial^2 y}{\partial x_i \partial x_j} x_i' x_j'.$$

Hence the equation for $\bar{Y}\Pi$ gives

$$\begin{aligned}\bar{Y}\Pi &= y'\delta + \tfrac{1}{2} y_c'' \delta^2 - \sum_\mu a_\mu \frac{\partial y}{\partial x_\mu} \\ &= \tfrac{1}{2}\delta^2 \sum_i \sum_j \left[\frac{\partial^2 y}{\partial x_i \partial x_j} - \sum_\mu \frac{\partial y}{\partial x_\mu} \{ij, \mu\} \right] x_i' x_j' + \tfrac{1}{2}\delta^2 \sum c_k \frac{dy}{dp_k} \\ &= \tfrac{1}{2}\delta^2 \left\{ \left(\sum_i \sum_j \eta_{ij} x_i' x_j' \right) + \sum_k c_k \frac{\partial y}{\partial p_k} \right\}.\end{aligned}$$

We write

$$\frac{1}{\rho} = \frac{2\Pi}{\delta^2} ;$$

and we have

$$\sum_i \sum_j \eta_{ij} x_i' x_j' = \frac{Y}{\rho},$$

where $1/\rho$ is the circular curvature of the amplitudinal geodesic in the direction $x_1', \dots, x_n'$. Hence, in the limiting position of Π as δ tends to zero, we have

$$\frac{\bar{Y}}{\bar{\rho}} = \frac{Y}{\rho} + \sum_k c_k \frac{\partial y}{\partial p_k},$$

for each of the space-coordinates y and the associated spatial direction-cosine $\bar{Y}$; and the m quantities c in this equation satisfy (or are determined by) the m equations

$$\sum_k c_k \cos \zeta_{k\beta} = \frac{1}{\gamma_\beta}, \quad (\beta = 1, \dots, m).$$

Now consider the circular curvature and the spatial direction-cosines of the prime normal of a geodesic belonging to the sub-amplitude of $n-m$ dimensions. Denoting the circular curvature of the geodesic by $1/\rho_0$, the typical direction-cosine of its prime normal by Y_0 (that is, the direction-cosines are $Y_{01}, Y_{02}, \dots$), and second variations of the space-coordinates along the geodesic by $\bar{y}_1'', \bar{y}_2'', \dots$ in full, and by $\bar{y}''$ as typical of all these quantities, we have

$$\bar{y}'' \rho_0 = Y_0,$$

while

$$\bar{y}'' = \sum_t \frac{\partial y}{\partial x_t} \bar{x}_t'' + \sum_i \sum_j \frac{\partial^2 y}{\partial x_i \partial x_j} x_i' x_j',$$

$$y'' = \sum_t \frac{\partial y}{\partial x_t} x_t'' + \sum_i \sum \frac{\partial^2 y}{\partial x_i \partial x_j} x_i' x_j'.$$

Hence

$$\bar{y}'' - y'' = \sum_t \frac{\partial y}{\partial x_t}(\bar{x}_t'' - x_t'')$$

$$= \sum_t \frac{\partial y}{\partial x_t}[\bar{x}_t'' + \sum_i \sum_j \{ij, t\} x_i' x_j']$$

$$= \sum_t \sum_\mu \frac{\partial y}{\partial x_t} c_\mu \frac{dx_t}{dp_\mu}$$

$$= \sum_\mu c_\mu \frac{dy}{dp_\mu},$$

where the quantities c are determined by the m equations

$$\sum_{\mu=1}^{m} c_\mu \cos \zeta_{\mu r} = \frac{1}{\gamma_r},$$

and therefore are identical with the quantities c on p. 174. Now

$$y'' = \frac{Y}{\rho}, \quad \bar{y}'' = \frac{Y_0}{\rho_0};$$

and therefore

$$\frac{\bar{Y}}{\bar{\rho}} = \frac{Y_0}{\rho_0},$$

a result typical of the N results corresponding to the full tale of dimensions of the plenary space. Hence

$$\bar{\rho} = \rho_0,$$

and, for all the direction-cosines,

$$\bar{Y} = Y_0.$$

Hence, as δ tends to vanish, the limiting position of the perpendicular drawn upon the tangent homaloid of the sub-amplitude at $y_1, y_2, \ldots$ from a contiguous point of the sub-amplitude at a small arc-distance δ from $y_1, y_2, \ldots$ in the direction $x_1', \ldots, x_n'$, is the prime normal of the geodesic of that sub-amplitude drawn in the specified direction. Further, when Y_0 denotes the typical direction-cosine of the prime normal of the geodesic of the sub-amplitude corresponding to the typical direction-cosine Y of the prime normal of the geodesic of the plenary amplitude with the same initial direction, this typical direction-cosine Y_0 and the circular curvature $1/\rho_0$ of the geodesic of the sub-amplitude are given by the equation

$$\frac{Y_0}{\rho_0} = \frac{Y}{\rho} + \sum_{\mu=1}^{m} c_\mu \frac{dy}{dp_\mu},$$

where $\dfrac{dy}{dp_\mu}$ is the typical direction-variable of the amplitudinal normal to the simple sub-amplitude $\theta_\mu(x_1, \ldots, x_n) = 0$, and where the magnitudes $c_1, \ldots, c_m$, are determined by the m equations

$$\sum_{\mu=1}^{m} c_\mu \cos \zeta_{\mu r} = \frac{1}{\gamma_r},$$

the quantity $\zeta_{\mu r}$ being the inclination of the two simple sub-amplitudes $\theta_\mu = 0$ and $\theta_r = 0$ to one another.

It remains to identify the quantities denoted by the symbols $\gamma_1, \ldots, \gamma_m$.

Amplitudinal flexure of a geodesic in a sub-amplitude.

70. The relative march of any geodesic curve can be estimated in two ways. In one of these ways, it is regarded mainly as a curve in a plenary homaloidal space: the march is influenced by properties of multiple non-homaloidal spaces in which its own amplitude may lie: its variations are estimated, ultimately, by reference to the plenary homaloidal space. In the other way, the variations of the curve are estimated by reference to any plenary non-homaloidal space which, containing the amplitude of the geodesic, therefore contains the geodesic. In both of them, the variations are measured as arc-rates of deviation from geodesics in the manifold of reference: when this manifold is a plenary homaloidal space, the geodesics of reference are straight lines: when the manifold is a plenary non-homaloidal amplitude, the geodesics of reference are the geodesics of that containing amplitude. The measures, thus obtained in the first description, are in effect the successive curvatures of a curve in some homaloidal space: they are mathematically expressed in the Frenet equations of a curve, referred to its orthogonal frame in the homaloidal space: and they constitute what may be called a canonical description of the curve—canonical, that is, for purposes of central reference. The measures, obtained in the second description, are estimated relative to the geodesics of the plenary amplitude: but those amplitudinal geodesics, being curves, have curvatures of their own; and therefore the second description has a somewhat relative and local character, undoubtedly significant and informing, yet varying from one amplitude to another when there are distinct plenary manifolds each completely containing the sub-amplitude to which the geodesic belongs.

It is to the second range of description that the magnitudes $\gamma_1, \ldots, \gamma_m$, belong; they arise in the following manner. Consider, initially, a geodesic belonging to any $(n-1)$-fold sub-amplitude $\epsilon(x_1, \ldots, x_n) = 0$ contained in the general amplitude of n dimensions. We take two geodesics through O, the initial point $y_1, y_2, \ldots$, in the direction $x_1', x_2', \ldots$, one of them belonging to the general amplitude, the other to the sub-amplitude; and, on these respectively, we take contiguous points P and Q at the same small arc-distance δ from O. The representative variable η of the point P on the amplitudinal geodesic is

$$\eta = y + y'\delta + \tfrac{1}{2}y''\delta^2 + \ldots,$$

while the corresponding variable $\bar{\eta}$ of the point Q on the geodesic in the simple sub-amplitude is

$$\bar{\eta} = y + y'\delta + \tfrac{1}{2}\bar{y}''\delta^2 + \ldots,$$

the coefficients of δ in η and in $\bar{\eta}$ being the same, because the geodesics are drawn in the same direction. Hence, when we neglect powers of δ higher than the second, we have

$$\bar{\eta}-\eta=\tfrac{1}{2}(\bar{y}''-y'')\,\delta^2.$$

The magnitude $\bar{\eta}-\eta$ is the projection, upon the typical axis in the plenary homaloidal space, of the distance $\varDelta$ between the points on the two geodesics—that is, of the deviation of the geodesic in the sub-amplitude $\epsilon=0$ from the amplitudinal geodesic. Moreover, we have

$$\sum(\bar{\eta}-\eta)\,y'=\tfrac{1}{2}\{(\sum y'\bar{y}'')-(\sum y'y'')\}\,\delta^2=0,$$

because $\sum y'^2=1$ along both geodesics, and therefore

$$\sum y'\bar{y}''=0,\quad \sum y'y''=0,$$

along the respective geodesics; hence the foregoing deviation $\varDelta$ is at right angles to the common direction of the two geodesics. Accordingly, we take a quantity $\bar{\gamma}_\epsilon$ such that

$$2\varDelta\bar{\gamma}_\epsilon=\delta^2,$$

and then we have

$$\frac{\bar{\eta}-\eta}{\varDelta}\,\frac{1}{\bar{\gamma}_\epsilon}=\bar{y}''-y''.$$

But $(\bar{\eta}-\eta)/\varDelta$ is the direction-cosine of the deviation with respect to the typical axis: denoting it momentarily by l, we have

$$\frac{l}{\bar{\gamma}_\epsilon}=\bar{y}''-y''.$$

Now for the simple sub-amplitude $\epsilon=0$, we have

$$\begin{aligned}\bar{y}''-y''&=\sum_\mu \frac{\partial y}{\partial x_\mu}(\bar{x}_\mu''-x_\mu'')\\ &=\sum_\mu \frac{\partial y}{\partial x_\mu}\frac{1}{\gamma_\epsilon}\frac{dx_\mu}{dp_\epsilon}=\frac{1}{\gamma_\epsilon}\frac{dy}{dp_\epsilon},\end{aligned}$$

where dp_ϵ denotes an elementary amplitudinal arc normal to the sub-amplitude. Hence we have

$$l=\frac{dy}{dp_\epsilon},\quad \bar{\gamma}_\epsilon=\gamma_\epsilon,$$

the first relation being typical of all the direction-cosines of the deviation. It follows that, in the limiting position as δ leads to zero, the direction of the deviation is in the line of the amplitudinal normal to the simple sub-amplitude, the direction-cosines of which are

$$\frac{dy_1}{dp_\epsilon},\quad \frac{dy_2}{dp_\epsilon},\ \dots\ .$$

Also, because the deviation Δ and the arc-distance δ are connected by the relation $2\Delta\bar{\gamma}_\epsilon=\delta^2$, the magnitude $\bar{\gamma}_\epsilon$ is a radius of curvature or flexure; and $\bar{\gamma}_\epsilon=\gamma_\epsilon$. Hence the quantity γ_ϵ also is a radius of curvature or flexure; we shall call $1/\gamma_\epsilon$ the *amplitudinal flexure* of the geodesic of the sub-amplitude $\epsilon=0$. Moreover, the deviation has been estimated along the positive direction of the prime normal of the amplitudinal geodesic, whereas the amplitudinal normal has been drawn in the opposite sense; and we therefore can expect the analytical measure of the amplitudinal flexure of a simple sub-amplitude to have formally a negative sign. In fact, we have taken (p. 168)

$$-\frac{1}{\gamma_\epsilon}=\left(\frac{\Omega}{D_\epsilon}\right)^{\frac{1}{2}}\sum_i\sum_j\left[\frac{\partial^2\epsilon}{\partial x_i\partial x_j}-\sum_l\frac{\partial\epsilon}{\partial x_l}\{ij,\, l\}\right]x_i'x_j'.$$

The foregoing result, in its initial form

$$\frac{l}{\bar{\gamma}_\epsilon}=\bar{y}''-y'',$$

can be obtained simply by a different (but equivalent) consideration. An estimate of the flexure of a geodesic in a simple (or in any) sub-amplitude can be framed, by regarding the flexure as the arc-rate of angular deviation of that geodesic from the geodesic of the plenary amplitude in the same direction: so that, if $d\theta$ be this angular deviation between the next succeeding tangents of the respective geodesics, we have

$$\frac{1}{\bar{\gamma}_\epsilon}=\frac{d\theta}{ds}.$$

Now the typical direction-cosines, in the plenary homaloidal space, of these respective consecutive tangents are

$$y'+\bar{y}''\,ds+\ldots,\quad y'+y''\,ds+\ldots.$$

Let an orb of reference be taken for the representation of directions in that homaloidal space, such that its equation becomes

$$\sum y'^2=1.$$

In this orb, the consecutive tangents are represented by two points; there, the distance between the two points is the foregoing angular deviation $d\theta$, while the direction-cosines of this distance (defining the earlier direction of the distance-displacement) are typified by the quantity l. Consequently,

$$l\,d\theta=(\bar{y}''\,ds+\ldots)-(y''\,ds+\ldots),$$

that is, in the limit as ds tends to zero,

$$l\frac{d\theta}{ds}=\bar{y}''-y'',$$

confirming the former result.

The quantities γ_μ thus measure the amplitudinal flexure of the geodesics in the respective simple sub-amplitudes $\theta_\mu(x_1, \ldots, x_n)=0$, contained in the plenary amplitude, all the individual geodesics being drawn in the same direction $x_1', \ldots, x_n'$. We proceed to the discussion of the same matter in connection with geodesics in the more restricted m-fold sub-amplitude represented by the set of m parametric equations

$$\theta_\mu(x_1, \ldots, x_n)=0, \quad (\mu=1, \ldots, m).$$

As before, we take two geodesics drawn in the direction $x_1', \ldots, x_n'$ through O, the point $y_1, y_2, \ldots$; one of them is the geodesic in the amplitude, the other is the geodesic in the restricted m-fold sub-amplitude. Whether we take contiguous points P and Q along these two geodesics respectively at the same arc-distance δ from O, according to the first mode of estimating the flexure ; or we consider the angular deviation between the tangents to the respective geodesics at these consecutive points ; by both modes, we are led to the relation

$$\frac{l}{\gamma}=\bar{y}''-y'',$$

where now $1/\gamma$ is the measure of the arc-rate of deviation, l is the typical spatial direction-cosine to be associated with the direction of this flexural deviation, y'' belongs to the amplitudinal geodesic, and $\bar{y}''$ belongs to the geodesic in the general sub-amplitude. Now, as before, we have

$$\begin{aligned}\bar{y}''-y''&=\sum_t \frac{\partial y}{\partial x_t}(\bar{x}_t''-x_t'')\\ &=\sum_t \frac{\partial y}{\partial x_t} c_\mu \frac{dx_t}{dp_\mu}=\sum_\mu c_\mu \frac{dy}{dp_\mu},\end{aligned}$$

where the m quantities c_μ are determined, by the m equations

$$\sum_{\mu=1}^{m} c_\mu \cos\zeta_{\mu r}=\frac{1}{\gamma_r}, \quad (r=1, \ldots, m),$$

in terms of the amplitudinal flexures of the respective geodesics belonging to the several simple sub-amplitudes $\theta_\mu(x_1, \ldots, x_n)=0$. Consequently, the equations determining the magnitude and the direction of the amplitudinal flexure of geodesics, in a restricted $(n-m)$-fold sub-amplitude represented by the equations

$$\theta_\mu(x_1, \ldots, x_n)=0, \quad (\mu=1, \ldots, m),$$

are the set symbolised by the typical equation

$$\frac{l}{\gamma}=\sum_\mu c_\mu \frac{dy}{dp_\mu};$$

the m quantities c_μ in these equations are determined by means of the m equations

$$\sum_\mu c_\mu \cos\zeta_{\mu r}=\frac{1}{\gamma_r}, \quad (r=1, \ldots, m),$$

and the spatial direction-cosines typified by $\frac{dy}{dp_\mu}$ are those of the respective amplitudinal normals to the simple sub-amplitudes $\theta_\mu=0$, for $\mu=1, \ldots, m$; and in the latter set of equations, the quantities $1/\gamma_r$ denote the respective amplitudinal flexures of geodesics in those simple sub-amplitudes, while $\zeta_{\mu r}$ denotes the inclination of the two simple sub-amplitudes $\theta_\mu=0$, $\theta_r=0$, as measured by the inclination of their unique amplitudinal normals.

Relations between radii of flexure and prime normals.

71. Further, the circular curvature of the geodesic in the restricted $(n-m)$-fold sub-amplitude, the circular curvature being always a measure of the deviation of the geodesic from the tangent line in the plenary homaloidal space of the sub-amplitude, has been given (§ 69), as to magnitude $1/\rho_0$ and spatial direction-cosines Y_0, in the aggregate of equations symbolised by

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\sum_\mu c_\mu \frac{dy}{dp_\mu},$$

where $1/\rho$ and Y denote the corresponding quantities for the amplitudinal geodesic in the same direction $x_1', \ldots, x_n'$. Hence

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{l}{\gamma}.$$

Manifestly, the three lines, the spatial direction-cosines of which are typified by Y_0, Y, l, lie in one plane, because the relations

$$\| \; Y_0, \;\; Y, \;\; l \; \|=0$$

are satisfied: and therefore the prime normal of the geodesic in the restricted $(n-m)$-fold sub-amplitude in the direction $x_1', \ldots, x_n'$, the radius of flexure of that geodesic, and the prime normal of the amplitudinal geodesic in the same direction, lie in one plane.

Let ψ denote the angle between the two prime normals of the respective geodesics in the amplitude and in the $(n-m)$-fold sub-amplitude, so that

$$\cos\psi=\sum YY_0,$$

the summation being taken over all the spatial direction-cosines. Also, each of the directions typified by $\frac{dy}{dp_\mu}$ is a direction in the amplitude, to which the prime normal of the amplitudinal geodesic is orthogonal, so that

$$\sum Y\frac{dy}{dp_\mu}=0$$

for $\mu=1, \ldots, m$; and therefore

$$\frac{1}{\gamma}\sum lY=\sum_{\mu} c_\mu\left(\sum Y\frac{dy}{dp_\mu}\right)=0,$$

or, on the assumption that the flexure $1/\gamma$ is not zero (we then should have a geodesically restricted sub-amplitude), we infer that the radius of flexure is at right angles to the prime normal of the amplitudinal geodesic. We therefore have

$$\sin\psi=\sum lY_0.$$

Now multiply the equation

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{l}{\gamma}$$

by Y, and add for all the equations; then

$$\frac{\cos\psi}{\rho_0}=\frac{1}{\rho}.$$

Multiply the same equation by l, and add for all the equations; then

$$\frac{\sin\psi}{\rho_0}=\frac{1}{\gamma}.$$

When these equations are combined, we have the following relations between the circular curvature of the geodesic in the restricted $(n-m)$-fold sub-amplitude, the flexure of that geodesic, and the circular curvature of the amplitudinal geodesic in the same spatial direction:

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{l}{\gamma},$$

$$Y_0=Y\cos\psi+l\sin\psi,$$

$$\frac{1}{\rho_0^2}=\frac{1}{\rho^2}+\frac{1}{\gamma^2},$$

$$\frac{1}{\rho}=\frac{\cos\psi}{\rho_0},\quad \frac{1}{\gamma}=\frac{\sin\psi}{\rho_0}.$$

These equations persistently recur for any sub-amplitude completely contained in a plenary amplitude.

The aggregate of equations

$$\frac{l}{\gamma}=\sum_{\mu} c_\mu\frac{dy}{dp_\mu},\quad \sum_{\mu} c_\mu\cos\zeta_{\mu r}=\frac{1}{\gamma_r},\quad (r=1, \ldots, m),$$

gives the analytical relations between the composite flexure (in magnitude and direction) of the geodesic in the restricted sub-amplitude

$$\theta_\mu(x_1, \ldots, x_n)=0,\quad (\mu=1, \ldots, m),$$

and the individual flexures (in magnitude and direction) of the respective geodesics

drawn, in the common direction $x_1', \ldots, x_n'$, in each of the several simple sub-amplitudes $\theta_\mu=0$. The relations can be exhibited in a more direct geometrical description.

Through O, the point $y_1, y_2, \ldots$, draw the m directions which are the respective amplitudinal normals to the several simple sub-amplitudes $\theta_\mu=0$; and let l_μ denote a typical spatial direction-cosine of the radius of flexure for a geodesic in the same simple sub-amplitude. Along the direction, typified by l_μ, measure a length γ_μ from O; and let such a measurement be made for each of the m simple sub-amplitudes $\theta_\mu=0$. The m points so selected determine a homaloid of $m-1$ dimensions, which passes through them all, its equations being typified by the relation

$$\bar{y}-y=\mu_1 \,.\, l_1\gamma_1+\mu_2 \,.\, l_2\gamma_2+\ldots+\mu_m \,.\, l_m\gamma_m,$$

with the parameters μ subject to the single condition

$$\mu_1+\mu_2+\ldots+\mu_m=1.$$

Let a perpendicular from O be drawn upon this $(m-1)$-fold homaloid, its length being denoted by p_0, and its typical direction-cosine by λ, so that, as it lies in the m-fold homaloid, with its vertex at O and with $l_1, \ldots, l_m$ for its leading lines, we have relations

$$\| \ \lambda, \ l_1, \ l_2, \ \ldots, \ l_m \ \|=0.$$

Let the foot of this perpendicular be the foregoing point $\bar{y}$, so that

$$\lambda p_0=\bar{y}-y=\sum_r \mu_r l_r \gamma_r.$$

In order that the line may be a perpendicular to the homaloid, the quantity

$$\begin{aligned} p_0^2, &= \sum (\lambda p_0)^2 \\ &= \sum \left(\sum_r \mu_r l_r \gamma_r\right)^2, \end{aligned}$$

(the external summation $\sum$ being for all the space-dimensions), must be a minimum, for all possible values of the parameters μ that are subject to the relation $\sum \mu_r=1$. The equations, critical for this minimum, are

$$\sum \left\{ l_a\gamma_a \left(\sum_r \mu_r l_r \gamma_r \right) \right\}=\theta, \quad (a=1, \ldots, m),$$

where θ is undetermined in forming these equations; and the external summation $\sum$ extends over all the space-dimensions, as before.

In the first place, these equations can be written

$$\sum l_a\gamma_a\lambda p_0=\theta, \quad (a=1, \ldots, m).$$

Let χ_a denote the angle between the direction l_a and the perpendicular, so that

$$\cos\chi_a=\sum l_a\lambda;$$

thus the critical equations become

$$\gamma_1 \cos\chi_1 = \gamma_2 \cos\chi_2 = \ldots = \gamma_m \cos\chi_m = \frac{\theta}{p_0}.$$

In the second place, multiply the original typical critical equation by μ_a, and add for all the values of a; then

$$\sum\left(\sum_a \mu_a l_a \gamma_a\right)\left(\sum_r \mu_r l_r y_r\right) = \theta \sum \mu_a,$$

that is,

$$\sum \lambda p_0 \,.\, \lambda p_0 = \theta,$$

or

$$p_0^2 = \theta.$$

Thus the aggregate of resulting equations can be typified by

$$\begin{aligned}
\lambda p_0 &= \mu_1 \,.\, l_1\gamma_1 + \mu_2 \,.\, l_2\gamma_2 + \ldots + \mu_m \,.\, l_m\gamma_m, \\
1 &= \mu_1 + \mu_2 + \ldots + \mu_m, \\
\frac{p_0^2}{\gamma_a} &= \sum l_a \left(\sum_r \mu_r l_r \gamma_r\right) \\
&= \mu_1\gamma_1 \cos\zeta_{1a} + \ldots + \mu_m\gamma_m \cos\zeta_{ma} \\
&= \sum_r \mu_r\gamma_r \cos\zeta_{ra}.
\end{aligned}$$

Take new quantities $c_1, \ldots, c_m$, such that

$$\mu_r\gamma_r = c_r p_0^2, \quad (r = 1, \ldots, m);$$

these new quantities c_r are determined by the relations

$$\frac{1}{\gamma_a} = \sum_r c_r \cos\zeta_{ra},$$

that is, they are the same as the quantities c in § 69. Also

$$\begin{aligned}
\lambda p_0 &= \sum \mu_r l_r \gamma_r \\
&= \sum p_0^2 c_r l_r,
\end{aligned}$$

and

$$l_r = \frac{dy}{dp_r};$$

hence

$$\begin{aligned}
\frac{\lambda}{p_0} &= \sum c_r \frac{dy}{dp_r} \\
&= \frac{l}{\gamma},
\end{aligned}$$

in the former investigation. As the quantities l and λ are typical, we have

$$\lambda = l, \quad \gamma = p_0:$$

that is, the radius of flexure γ of the general sub-amplitude is the length of the perpendicular from the central point O on the foregoing $(m-1)$-fold homaloid,

and the direction of that radius lies along the perpendicular. Moreover, we now have

$$\gamma_1 \cos \chi_1 = \gamma_2 \cos \chi_2 = \ldots = \gamma_m \cos \chi_m = \frac{\theta}{p_0} = \gamma,$$

a set of results obvious from the properties just established concerning the perpendicular upon the $(m-1)$-fold homaloid through the m centres of flexure of the simple homaloids.

These last results follow at once from the equations

$$\frac{l}{\gamma} = \sum_\mu c_\mu \frac{dy}{dp_\mu}, \quad \sum_\mu c_\mu \cos \zeta_{\mu r} = \frac{1}{\gamma_r}.$$

For we have

$$\cos \zeta_{\mu r} = \sum_y \frac{dy}{dp_\mu} \frac{dy}{dp_r},$$

$$\cos \chi_r = \sum l \frac{dy}{dp_r};$$

and therefore

$$\frac{1}{\gamma} \cos \chi_r = \frac{1}{\gamma} \sum l \frac{dy}{dp_r} = \sum_\mu \sum_y c_\mu \frac{dy}{dp_\mu} \frac{dy}{dp_r}$$

$$= \sum c_\mu \cos \zeta_{\mu r} = \frac{1}{\gamma_r},$$

typifying the relations in question. They admit an obvious interpretation affecting flexures alone, viz. the centres of amplitudinal flexure of the geodesics of the several simple sub-amplitudes $\theta_\mu = 0$ all project into the centre of amplitudinal flexure of the geodesic of the restricted sub-amplitude.

When we square the typical equation

$$\frac{l}{\gamma} = \sum_\mu c_\mu \frac{dy}{dp_\mu},$$

and add for all the equations typified, we have

$$\frac{1}{\gamma^2} = \sum_y \sum_\lambda \sum_\mu c_\mu \frac{dy}{dp_\mu} c_\lambda \frac{dy}{dp_\lambda}$$

$$= \sum_\lambda \sum_\mu c_\mu c_\lambda \cos \zeta_{\lambda\mu}$$

$$= \sum_\mu \frac{c_\mu}{\gamma_\mu}.$$

We obtain the same result after multiplying the equation by l, and adding for all the equations; for then

$$\frac{1}{\gamma} = \sum_y \sum_\mu c_\mu l \frac{dy}{dp_\mu}$$

$$= \sum_\mu c_\mu \cos \chi_\mu = \sum_\mu c_\mu \frac{\gamma}{\gamma_\mu},$$

which is the same result. We retain both the forms

$$\frac{1}{\gamma}=\sum_\mu c_\mu \cos\chi_\mu, \quad \frac{1}{\gamma^2}=\sum\frac{c_\mu}{\gamma_\mu}.$$

When the relations

$$\sum c_\mu \cos\zeta_{\mu r}=\frac{1}{\gamma_r}, \quad (r=1, \dots, m),$$

are used to eliminate the quantities c, we have the two equations

$$\begin{vmatrix} \frac{1}{\gamma}, & \cos\chi_1, & \cos\chi_2, & \dots, & \cos\chi_m \\ \frac{1}{\gamma_1}, & 1, & \cos\zeta_{12}, & \dots, & \cos\zeta_{1m} \\ \frac{1}{\gamma_2}, & \cos\zeta_{21}, & 1, & \dots, & \cos\zeta_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{\gamma_m}, & \cos\zeta_{m1}, & \cos\zeta_{m2}, & \dots, & 1, \end{vmatrix}=0,$$

$$\begin{vmatrix} \frac{1}{\gamma^2}, & \frac{1}{\gamma_1}, & \frac{1}{\gamma_2}, & \dots, & \frac{1}{\gamma_m} \\ \frac{1}{\gamma_1}, & 1, & \cos\zeta_{12}, & \dots, & \cos\zeta_{1m} \\ \frac{1}{\gamma_2}, & \cos\zeta_{21}, & 1, & \dots, & \cos\zeta_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{\gamma_m}, & \cos\zeta_{m1}, & \cos\zeta_{m2}, & \dots, & 1 \end{vmatrix}=0,$$

each providing an expression for the magnitude of the flexure of a geodesic of the restricted sub-amplitude, in terms of the known flexures of the several simple sub-amplitudes.

72. A comparatively simple geometrical construction for the prime normal of any geodesic in a restricted sub-amplitude can be framed, on the foundation of the foregoing geometrical statement of the analytical results.

For any simple sub-amplitude $\theta_t=0$, let OF_t represent the actual radius of flexure of its geodesic through O in a direction $x_1', \dots, x_n'$; and let OY represent the radius of circular curvature of the amplitudinal geodesic in the same direction, OY and OF_t being at right angles. The centre of circular curvature of the geodesic of the simple sub-amplitude $\theta_t=0$ is C_t, the foot of the perpendicular from O upon the line YF_t. And t can have the values $1, \dots, m$.

In the restricted sub-amplitude, the direction of the prime normal of its geodesic through O in the direction $x_1', \ldots, x_n'$, its circular curvature $1/\rho_0$, and its flexure $1/\gamma$, are connected by relations typified by the single equation

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{l}{\gamma}.$$

Let OF represent its actual radius of flexure, so that $OF=\gamma$. Then l denotes the typical direction-cosine; also, OF and OY are at right angles. Hence the centre of circular curvature of the specified geodesic of the general sub-amplitude is C, the foot of the perpendicular from O upon the line YF. Moreover, the line YF is perpendicular to the $(m-1)$-fold homaloid through the flexural centres $F_1, \ldots, F_m$; and the prime normal OY of the amplitudinal geodesic, being at right angles to all the lines OF_t, is also at right angles to that $(m-1)$-fold homaloid; that is, the plane YOF is orthogonal to that homaloid.

We have seen the point C_t lies on the line YF_t, for $t=1, \ldots, m$. Take all the points $C_1, \ldots, C_m, C$; as these are the feet of the perpendiculars from O upon the respective lines $YF_1, \ldots, YF_m, YF$, all these $(m+1)$ points, together with O and Y, lie on an orbicular m-fold configuration on the line OY as diameter; and the $m+2$ points $C_1, \ldots, C_m, C, Y$, lie on an orbicular configuration of $m-1$ dimensions. We therefore postulate an orbicular configuration of $m-1$ dimensions through the point Y, and the m points $C_1, \ldots, C_m$, the centres of circular curvature of the geodesics belonging to the respective simple sub-amplitudes $\theta_1=0, \ldots, \theta_m=0$. The centre of circular curvature C of the geodesic in the general sub-amplitude, in the same initial direction $x_1', \ldots, x_n'$, is the intersection of this orbicular configuration of $m-1$ dimensions by the line drawn from Y perpendicular to the $(m-1)$-fold homaloid through the m centres $F_1, \ldots, F_m$, of flexure of geodesics in that same direction belonging to the m respective simple sub-amplitudes.

Thus in the case of a secondary sub-amplitude, defined by two parametric equations $\theta_1=0$, $\theta_2=0$, let OF_1 and OF_2 be the radii of flexure of geodesics in the direction $x_1', \ldots, x_n'$, belonging to the simple sub-amplitudes $\theta_1=0$ and $\theta_2=0$ respectively. Let OF be the perpendicular from O on the line F_1F_2; then OF, in magnitude and in direction, is the radius of flexure of the geodesic of the secondary sub-amplitude in the same direction. The line OY, in magnitude and direction, represents the radius of circular curvature of the amplitudinal geodesic; and OC_1, OC_2, OC, are the perpendiculars from O on the respective lines YF_1, YF_2, YF. The points C_1 and C_2 are the centres of circular curvature of geodesics of the simple sub-amplitudes $\theta_1=0$ and $\theta_2=0$ respectively; and the point C is the centre of circular

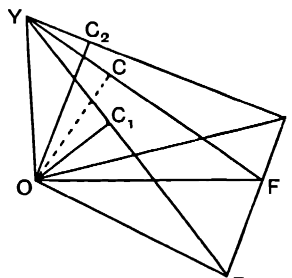

FIG. 3.

curvature of the geodesic of the secondary sub-amplitude, all the geodesics being drawn in the same direction $x_1', \dots, x_n'$. The five points, O, C_1, C, C_2, Y, lie on a sphere on OY as diameter. The four points Y, C_1, C, C_2, lie on a circle in the plane YF_1F_2; and the point C is the intersection of the circle through Y, C_1, C_2, by the line YF drawn perpendicular to the line F_1F_2, joining the centres of flexure of the geodesics belonging to the two simple sub-amplitudes respectively. The aggregate of analytical relations is

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{l}{\gamma}, \quad \frac{Y_1}{\rho_1}=\frac{Y}{\rho}+\frac{l_1}{\gamma_1}, \quad \frac{Y_2}{\rho_2}=\frac{Y}{\rho}+\frac{l_2}{\gamma_2},$$

$$\frac{l}{\gamma}=c_1l_1+c_2l_2, \quad \frac{l}{\gamma_1}=c_1+c_2\cos\zeta, \quad \frac{l}{\gamma_2}=c_1\cos\zeta+c_2,$$

$$\gamma_1\cos\chi_1=\gamma_2\cos\chi_2=\gamma, \quad \chi_1+\chi_2=\zeta,$$

$$\frac{\cos\psi_0}{\rho_0}=\frac{\cos\psi_1}{\rho_1}=\frac{\cos\psi_2}{\rho_2}=\frac{1}{\rho},$$

$$\frac{\sin\psi_0}{\rho_0}=\frac{1}{\gamma}, \quad \frac{\sin\psi_1}{\rho_1}=\frac{1}{\gamma_1}, \quad \frac{\sin\psi_2}{\rho_2}=\frac{1}{\gamma_2},$$

where $FOF_1=\chi_1$, $FOF_2=\chi_2$, $YOC_1=\psi_1$, $YOC_2=\psi_2$, $YOC=\psi_0$.

Similarly for a tertiary sub-amplitude, defined by three equations $\theta_1=0$, $\theta_2=0$, $\theta_3=0$, collectively. The six points O, Y, C_1, C_2, C_3, C, lie on a globe; the five points Y, C_1, C_2, C_3, C, lie on a sphere; and the point C is the intersection of a sphere through the points Y, C_1, C_2, C_3, by a line YF drawn perpendicular to the plane $F_1F_2F_3$, through the centres of flexure of geodesics belonging respectively to the three simple sub-amplitudes $\theta_1=0$, $\theta_2=0$, $\theta_3=0$, all the geodesics being drawn in the same direction.

73. When a configuration of $n-r$ dimensions is given in the n-fold amplitude by r parametric equations

$$\theta_i(x_1, \dots, x_n)=0, \quad (i=1, \dots, r),$$

the curvature of a geodesic in the configuration is given, in direction and in magnitude, by the typical equation

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{l}{\gamma},$$

where $1/\rho$ is the circular curvature of an amplitudinal geodesic in the same direction, and where

$$\frac{l}{\gamma}=k_1\frac{dy}{dn_1}+k_2\frac{dy}{dn_2}+\dots+k_r\frac{dy}{dn_r},$$

while $\frac{dy}{dn_i}$ is a typical spatial direction-cosine of the amplitudinal normal to the sub-amplitude $\theta_i=0$, and

$$k_1+k_2\cos\widehat{12}+\ldots+k_r\cos\widehat{1r}=\frac{1}{\gamma_1},$$

$$k_1\cos\widehat{21}+k_2+\ldots+k_r\cos\widehat{2r}=\frac{1}{\gamma_2},$$

....

$$k_1\cos\widehat{r1}+k_2\cos\widehat{r2}+\ldots+k_r=\frac{1}{\gamma_r},$$

the quantities $\gamma_1, \ldots, \gamma_r$ being the radii of amplitudinal flexure of geodesics in the respective sub-amplitudes $\theta_1=0, \ldots, \theta_r=0$.

In the configuration of $m(=n-r)$ dimensions, let the parametric variables be taken to be $x_1, \ldots, x_m$, so that the quantities $x_{m+1}, \ldots, x_n$, are then given by the r parametric equations defining the configuration. With the customary expressions for the radii of amplitudinal flexure, we shall have expressions of the form

$$\frac{1}{\gamma_1}=\sum_i^n\sum_j^n a_{ij}x_i'x_j'=\sum_i^m\sum_j^m \bar{a}_{ij}x_i'x_j',$$

$$\frac{1}{\gamma_2}=\sum_i^n\sum_j^n b_{ij}x_i'x_j'=\sum_i^m\sum_j^m \bar{b}_{ij}x_i'x_j',$$

....

$$\frac{1}{\gamma_r}=\sum_i^n\sum_j^n k_{ij}x_i'x_j'=\sum_i^m\sum_j^m \bar{k}_{ij}x_i'x_j',$$

with the customary relations of the type

$$\bar{a}_{ij}=a_{ij}+\sum_\alpha a_{m+\alpha,j}\frac{\partial x_{m+\alpha}}{\partial x_i}+\sum_\beta a_{i,m+\beta}\frac{\partial x_{m+\beta}}{\partial x_j}$$
$$+\sum_\alpha\sum_\beta a_{m+\alpha,m+\beta}\frac{\partial x_{m+\alpha}}{\partial x_i}\frac{\partial x_{m+\beta}}{\partial x_j},$$

while the quantities $\frac{\partial x_{m+\alpha}}{\partial x_i}$ are given by the equations

$$\frac{\partial\theta_p}{\partial x_i}+\sum_\alpha\frac{\partial\theta_p}{\partial x_{m+\alpha}}\frac{\partial x_{m+\alpha}}{\partial x_i}=0,$$

for the values $p=1, \ldots, r$, and $i=1, \ldots, m$. Also we take

$$\frac{Y}{\rho}=\sum_i^n\sum_j^n \eta_{ij}x_i'x_j'$$
$$=\sum_i^m\sum_j^m \bar{\eta}_{ij}x_i'x_j'$$

with the same law of relation for the quantities $\bar{\eta}_{ij}$ in terms of η_{ij} as connects $\bar{a}_{ij}$ with the magnitudes a_{ij}. Further, we write

$$\left.\begin{aligned} k_1 &= \sum_i^m \sum_j^m a_{ij} x_i' x_j' \\ k_2 &= \sum_i^m \sum_j^m \beta_{ij} x_i' x_j' \\ &\dots\dots\dots\dots \\ k_r &= \sum_i^m \sum_j^m \kappa_{ij} x_i' x_j' \end{aligned}\right\},$$

so that the coefficients in k_p, for $p=1, \dots, r$, are given by relations

$$\begin{aligned} a_{ij} \qquad\qquad &+ \beta_{ij} \cos \widehat{12} + \dots + \kappa_{ij} \cos \widehat{1r} = \bar{a}_{ij}, \\ a_{ij} \cos \widehat{21} + \beta_{ij} \qquad &+ \dots + \kappa_{ij} \cos \widehat{2r} = \bar{b}_{ij}, \\ &\dots\dots\dots\dots\dots\dots \\ a_{ij} \cos \widehat{r1} + \beta_{ij} \cos \widehat{r2} &+ \dots + \kappa_{ij} \qquad\qquad = \bar{k}_{ij}. \end{aligned}$$

Let

$$\frac{Y_0}{\rho_0} = \sum_i^m \sum_j^m Z_{ij} x_i' x_j' ;$$

and, for all values of i and j, let

$$\xi_{ij} = a_{ij} \frac{dy}{dn_1} + \beta_{ij} \frac{dy}{dn_2} + \dots + \kappa_{ij} \frac{dy}{dn_r},$$

so that

$$\frac{l}{\gamma} = \sum_i^m \sum_j^m \xi_{ij} x_i' x_j'.$$

As the only relation affecting the magnitudes $x_1', \dots, x_m'$, is the permanent arc-relation which now has the form

$$\sum_i^m \sum_j^m \bar{A}_{ij} x_i' x_j' = 1,$$

where the magnitudes $\bar{A}_{ij}$ bear to the primary magnitudes of the whole amplitude the same algebraical relation as the magnitudes $\bar{a}_{ij}$ in $1/\gamma_1$ bear to the magnitudes a_{ij}, there is no homogeneous relation of the second order among the magnitudes $x_1', \dots, x_m'$. Hence a comparison of the coefficients of the various terms in the equation

$$\frac{Y_0}{\rho_0} = \frac{Y}{\rho} + \frac{l}{\gamma},$$

after substitution for the members in terms of $x_1', \dots, x_m'$, leads to an aggregate of relations

$$Z_{ij} = \bar{\eta}_{ij} + \xi_{ij},$$

for $i, j, =1, \ldots, m$, taken independently of one another. Moreover, as $\bar{\eta}_{ij}$ is a homogeneous linear combination of quantities η_{hk}, and as (p. 29)

$$\sum \frac{\partial y}{\partial x_l} \eta_{hk}=0,$$

for all values of h, k, l, we have

$$\sum \frac{\partial y}{\partial x_l} \bar{\eta}_{ij}=0.$$

The quantities ξ_{hk} are linear combinations of all the n quantities $\frac{\partial y}{\partial x_l}$; and therefore

$$\sum \xi_{hk} \bar{\eta}_{ij}=0,$$

for all the combinations $h, k, i, j, =1, \ldots, m$, independently of one another, the summation being over the dimension-range of the plenary homaloidal space.

Geodesic surfaces of an amplitude.

74. The least extensive sub-amplitude contained in any n-fold amplitude is a curve, as given by $n-1$ equations

$$\theta_\mu(x_1, \ldots, x_n)=0, \quad (\mu=1, \ldots, n-1),$$

among the n parameters of the amplitude. The relations of a curve, thus defined, to the characteristic magnitudes of the containing amplitude will be sufficiently illustrated later, by considering the properties of a curve in specific amplitudes such as a surface and a region.

The next most restricted sub-amplitudes thus contained in any n-fold amplitude constitute surfaces. In the general aspect, they can be regarded as given by $n-2$ relations affecting the n parameters of the amplitude. Viewed in this analytical aspect, their relations to the characteristic magnitudes of the containing amplitude will be illustrated later, when we consider parametric surfaces in a free region and parametric surfaces in a free domain.

But there is one aggregate of contained surfaces which have a special organic relation to the containing amplitude; usually, they are styled *geodesic surfaces*, and they occur in the following manner. Within the amplitude, we select any arbitrary superficial orientation at O, the central point $y_1, y_2, \ldots$, of reference: the orientation can be regarded as determined by two different directions with sets of direction-variables $x_1', \ldots, x_n'$, and $z_1', \ldots, z_n'$, so that all directions in the orientation have direction-variables included in the set

$$\alpha x_1'+\beta z_1', \quad \alpha x_2'+\beta z_2', \ldots, \quad \alpha x_n'+\beta z_n',$$

where α and β are arbitrary parameters. Amplitudinal geodesics are drawn through all these directions in the orientation; and they generate a surface. Every geodesic of the surface through O is a geodesic of the amplitude, and each

amplitudinal geodesic through O, with a direction originating in the orientation, is a superficial geodesic; the surface is geodesic to the amplitude at O. Such a surface, however, is not necessarily (nor is it generally) geodesic to the amplitude over its whole extent; thus, if P and Q be two points on geodesics through O, so that OP and OQ are alike superficial geodesics and amplitudinal geodesics, the superficial geodesic PQ is not necessarily (nor is it generally) an amplitudinal geodesic, though of course there will be amplitudinal geodesic tangents to PQ at every point of its course on the surface. If, however, the superficial geodesic PQ is also an amplitudinal geodesic, for all choices of P and Q, the surface is said to be everywhere geodesic to the amplitude. It will appear that, even in the least extensive amplitudes such as regions and domains, the equations of surfaces everywhere geodesic are bound to satisfy a number of simultaneous partial differential equations of the second order; and it is a matter of investigation to determine the geodesic surfaces which an amplitude does contain and which of these surfaces are everywhere geodesic to the amplitude.

Surfaces, geodesic to the amplitude at a point, can be drawn through any orientation at the point; and their importance lies in the fact that a certain measure, often called the Riemann measure, of curvature of the surface so constructed is taken to be the measure of curvature of the amplitude in that orientation. The measure in question was propounded *, without proof, by Riemann in the form

$$\frac{\sum_i \sum_j \sum_k \sum_l (ij,\, kl)(x_i' z_j' - x_j' z_i')(x_k' z_l' - x_l' z_k')}{(\sum_i \sum_j A_{ij} x_i' x_j')(\sum_i \sum_j A_{ij} z_i' z_j') - (\sum_i \sum_j A_{ij} x_i' z_j')^2},$$

and in a note by Dedekind and Weber, based upon manuscript fragments of Riemann's, it is shewn † that the measure bears the same relation to the geodesic surface of the amplitude as does its simplest form

$$\frac{(12,\, 12)}{A_{11} A_{22} - A_{12}^2}$$

to a surface in free space. The significance of the measure in the last instance, when the free homaloidal space is triple, is what sometimes is called the Gauss measure, being the product of the two principal circular curvatures of the surface. But surfaces, existing only in more extensive plenary spaces, do not admit this particular interpretation: for instance, all such surfaces have four principal circular curvatures, and there is no obvious relation between these circular curvatures and the Riemann measure.

A discussion of the matter, and in particular the derivation of a geometrical significance for the Riemann measure, will be deferred until the measure is

* *Ges. math. Werke*, (1892), p. 403.

† *l.c.* pp. 405-412.

considered for its simplest form, this arising in the instance of a free surface (Chap. IX), where it will be proved that the magnitude, obviously of dimensions minus two in length, is the same, at the point, as the measure for a sphere, being the reciprocal of the square of the radius; and, because of the relation to the area of a small geodesic triangle on the surface, the measure is called the *sphericity*. Meanwhile, one property (based upon the analysis of § 73) may be noted. With that notation (p. 190), we have

$$(ij,\, kl) = \sum (Z_{ik}Z_{jl} - Z_{il}Z_{jk}),$$

when the circular curvature of a geodesic in a sub-amplitude and the direction of its prime normal are given by

$$\frac{Y_0}{\rho_0} = \sum_i \sum_j Z_{ij} x_i' x_j'.$$

Accordingly, we denote the value of the four-index symbol for the sub-amplitude itself, relative to the plenary homaloidal space of the whole configuration, by $(ij,\, kl)_s$; its value for the amplitude, of course relative to that plenary space, by $(ij,\, kl)_a$; and its value for the sub-amplitude, relative to the amplitude solely, by $(ij,\, kl)_f$. Thus $(ij,\, kl)_s$ is related to the circular curvature of a geodesic in the sub-amplitude, $(ij,\, kl)_f$ to the amplitudinal flexure of that geodesic, and $(ij,\, kl)_a$ to the circular curvature of the amplitudinal geodesic. The corresponding Riemann measures are denoted by R_s, R_a, R_f. Now we have

$$\begin{aligned}(ij,\, kl)_s &= \sum (Z_{ik}Z_{jl} - Z_{il}Z_{jk}) \\ &= \sum (\xi_{ik}\xi_{jl} - \xi_{il}\xi_{jk}) + \sum (\bar{\eta}_{ik}\bar{\eta}_{jl} - \bar{\eta}_{il}\bar{\eta}_{jk}),\end{aligned}$$

because of the relations

$$\sum \bar{\eta}_{ij}\xi_{kl} = 0$$

for all combinations of i, j, k, l: that is,

$$(ij,\, kl)_s = (ij,\, kl)_a + (ij,\, kl)_f.$$

The direction-variables in all the Riemann measures are the same for the geodesic of the sub-amplitude as for the geodesic of the amplitude, because the originating direction is common to the two geodesics; and the denominator in the expression for the Riemann measures R_s, R_a, R_f, is therefore the same for all three. Hence we have

$$R_s = R_a + R_f.$$

Two simple instances may be adduced. The simplest of all occurs, when we have to deal with a surface in a triple plenary space, being the customary Gauss surface. We may regard the triple space as a section of a quadruple space. Then R_a vanishes, and therefore

$$R_s = R_j\,;$$

that is, the measure relative to the triple space is the measure relative to the quadruple space.

The other example occurs when a surface exists in a region for which the plenary homaloidal space is quadruple. We now have R_a as the Riemann measure of curvature of the region in the orientation; and, in fact, R_a is equal to the product of the two principal circular curvatures of regional geodesics originating in that orientation, so that we can take

$$R_a = \frac{1}{\rho_1 \rho_2}$$

in this instance. Also, it is found that

$$R_f = \frac{1}{\gamma_1 \gamma_2},$$

being the Gauss (or Riemann) measure of regional flexure of the surface; and so we have

$$R_s = \frac{1}{\rho_1 \rho_2} + \frac{1}{\gamma_1 \gamma_2},$$

as an interpretation of the measure of curvature of the specified surface *.

Minimal surfaces in an amplitude.

75. The intrinsic equations, characteristic of minimal surfaces in an amplitude, are obtainable in a manner similar to that which is used for the formation of the intrinsic equations of geodesic curves. Any surface in an amplitude can be represented by formulating the parameters of the amplitude as functions of two new independent variables (say u and v) which become the parameters of the surface. Under this formulation, the expression for the superficial arc (which is the specialised arc of the amplitude) is

$$ds^2 = A_0 du^2 + 2H_0 du\, dv + B_0 dv^2,$$

where

$$A_0 = \sum_i \sum_j A_{ij} \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial u}, \quad B_0 = \sum_i \sum_j A_{ij} \frac{\partial x_i}{\partial v} \frac{\partial x_j}{\partial v},$$

$$H_0 = \sum_i \sum_j A_{ij} \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial v}.$$

(It will be convenient to write

$$x_m^{(1)} = \frac{\partial x_m}{\partial u}, \quad x_m^{(2)} = \frac{\partial x_m}{\partial v},$$

for all values of m.) Then the area of the surface, between any assigned limits, is

$$S = \iint V du\, dv,$$

where

$$V = (A_0 B_0 - H_0^2)^{\frac{1}{2}}.$$

* See *G.F.D.*, vol. ii., § 368.

To obtain a minimal surface, this double integral has to be made a minimum, the dependent variables being the original parameters $x_1, \dots, x_n$; and therefore * the critical equations are

$$-\frac{\partial V}{\partial x_m}+\frac{d}{du}\left\{\frac{\partial V}{\partial x_m^{(1)}}\right\}+\frac{d}{dv}\left\{\frac{\partial V}{\partial x_m^{(2)}}\right\}=0,$$

for all the values $m=1, \dots, n$.

In the first place, it is to be noted that the nul-surfaces of the amplitude satisfy these characteristic critical equations, a result analogous to the property (§ 18) that the nul-lines of an amplitude satisfy the intrinsic equations of its geodesics. To establish the statement, we have

$$\frac{\partial V}{\partial x_m}=\frac{1}{2V}\left(B_0\frac{\partial A_0}{\partial x_m}-2H_0\frac{\partial H_0}{\partial x_m}+A_0\frac{\partial B_0}{\partial x_m}\right)=\frac{R_m}{V},$$

$$\frac{\partial V}{\partial x_m^{(1)}}=\frac{1}{V}\left\{B_0\left(\sum_j A_{mj}\frac{\partial x_j}{\partial u}\right)-H_0\sum_j\left(A_{mj}\frac{\partial x_j}{\partial v}\right)\right\}=\frac{P_m}{V},$$

$$\frac{\partial V}{\partial x_m^{(2)}}=\frac{1}{V}\left\{A_0\left(\sum_j A_{mj}\frac{\partial x_j}{\partial v}\right)-H_0\sum_j\left(A_{mj}\frac{\partial x_j}{\partial u}\right)\right\}=\frac{Q_m}{V},$$

and therefore the critical equation, specially associated with x_m, is

$$\frac{d}{du}\left(\frac{P_m}{V}\right)+\frac{d}{dv}\left(\frac{Q_m}{V}\right)=\frac{R_m}{V},$$

so that

$$P_m\frac{dV}{du}+Q_m\frac{dV}{dv}=V\left(\frac{dP_m}{du}+\frac{dQ_m}{du}-R_m\right).$$

There is one such equation for each value of m, $=1, \dots, n$.

Manifestly, all the equations are satisfied by

$$V=0,$$

as a permanent relation: that is, the nul-surfaces of the amplitude satisfy the intrinsic equations of the minimal surfaces of the amplitude.

The nul-surfaces may now be set on one side: we return to the m critical equations in their initial form. As yet, no choice of the superficial parameters has been made; for convenience of analysis, we choose them to be the parameters of the nul-lines of the surface, so that

$$A_0=\sum_i\sum_j A_{ij}\frac{\partial x_i}{\partial u}\frac{\partial x_j}{\partial u}=0,\quad B_0=\sum_i\sum_j A_{ij}\frac{\partial x_i}{\partial v}\frac{\partial x_j}{\partial v}=0.$$

(There are two such lines in every superficial orientation.) Thus

$$V=iH_0,$$

so that

$$\frac{\partial V}{\partial x_m}=i\frac{\partial H_0}{\partial x_m},\quad \frac{\partial V}{\partial x_m^{(1)}}=i\sum_j\left(A_{mj}\frac{\partial x_j}{\partial v}\right),\quad \frac{\partial V}{\partial x_m^{(2)}}=i\sum_j\left(A_{mj}\frac{\partial x_j}{\partial u}\right);$$

* See my *Calculus of Variations*, chap. ix.

and now the x_m-critical equation is

$$\frac{d}{du}\left\{\sum_j\left(A_{mj}\frac{\partial x_j}{\partial v}\right)\right\}+\frac{d}{dv}\left\{\sum_j\left(A_{mj}\frac{\partial x_j}{\partial u}\right)\right\}=\frac{\partial H_0}{\partial x_m}.$$

Consequently, we have

$$2\sum_j\left(A_{mj}\frac{\partial^2 x_j}{\partial u\,\partial v}\right)=\frac{\partial H_0}{\partial x_m}-\sum_\beta\sum_l\left(\frac{\partial A_{m\beta}}{\partial x_l}\frac{\partial x_\beta}{\partial v}\frac{\partial x_l}{\partial u}\right)-\sum_a\sum_k\left(\frac{\partial A_{ma}}{\partial x_k}\frac{\partial x_a}{\partial u}\frac{\partial x_k}{\partial v}\right).$$

Also

$$\frac{\partial H_0}{\partial x_m}=\sum_\lambda\sum_\mu\frac{\partial A_{\lambda\mu}}{\partial x_m}\frac{\partial x_\lambda}{\partial u}\frac{\partial x_\mu}{\partial v}.$$

The aggregate of terms on the right-hand side, involving derivatives of x_λ with regard to u and of x_μ with regard to v, is

$$\begin{aligned}&=\frac{\partial A_{\lambda\mu}}{\partial x_m}\frac{\partial x_\lambda}{\partial u}\frac{\partial x_\mu}{\partial v}-\frac{\partial A_{m\mu}}{\partial x_\lambda}\frac{\partial x_\mu}{\partial v}\frac{\partial x_\lambda}{\partial u}-\frac{\partial A_{m\lambda}}{\partial x_\mu}\frac{\partial x_\lambda}{\partial u}\frac{\partial x_\mu}{\partial v}\\ &=\left(\frac{\partial A_{\lambda\mu}}{\partial x_m}-\frac{\partial A_{m\lambda}}{\partial x_\mu}-\frac{\partial A_{m\mu}}{\partial x_\lambda}\right)\frac{\partial x_\lambda}{\partial u}\frac{\partial x_\mu}{\partial v}\\ &=-2\frac{\partial x_\lambda}{\partial u}\frac{\partial x_\mu}{\partial v}\sum_j\left[A_{mj}\{\lambda\mu,j\}\right],\end{aligned}$$

by the result in § 12. Hence the x_m-critical equation becomes

$$\sum_j A_{mj}\left[\frac{\partial^2 x_j}{\partial u\,\partial v}+\sum_\lambda\sum_\mu\{\lambda\mu,j\}\frac{\partial x_\lambda}{\partial u}\frac{\partial x_\mu}{\partial v}\right]=0.$$

This result holds for $m=1, \ldots, n$; and the quantity Ω does not vanish. When these equations are resolved, we have

$$\frac{\partial^2 x_j}{\partial u\,\partial v}+\sum_\lambda\sum_\mu\{\lambda\mu,j\}\frac{\partial x_\lambda}{\partial u}\frac{\partial x_\mu}{\partial v}=0,$$

holding for $j=1, \ldots, n$.

These accordingly are the equations, which determine the amplitudinal parameters x in terms of the two variables u and v so that the resulting surface becomes a minimal surface in the amplitude. The equations, apparently, are n in number; but there are two integrals of the first order,

$$\sum_i\sum_j A_{ij}\frac{\partial x_i}{\partial u}\frac{\partial x_j}{\partial u}=0,\quad \sum_i\sum_j A_{ij}\frac{\partial x_i}{\partial v}\frac{\partial x_j}{\partial v}=0;$$

when these are retained, the system contains $n-2$ independent equations.

CHAPTER VII

Primary General Amplitudes

Normal to the amplitude.

76. Considerable simplifications in the analytical expressions of many properties of an amplitude of n dimensions appear when the amplitude is primary to its plenary space: that is, when the plenary space has $n+1$ dimensions, one greater in number than those of the amplitude.

In the preliminary investigations, no new relations or special limitations are requisite beyond those already specified. The quantities A_{ij}, a_{ij}, $[ij, k]$, $\{ij, k\}$, (ij, kl), remain with unaltered significance of definition. The element of arc is still given by

$$ds^2 = \sum_i \sum_j A_{ij} dx_i\, dx_j\, ;$$

and the n intrinsic equations of amplitudinal geodesics still are

$$x_k'' + \sum_i \sum_j \{ij, k\} x_i' x_j' = 0,$$

for $k=1, \ldots, n$, being equivalent to only $n-1$ independent equations owing to the permanent relation arising from the arc-element

$$\sum_i \sum_j A_{ij} x_i' x_j' = 1.$$

The first significant simplification occurs in connection with the perpendicular from a neighbouring point of the amplitude upon the tangent homaloid. Instead of depending for direction upon the direction of a geodesic, for which it becomes a prime normal when the neighbouring points coincide, it now is a unique line at a point O of the amplitude, and is the direction common to the prime normals of all amplitudinal geodesics through O. Its direction-cosines $Y_1, \ldots, Y_{n+1}$, are magnitudes of position only, being independent of any set of direction-variables $x_1', \ldots, x_n'$, at O. It remains, of course, orthogonal to the n-fold tangent homaloid of the amplitude, and so remains at right angles to every direction in that n-fold homaloid; we therefore have n relations

$$\sum_m Y_m \frac{\partial y_m}{\partial x_r} = \sum Y \frac{\partial y}{\partial x_r} = 0,$$

for $r=1, \ldots, n$, the summation on the left-hand side being over the values

$m=1, \ldots, n+1$, for the dimensions of the plenary space. As there are $n+1$ such quantities Y in these n relations, we have

$$Y_1, \ldots, Y_{n+1} = \frac{1}{\Omega^{\frac{1}{2}}} \left\| \begin{array}{cccc} \dfrac{\partial y_1}{\partial x_1}, & \dfrac{\partial y_2}{\partial x_1}, & \ldots, & \dfrac{\partial y_{n+1}}{\partial x_1} \\ \dfrac{\partial y_1}{\partial x_2}, & \dfrac{\partial y_2}{\partial x_2}, & \ldots, & \dfrac{\partial y_{n+1}}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial y_1}{\partial x_n}, & \dfrac{\partial y_2}{\partial x_n}, & \ldots, & \dfrac{\partial y_{n+1}}{\partial x_n} \end{array} \right\|,$$

where Ω is the determinant of the primary magnitudes A_{ij}, and the common factor $\Omega^{-\frac{1}{2}}$ is determined by the requirement

$$\sum_m Y_m^2 = 1.$$

The n-fold tangent homaloid of the amplitude now can be represented by the single equation

$$\sum_m \{(\bar{y}_m - y_m) Y_m\} = 0,$$

with the foregoing values of $Y_1, \ldots, Y_{n+1}$; and the point-representation

$$\bar{y}_m - y_m = \sum_r \lambda_r \frac{\partial y_m}{\partial x_r} \quad (r=1, \ldots, n),$$

is still valid, with parametric quantities $\lambda_1, \ldots, \lambda_n$.

The $(n+1)$-fold homaloid, osculating the geodesic, now becomes the plenary space itself; and a set of axes of spatial reference at O is constituted by the n parametric lines, having direction-cosines proportional to

$$\frac{\partial y_1}{\partial x_r}, \quad \frac{\partial y_2}{\partial x_r}, \quad \ldots, \quad \frac{\partial y_{n+1}}{\partial x_r}, \quad (r=1, \ldots, n),$$

together with the normal to the amplitude.

The next simplification arises in connection with the secondary magnitudes L_{ij} of the amplitude, defined originally in connection with a specified geodesic. They now become magnitudes of position only, no longer having any intrinsic association with any specific direction. The original definition of a magnitude L_{ij} is unaltered in form, still being

$$L_{ij} = \sum_m Y_m \frac{\partial^2 y_m}{\partial x_i \, \partial x_j},$$

for all values of $i, j, =1, \ldots, n$; the independence of direction is due to the fact that the direction-cosines of the prime normal are independent of directions at the point. As before, the circular curvature of a geodesic through the direction $x_1', \ldots, x_n'$, is given by the unchanged formula

$$\frac{1}{\rho} = \sum_i \sum_j L_{ij} x_i' x_j';$$

equally unchanged are the formulæ

$$\frac{Y_m}{\rho}=\sum_i \sum_j \left[\frac{\partial^2 y_m}{\partial x_i \partial x_j}-\sum_k \frac{\partial y_m}{\partial x_k}\{ij,\, k\}\right] x_i' x_j' = \sum_i \sum_j \eta_{ij}{}^{(m)} x_i' x_j'.$$

But in estimating the circular curvature of the geodesic, the foregoing formula involving the undirected magnitudes L_{ij} suffices : there is no longer a necessity to take the magnitude $\sum_m (Y_m/\rho)^2$.

77. Further, the curves of curvature become simplified, as regards their determination. Assuming one of the customary definitions—that they are the curves along which the circular curvature of a tangential geodesic is a maximum or a minimum among the circular curvatures for all possible directions of geodesics —we have to make the magnitude of $1/\rho$, as already given, a maximum or a minimum for all admissible values of $x_1', \ldots, x_n'$, that is, for all values of $x_1', \ldots, x_n'$, satisfying the relation

$$\sum_i \sum_j A_{ij} x_i' x_j' = 1.$$

The critical conditions are

$$\sum_j L_{ij} x_j' = \mu \sum_j A_{ij} x_j',$$

for $i=1, \ldots, n$, the quantity μ being undetermined in the formation of the critical equations. To determine μ, we multiply this typical equation by x_i' and add for all the values of i; then

$$\frac{1}{\rho}=\mu,$$

and the n critical equations become

$$\sum_j L_{ij} x_j' = \frac{1}{\rho} \sum_j A_{ij} x_j', \quad (i=1, \ldots, n).$$

The elimination of the direction-variables $x_1', \ldots, x_n'$, leads to the equation

$$\begin{vmatrix} L_{11}-\frac{1}{\rho}A_{11}, & L_{12}-\frac{1}{\rho}A_{12}, & \ldots, & L_{1n}-\frac{1}{\rho}A_{1n} \\ L_{21}-\frac{1}{\rho}A_{21}, & L_{22}-\frac{1}{\rho}A_{22}, & \ldots, & L_{2n}-\frac{1}{\rho}A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ L_{n1}-\frac{1}{\rho}A_{n1}, & L_{n2}-\frac{1}{\rho}A_{n2}, & \ldots, & L_{nn}-\frac{1}{\rho}A_{nn} \end{vmatrix} = 0,$$

of degree n, determining what are called the principal linear curvatures of the amplitude. If the n values of ρ, satisfying the equation, are denoted by $\rho_1, \rho_2, \ldots, \rho_n$, the symmetric combinations

$$\sum \frac{1}{\rho_1}, \quad \sum \frac{1}{\rho_1 \rho_2}, \ldots, \quad \sum \frac{1}{\rho_1 \rho_2 \cdots \rho_{n-1}}, \quad \frac{1}{\rho_1 \rho_2 \cdots \rho_n},$$

are called the principal measures of linear curvature of the amplitude. Let Υ denote the determinant

$$\begin{vmatrix} L_{11}, & L_{12}, & \dots, & L_{1n} \\ L_{21}, & L_{22}, & \dots, & L_{2n} \\ \dots & \dots & \dots & \dots \\ L_{n1}, & L_{n2}, & \dots, & L_{nn} \end{vmatrix};$$

then

$$\frac{1}{\rho_1\rho_2\dots\rho_n}=\frac{\Upsilon}{\Omega},$$

leading to the geometric mean of the principal linear curvatures. It is a quantity that may be regarded as a corporate measure of curvature of the amplitude; but it is only a single one out of the n measures. Also we have

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}+\dots+\frac{1}{\rho_n}=\frac{1}{\Omega}\sum_i\sum_j L_{ij}a_{ij},$$

leading to the arithmetic mean of the principal linear curvatures. It is a magnitude that vanishes for n-fold amplitudes in homaloidal $(n+1)$-fold space which have minimal content.

Further, for each of the principal linear curvatures, that is, for each of the values $\rho=\rho_1, \rho_2, \dots, \rho_n$, the original critical equations determine a unique set of values of the quantities $x_1', x_2', \dots, x_n'$, in general: that is, when no two of the principal linear curvatures are equal. Thus there is a principal direction, corresponding to each of the values $\rho_1, \dots, \rho_n$: when continued from point to point in one of these directions, it leads to a curve of curvature: and, consequently, there are n curves of curvature at any point of the primary amplitude.

Moreover, these directions are at right angles to one another in pairs. Selecting any two directions; and denoting them by $x_1', x_2', \dots, x_n'$, for a principal linear curvature $1/\rho_1$, and $\bar{x}_1', \bar{x}_2', \dots, \bar{x}_n'$, for a principal linear curvature $1/\rho_2$, we have

$$\sum_j L_{ij}x_j'=\frac{1}{\rho_1}\sum_j A_{ij}x_j', \quad \sum_j L_{ij}\bar{x}_j'=\frac{1}{\rho_2}\sum_j A_{ij}\bar{x}_j', \quad (i=1, \dots, n).$$

Multiply the typical equation in the first set by $\bar{x}_i'$, and add for all values of i: then

$$\sum_i\sum_j L_{ij}\bar{x}_i'x_j'=\frac{1}{\rho_1}\sum_i\sum_j A_{ij}\bar{x}_i'x_j'.$$

Multiply the typical equation in the second set by x_i', and add for all values of i: then

$$\sum_i\sum_j L_{ij}x_i'\bar{x}_j'=\frac{1}{\rho_2}\sum_i\sum_j A_{ij}x_i'\bar{x}_j'.$$

But

$$\sum_i\sum_j L_{ij}\bar{x}_i'x_j'=\sum_i\sum_j L_{ij}x_i'\bar{x}_j',$$

because each sum in the equation is homogeneous and linear in the two sets of variables x', $\bar{x}'$; and similarly

$$\sum_i \sum_j A_{ij}\bar{x}_i' x_j' = \sum_i \sum_j A_{ij} x_i' \bar{x}_j'.$$

Hence

$$\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \sum_i \sum_j A_{ij} x_i' \bar{x}_j' = 0:$$

or, on the assumption that ρ_1 and ρ_2 are unequal, that is, on the assumption that no two of the principal linear curvatures of the amplitude are equal, we have

$$\sum_i \sum_j A_{ij} x_i' \bar{x}_j' = 0.$$

Consequently, the two directions $x_1', \ldots, x_n'$, and $\bar{x}_1', \ldots, \bar{x}_n'$, are at right angles. They are chosen as the direction-variables of any two curves of curvature at the point; and therefore the directions of the n curves of curvature at any point of the amplitude are at right angles to one another in pairs.

Further, the normal to the whole amplitude is at right angles to each curve of curvature, because it is at right angles to the direction of any curve in the amplitude.

It follows that the directions of the n curves of curvature and the direction of the normal to the amplitude constitute an orthogonal system of $n+1$ directions in the plenary homaloidal space. Moreover, when the parametric curves for the primary amplitude are its curves of curvature, we have

$$A_{ij} = 0, \quad L_{ij} = 0,$$

if i and j are different from one another: the former, because the directions are orthogonal: the latter, because the critical equations (p. 199) of the lines of curvature must be satisfied. Consequently, for such parametric curves, the permanent arc-relation and the equation for the circular curvature of a geodesic in the primary amplitude respectively become

$$\sum_i A_{ii} x_i'^2 = 1, \quad \frac{1}{\rho} = \sum_i L_{ii} x_i'^2.$$

Partial equations of the second order, satisfied by point-coordinates.

78. The space-coordinates of any point in the amplitude satisfy a number of partial differential equations of the second order, the amplitudinal parameters being the independent variables.

For all values of $i, j, k, = 1, \ldots, n$, in all selections, we have

$$\begin{aligned}\sum_m \frac{\partial^2 y_m}{\partial x_i \partial x_j} \frac{\partial y_m}{\partial x_k} &= [ij, k] \\ &= \sum_\mu A_{k\mu} \{ij, \mu\} \\ &= \sum_m \frac{\partial y_m}{\partial x_k} \left(\sum_\mu \frac{\partial y_m}{\partial x_\mu} \{ij, \mu\}\right),\end{aligned}$$

so that

$$\sum_m \frac{\partial y_m}{\partial x_k}\left[\frac{\partial^2 y_m}{\partial x_i \partial x_j} - \sum_\mu \frac{\partial y_m}{\partial x_\mu}\{ij, \mu\}\right]=0,$$

or, with the notation of § 20,

$$\sum_m \frac{\partial y_m}{\partial x_k}\eta_{ij}^{(m)}=0.$$

Hence for any selected pair of values of i and j, there are n equations (arising from the n values of k) linear in the $n+1$ quantities $\frac{\partial^2 y_m}{\partial x_i \partial x_j}$ for $m=1, \ldots, n+1$. Also, we have had the equation

$$L_{ij}=\sum Y_m \frac{\partial^2 y_m}{\partial x_i \partial x_j}=\sum Y_m \eta_{ij}^{(m)},$$

because

$$\sum Y_m \frac{\partial y_m}{\partial x_r}=0,$$

for all values of r. We now have $n+1$ equations, linear in the $n+1$ quantities $\eta_{ij}^{(m)}$; and these equations have a unique set of solutions because the determinant

$$\begin{vmatrix} Y_1, & Y_2, & \ldots, & Y_{n+1} \\ \frac{\partial y_1}{\partial x_1}, & \frac{\partial y_2}{\partial x_1}, & \ldots, & \frac{\partial y_{n+1}}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_1}{\partial x_n}, & \frac{\partial y_2}{\partial x_n}, & \ldots, & \frac{\partial y_{n+1}}{\partial x_n} \end{vmatrix}$$

has a value $\Omega^{\frac{1}{2}}$, distinct from zero. The equations are clearly satisfied by

$$\eta_{ij}^{(m)}=Y_m L_{ij},$$

for $m=1, \ldots, n+1$, which accordingly provide the solutions: that is,

$$\frac{\partial^2 y_m}{\partial x_i \partial x_j}=Y_m L_{ij}+\{ij, 1\}\frac{\partial y_m}{\partial x_1}+\{ij, 2\}\frac{\partial y_m}{\partial x_2}+\ldots+\{ij, n\}\frac{\partial y_m}{\partial x_n}.$$

This equation holds for all the combinations $i, j, =1, \ldots, n$, taken for each integer independently of the other; and it holds for each of the space-coordinates, as given by $m=1, \ldots, n+1$.

The Riemann four-index symbols take a specially simple form for a primary amplitude. We have seen that, in general, they are given by the expression (§ 14)

$$(ij, kl)=\sum_m \left(\frac{\partial^2 y_m}{\partial x_i \partial x_k}\frac{\partial^2 y_m}{\partial x_j \partial x_l}-\frac{\partial^2 y_m}{\partial x_j \partial x_k}\frac{\partial^2 y_m}{\partial x_i \partial x_l}\right)$$
$$+\sum_\lambda \sum_\mu A_{\lambda\mu}[\{il, \mu\}\{jk, \lambda\}-\{ik, \mu\}\{jl, \lambda\}]$$

for any n-fold amplitude, the summations being for $\lambda, \mu, = 1, \dots, n$, independently of one another. For a primary amplitude, we have

$$\sum_m \frac{\partial^2 y_m}{\partial x_i \partial x_k} \frac{\partial^2 y_m}{\partial x_j \partial x_l} = L_{ik} L_{jl} + \sum_\lambda \sum_\mu A_{\lambda\mu} \{ik, \lambda\} \{jl, \lambda\},$$

$$\sum_m \frac{\partial^2 y_m}{\partial x_j \partial x_k} \frac{\partial^2 y_m}{\partial x_i \partial x_l} = L_{jk} L_{il} + \sum_\lambda \sum_\mu A_{\lambda\mu} \{jk, \lambda\} \{il, \mu\},$$

with the same summations for λ and μ, these values being obtained by direct substitution of the second derivatives of the point-coordinates from the foregoing characteristic equations. Hence

$$(ij, kl) = L_{ik} L_{jl} - L_{il} L_{jk},$$

the simple form of expression indicated; and it holds for all possible combinations of the values of i, j, k, l.

Next, for all values $\alpha, \beta, = 1, \dots, n$, we have

$$A_{\alpha\beta} = \sum_m \frac{\partial y_m}{\partial x_\alpha} \frac{\partial y_m}{\partial x_\beta},$$

and therefore, on substitution and reduction,

$$\tfrac{1}{2}\left(\frac{\partial^2 A_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 A_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 A_{jl}}{\partial x_i \partial x_k} + \frac{\partial^2 A_{jk}}{\partial x_i \partial x_l} \right) = \sum_m \left(\frac{\partial^2 y_m}{\partial x_i \partial x_k} \frac{\partial^2 y_m}{\partial x_j \partial x_l} - \frac{\partial^2 y_m}{\partial x_j \partial x_k} \frac{\partial^2 y_m}{\partial x_i \partial x_l} \right).$$

Consequently, when the expression of the right-hand side in terms of this combination of second derivatives of the magnitudes $A_{\alpha\beta}$ is used, it appears that the magnitude represented by the symbol

$$(ij, kl)$$

is completely expressible in terms of the first-order magnitudes: and this inference is valid for all combinations of the integers i, j, k, l. Hence, for all such combinations, every quantity

$$L_{ik} L_{jl} - L_{il} L_{jk}$$

is thus expressible.

Now consider a determinant

$$D = \begin{vmatrix} L_{ii'}, & L_{ij'}, & L_{ik'} \\ L_{ji'}, & L_{jj'}, & L_{jk'} \\ L_{ki'}, & L_{kj'}, & L_{kk'} \end{vmatrix},$$

and the determinant of the first minors of D, being

$$D^2 = \begin{vmatrix} L_{jj'}L_{kk'} - L_{jk'}L_{kj'}, & L_{jk'}L_{ki'} - L_{ji'}L_{kk'}, & L_{ji'}L_{kj'} - L_{jj'}L_{ki'} \\ L_{kj'}L_{ik'} - L_{kk'}L_{ij'}, & L_{kk'}L_{ii'} - L_{ki'}L_{ik'}, & L_{ki'}L_{ij'} - L_{kj'}L_{ii'} \\ L_{ij'}L_{jk'} - L_{ik'}L_{jj'}, & L_{ik'}L_{ji'} - L_{ii'}L_{jk'}, & L_{ii'}L_{jj'} - L_{ij'}L_{ji'} \end{vmatrix} = \begin{vmatrix} l_{ii'}, & l_{ij'}, & l_{ik'} \\ l_{ji'}, & l_{jj'}, & l_{kk'} \\ l_{ki'}, & l_{kj'}, & l_{kk'} \end{vmatrix}.$$

Every constituent of D^2 is expressible in terms of the first-order magnitudes; consequently D^2 is so expressible and therefore also D. Now

$$L_{ii'}D = l_{jj'}l_{kk'} - l_{jk'}l_{kj'}$$

with three like relations, and

$$L_{ij'}D = l_{jk'}l_{ki'} - l_{ji'}l_{kk'}$$

also with three like relations; hence, unless D should vanish, each of its constituents is expressible in terms of the first-order magnitudes.

It may be the fact that, for some particular selection of i, j, k, and some particular selection of i', j', k', the determinant D vanishes. But it cannot happen that, for the particular selection of i, j, k, and for all selections i', j', k', all the corresponding determinants D should vanish unless the complete determinant $|L_{ij}|$ vanishes, that is, unless the corporate measure of amplitude vanishes. This might occur at merely isolated places which would be of the nature of special points in the amplitude; if it did not otherwise occur, the inference would still be valid, and the contour of the amplitude in the immediate vicinity of the special point would be the subject of a separate investigation of a different character. If it happened that each determinant $|L_{ij}|$ vanished everywhere, the inference would be that at least one of the principal linear curvatures of the amplitude is everywhere zero, that at least one set of the curves of curvature is made up of straight lines, and that the amplitude can be developed about such lines in succession so as to lose one of its dimensions.

Excluding this possibility and assuming therefore that the corporate measure of the amplitudinal curvature is not zero, we infer that each of the magnitudes L_{ij}, that is, each of the secondary magnitudes of the amplitude, is expressible in terms of the primary magnitudes (and of the derivatives of such primary magnitudes).

Manifestly the inference cannot be drawn, if there is no determinant of the third degree in the magnitudes L_{ij}. Such is the fact when we have to deal with an ordinary surface in homaloidal triple space, that is, in the Gauss theory of such surfaces; and therefore the inference holds for primary amplitudes which are of more than two dimensions.

79. We shall require the first derivatives of the direction-cosines Y with respect to the parameters.

The relation

$$\sum_m Y_m \frac{\partial y_m}{\partial x_t} = 0$$

holds for all the n values of t. Differentiating with respect to x_k, we have

$$\sum_m \frac{\partial Y_m}{\partial x_k} \frac{\partial y_m}{\partial x_t} = -\sum_m Y_m \frac{\partial^2 y_m}{\partial x_t \partial x_k} = -L_{tk},$$

for all values of t and k. Now consider the expression

$$f(m, k)=\sum_i \sum_j \frac{\partial y_m}{\partial x_j} a_{ij} L_{ik}.$$

Let each side be multiplied by $\frac{\partial y_m}{\partial x_t}$, and let the results for all the values of m be added; then

$$\sum_m \frac{\partial y_m}{\partial x_t} f(m, k)=\sum_i \sum_j \sum_m \frac{\partial y_m}{\partial x_t} \frac{d y_m}{\partial x_j} a_{ij} L_{ik}$$
$$=\sum_i \sum_j A_{tj} a_{ij} L_{ik}.$$

The quantity $\sum_j A_{tj} a_{ij}$ is equal to zero when i differs from t, and it is equal to Ω when i is equal to t: thus the only emerging term on the right-hand side is ΩL_{tk}, and therefore

$$\frac{1}{\Omega} \sum_m \frac{\partial y_m}{\partial x_t} f(m, k)=L_{tk}.$$

Consequently

$$\sum_m \left\{\frac{\partial Y_m}{\partial x_k}+\frac{1}{\Omega} f(m, k)\right\} \frac{\partial y_m}{\partial x_t}=0.$$

Again, we have

$$\sum_m Y_m f(m, k)=\sum_i \sum_j \left(\sum_m Y_m \frac{\partial y_m}{\partial x_j}\right) a_{ij} L_{ik}=0;$$

and, because $\sum Y_m^2=1$, it follows that

$$\sum Y_m \frac{\partial Y_m}{\partial x_k}=0,$$

so that

$$\sum_m \left\{\frac{\partial Y_m}{\partial x_k}+\frac{1}{\Omega} f(m, k)\right\} Y_m=0.$$

Thus there are $m+1$ equations, homogeneous and linear in the $n+1$ magnitudes

$$\frac{\partial Y_m}{\partial x_k}+\frac{1}{\Omega} f(m, k)$$

for $m=1, \ldots, n+1$. The determinant of the coefficients of these magnitudes is equal to

$$\begin{vmatrix} Y_1, & Y_2, & \ldots, & Y_{n+1} \\ \frac{\partial y_1}{\partial x_1}, & \frac{\partial y_2}{\partial x_1}, & \ldots, & \frac{\partial y_{n+1}}{\partial x_1} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial y_1}{\partial x_n}, & \frac{\partial y_2}{\partial x_n}, & \ldots, & \frac{\partial y_{n+1}}{\partial x_n} \end{vmatrix}$$

that is, is equal to $\Omega^{\frac{1}{2}}$ so that it does not vanish; hence each of the $n+1$ magnitudes vanishes, so that

$$\begin{aligned}\frac{\partial Y_m}{\partial x_k} &= -\frac{1}{\Omega} f(m, k) \\ &= -\frac{1}{\Omega}\sum_i \sum_j a_{ij} L_{ik} \frac{\partial y_m}{\partial x_j},\end{aligned}$$

which holds for all the values $m=1, \dots, n+1$, and for all the values $k=1, \dots, n$.

The result may be established otherwise, as follows. The first of the quantities in question $\dfrac{\partial Y_1}{\partial x_k}$ is a linear vector magnitude, of a special form; for

$$Y_1 \Omega^{-\frac{1}{2}} = \begin{vmatrix} \dfrac{\partial y_2}{\partial x_1}, & \dots, & \dfrac{\partial y_{n+1}}{\partial x_1} \\ \dfrac{\partial y_2}{\partial x_2}, & \dots, & \dfrac{\partial y_{n+1}}{\partial x_2} \\ \dots & \dots & \dots \\ \dfrac{\partial y_2}{\partial x_n}, & \dots, & \dfrac{\partial y_{n+1}}{\partial x_n} \end{vmatrix}$$

so that, when we form the derivative

$$\frac{\partial}{\partial x_k}(Y_1 \Omega^{-\frac{1}{2}}),$$

the result is an aggregate of determinants of the type

$$\begin{vmatrix} \dfrac{\partial y_2}{\partial x_1}, & \dots, & \dfrac{\partial y_{n+1}}{\partial x_1} \\ \dots & \dots & \dots \\ \dfrac{\partial^2 y_2}{\partial x_i \partial x_k}, & \dots, & \dfrac{\partial^2 y_{n+1}}{\partial x_i \partial x_k} \\ \dots & \dots & \dots \\ \dfrac{\partial y_2}{\partial x_n}, & \dots, & \dfrac{\partial y_{n+1}}{\partial x_n} \end{vmatrix}.$$

When the values of the second derivatives in this determinant are substituted, there is a term

$$L_{ik} \begin{vmatrix} \dfrac{\partial y_2}{\partial x_1}, & \dots, & \dfrac{\partial y_{n+1}}{\partial x_1} \\ \dots & \dots & \dots \\ Y_2, & \dots, & Y_{n+1} \\ \dots & \dots & \dots \\ \dfrac{\partial y_2}{\partial x_n}, & \dots, & \dfrac{\partial y_{n+1}}{\partial x_n} \end{vmatrix},$$

which is proportional to $L_{ik}\dfrac{\partial y_1}{\partial x_i}$, the symmetrical coefficient not depending upon y_1, or Y_1, or derivatives of y_1, to the exclusion of other like quantities; there is a term

$$\{ik, i\}\begin{vmatrix} \dfrac{\partial y_2}{\partial x_1}, & \dots, & \dfrac{\partial y_{n+1}}{\partial x_1} \\ \dots & \dots & \dots \\ \dfrac{\partial y_2}{\partial x_i}, & \dots, & \dfrac{\partial y_{n+1}}{\partial x_i} \\ \dots & \dots & \dots \\ \dfrac{\partial y_2}{\partial x_n}, & \dots, & \dfrac{\partial y_{n+1}}{\partial x_n} \end{vmatrix},$$

which is proportional to Y_1, with a similar symmetrical coefficient; and these are vanishing terms: consequently the determinant provides terms in Y_1 and in $\dfrac{\partial y_1}{\partial x_i}$. Such determinants occur, in the full aggregate, for $i=1, \dots, n$. We therefore have an expression

$$\frac{\partial Y_1}{\partial x_k} = \text{a quantity linear and homogeneous in } Y_1, \frac{\partial y_1}{\partial x_1}, \dots, \frac{\partial y_1}{\partial x_n}.$$

The same form of result holds for the derivatives of each of the direction-cosines Y_m. Accordingly, we can take, for all values of m,

$$\frac{\partial Y_m}{\partial x_k} = ZY_m + P_1\frac{\partial y_m}{\partial x_1} + P_2\frac{\partial y_m}{\partial x_2} + \dots + P_n\frac{\partial y_m}{\partial x_n},$$

where the coefficients $Z, P_1, \dots, P_n$, which remain to be determined, are the same for all the equations arising from the $n+1$ values of m.

To determine these coefficients, first multiply by Y_m and add for all the values of m; then because $\sum\limits_m Y_m^2 = 1$, we have

$$\sum Y_m \frac{\partial Y_m}{\partial x_k} = 0;$$

also

$$\sum Y_m \frac{\partial y_m}{\partial x_i} = 0, \quad i = 1, \dots n, ;$$

hence

$$Z = 0.$$

Next, multiply by $\dfrac{\partial y_m}{\partial x_i}$ and add for all the values of m; then, as

$$\sum \frac{\partial Y_m}{\partial x_k}\frac{\partial y_m}{\partial x_i} = -\sum Y_m \frac{\partial^2 y_m}{\partial x_i \partial x_k} = -L_{ik},$$

for all values of i, we have

$$A_{1i}P_1 + A_{2i}P_2 + \dots + A_{ni}P_n = -L_{ik},$$

for $i=1, \ldots, n$. When these n equations are resolved, they yield

$$\Omega P_j = -\sum_i a_{ij} L_{ik},$$

for $j=1, \ldots, n$. Inserting the values of $Z, P_1, \ldots, P_n$, we find

$$\frac{\partial Y_m}{\partial x_k} = -\frac{1}{\Omega}\sum_i \sum_j a_{ij} L_{ik} \frac{\partial y_m}{\partial x_j},$$

in agreement with the expression already obtained.

In passing, it may be noted that the result is in formal accord with the less particular corresponding result for the amplitude when the plenary space is of more than $n+1$ dimensions. There (§ 34), it proved convenient to take the derivatives of $Y_1, Y_2, Y_3, \ldots$, along the geodesic. To compare the result just obtained with the earlier result, we take (for the present amplitude)

$$\begin{aligned} Y_m' &= \sum_k \frac{\partial Y_m}{\partial x_k} x_k' \\ &= -\frac{1}{\Omega}\sum_i \sum_j a_{ij} \left(\sum_k L_{ik} x_k'\right) \frac{\partial y_m}{\partial x_j} \\ &= -\frac{1}{\Omega}\sum_i \sum_j a_{ij} v_i \frac{\partial y_m}{\partial x_j}, \end{aligned}$$

with the former significance for v_i which is retained for the present more special expression for the circular curvature of the amplitudinal geodesic. The results are in accord; but the detailed expressions for the primary amplitude are simpler than for amplitudes existing only in more extensive plenary spaces.

Binormal of a geodesic: the torsion.

80. That earlier result (from § 34) leads immediately to expressions for the torsion and the direction-cosines of the binormal. The equation

$$Y_m' = -\frac{1}{\Omega}\sum_i \sum_j a_{ij} v_i \frac{\partial y_m}{\partial x_j}$$

is characteristic of the amplitude; and there is the typical Frenet equation

$$\frac{l_3}{\sigma} = \frac{y'}{\rho} + Y',$$

characteristic of any curve. Now

$$y_m' = \sum \frac{\partial y_m}{\partial x_j} x_j',$$

and therefore

$$\frac{l_3^{(m)}}{\sigma} = \sum_j \left\{\frac{x_j'}{\rho} - \frac{1}{\Omega}\left(\sum_i a_{ij} v_i\right)\right\} \frac{\partial y_m}{\partial x_j},$$

when $l_3^{(m)}$ is selected as the direction-cosine corresponding to y_m' for the tangent. Thus, if we write

$$l_3 = Q_1 \frac{\partial y}{\partial x_1} + Q_2 \frac{\partial y}{\partial x_2} + \dots + Q_n \frac{\partial y}{\partial x_n},$$

we have

$$\frac{1}{\sigma} Q_j = \frac{x_j'}{\rho} - \frac{1}{\Omega} \sum_i a_{ij} v_i,$$

so that the result already given (§ 34) is unaltered in form.

We verify, at once, that the line, thus represented in direction, is at right angles to the tangent. The necessary condition is

$$\sum y' l_3 = 0,$$

that is,

$$\left(\sum x_i' \frac{\partial y}{\partial x_i}\right)\left(\sum Q_j \frac{\partial y}{\partial x_j}\right) = 0,$$

that is,

$$\sum_j \sum_i Q_j A_{ij} x_i' = 0.$$

Now

$$\frac{1}{\sigma} \sum_j A_{ij} Q_j = \frac{1}{\rho} u_i - v_i,$$

and therefore

$$\frac{1}{\sigma} \sum_j \sum_i Q_j A_{ij} x_i' = \frac{1}{\rho}\left(\sum_i u_i x_i'\right) - \left(\sum_i v_i x_i'\right);$$

hence, because

$$\sum u_i x_i' = 1, \quad \sum v_i x_i' = \frac{1}{\rho},$$

the condition is satisfied.

Again,

$$\sum_m Y_m'^2 = \sum_m \frac{1}{\Omega^2} \left(\sum_i \sum_j a_{ij} v_i \frac{\partial y_m}{\partial x_j}\right)\left(\sum_k \sum_l a_{kl} v_k \frac{\partial y_m}{\partial x_l}\right)$$
$$= \frac{1}{\Omega^2} \sum_i \sum_j \sum_k \sum_l a_{ij} a_{kl} v_i v_k A_{jl}.$$

Also we have, as usual,

$$\sum_l a_{kl} A_{jl} = 0, \text{ when } j \text{ and } k \text{ are different,}$$
$$= \Omega, \text{ when } k = j;$$

and therefore

$$\sum_m Y_m'^2 = \frac{1}{\Omega} \sum_i \sum_j a_{ij} v_i v_j.$$

But by squaring the equation

$$\frac{l_3}{\sigma} - \frac{y'}{\rho} = Y',$$

and adding the results, we have

$$\frac{1}{\sigma^2}+\frac{1}{\rho^2}=\sum Y'^2=\frac{1}{\Omega}\sum_i\sum_j a_{ij}v_iv_j,$$

the customary formal result relating to the torsion (§ 35).

Extended Mainardi-Codazzi relations.

81. One set of relations among the secondary magnitudes L_{ij}, involving the magnitudes but not their derivatives, has already (§ 78) been obtained in the form

$$\begin{aligned} L_{ik}L_{jl}-L_{jk}L_{il} &= (ij, kl) \\ &= \tfrac{1}{2}\left(\frac{\partial^2 A_{il}}{\partial x_j \partial x_k}-\frac{\partial^2 A_{ik}}{\partial x_j \partial x_l}-\frac{\partial^2 A_{jl}}{\partial x_i \partial x_k}+\frac{\partial^2 A_{jk}}{\partial x_i \partial x_l}\right) \\ &\qquad +\sum_\lambda\sum_\mu A_{\lambda\mu}\{il, \mu\}\{jk, \lambda\}-\sum_\lambda\sum_\mu A_{\lambda\mu}\{ik, \mu\}\{jl. \lambda\}, \end{aligned}$$

these relations expressing particular combinations of the secondary magnitudes in terms of the first-order magnitudes. They are the extension of the single relation, commonly called the Gauss equation or the Gauss relation, belonging to surfaces in homaloidal triple space, and serving to express one of the two measures of curvature of such surfaces in terms of the first-order magnitudes alone.

But there are other relations, which involve the first derivatives of the secondary magnitudes. For surfaces in homaloidal triple space, there are the two Mainardi-Codazzi relations of this type: we proceed to obtain the extension of the Mainardi-Codazzi relations * for the secondary magnitudes in the n-fold primary amplitude.

In order to abbreviate the subsequent argument, an elementary lemma may be premised, as follows. When an equation

$$Y_mZ+\frac{\partial y_m}{\partial x_1}Z_1+\frac{\partial y_m}{\partial x_2}Z_2+\ldots+\frac{\partial y_m}{\partial x_n}Z_n=0$$

holds for $m=1, \ldots, n+1$, the quantities $Z, Z_1, \ldots, Z_n$, being the same throughout the $n+1$ equations, then

$$Z=0, \quad Z_1=0, \quad Z_2=0, \ldots, Z_n=0.$$

The property follows at once from the fact that the determinant of the coefficients of the $n+1$ quantities is equal to $\Omega^{\frac{1}{2}}$ and so does not vanish.

The equation

$$\frac{\partial^2 y_m}{\partial x_i \partial x_j}=Y_mL_{ij}+\sum_r\{ij, r\}\frac{\partial y_m}{\partial x_r},$$

holds for all the values $i, j=1, \ldots, n$, taken independently of one another, and for all the values $m=1, \ldots, n+1$.

* See my *Lectures on the Differential Geometry of Curves and Surfaces*, § 35.

Let the equation be differentiated with respect to x_k, where k is any one of the values $1, \ldots, n$, again taken independently, that is, without regard to the value of i or the value of j; then

$$\frac{\partial^3 y_m}{\partial x_i \partial x_j \partial x_k} = Y_m \frac{\partial L_{ij}}{\partial x_k} - \frac{1}{\Omega} L_{ij} \sum_\alpha \sum_\beta \frac{\partial y_m}{\partial x_\alpha} a_{\alpha\beta} L_{\beta k}$$
$$+ \sum_r \{ij, r\} \frac{\partial^2 y_m}{\partial x_r \partial x_k} + \sum_r \frac{\partial y_m}{\partial x_r} \frac{\partial}{\partial x_k} \{ij, r\}.$$

When the value of $\dfrac{\partial^2 y_m}{\partial x_r \partial x_k}$ is substituted, the resulting expression becomes linear in Y_m, and in the n derivatives $\dfrac{\partial y_m}{\partial x_1}, \ldots, \dfrac{\partial y_m}{\partial x_n}$; and, after re-arrangement of terms, the equation becomes

$$\frac{\partial^3 y_m}{\partial x_i \partial x_j \partial x_k} = Y_m \left[\frac{\partial L_{ij}}{\partial x_k} + \sum_r \{ij, r\} L_{rk}\right] + \sum_r \binom{ijk}{r} \frac{\partial y_m}{\partial x_r},$$

where

$$\binom{ijk}{r} = \frac{\partial}{\partial x_k} \{ij, r\} + \sum_p \{pk, r\}\{ij, p\} - \frac{1}{\Omega} L_{ij} \sum_q a_{rq} L_{qk}.$$

Now on the left-hand side, no alteration is caused by the free interchanges of the subscript indices i, j, k; consequently, the right-hand sides must equally remain unaltered by all such free interchanges. Accordingly, when such interchanges are effected and the resulting quantities are duly equated, there results an aggregate of equations, $n+1$ in number; they are homogeneous and linear in $Y_m, \dfrac{dy_m}{\partial x_1}, \ldots, \dfrac{\partial y_m}{\partial x_n}$; the coefficients in such equations are the same for all the equations; hence, by the preceding lemma, such coefficients vanish. But these coefficients are the differences among the coefficients in the expression for $\dfrac{\partial^3 y_m}{\partial x_i \partial x_j \partial x_k}$, arising through free interchanges of i, j, k; and therefore such coefficients themselves are unaltered in value by those interchanges.

We thus have one set of relations, arising out of the coefficient of Y_m; it has the form

$$\frac{\partial L_{ij}}{\partial x_k} + \sum_r \{ij, r\} L_{rk} = \frac{\partial L_{jk}}{\partial x_i} + \sum_r \{jk, r\} L_{ri} = \frac{\partial L_{ki}}{\partial x_j} + \sum \{ki, r\} L_{rj} = C_{ijk},$$

where the symbol C_{ijk} is used to denote the common value of the three expressions, and therefore denotes a magnitude which is unaltered by the free interchanges of i, j, k, among one another. These relations constitute the extension of the Mainardi-Codazzi relations for a surface in homaloidal triple space; their number is

$$\tfrac{1}{3} n (n^2 - 1).$$

The other set has the form

$$\frac{\partial}{\partial x_k}\{ij, r\} + \sum_p \{pk, r\}\{ij, p\} - \frac{1}{\Omega} L_{ij} \sum_q a_{rq} L_{qk}$$
$$= \frac{\partial}{\partial x_i}\{jk, r\} + \sum_p \{pi, r\}\{jk, p\} - \frac{1}{\Omega} L_{jk} \sum_q a_{rq} L_{qi}$$
$$= \frac{\partial}{\partial x_j}\{ki, r\} + \sum_p \{pj, r\}\{ki, p\} - \frac{1}{\Omega} L_{ki} \sum_q a_{rq} L_{qj} = \binom{ijk}{r}.$$

The symbol $\binom{ijk}{r}$, originally used to denote the first of these three equal quantities, now can denote each of the three; like C_{ijk}, it denotes a magnitude which is unaltered by the free interchange of i, j, k, among one another.

Our general result is therefore

$$\frac{\partial^3 y_m}{\partial x_i \partial x_j \partial x_k} = Y_m C_{ijk} + \sum_r \binom{ijk}{r} \frac{\partial y_m}{\partial x_r}.$$

Expression for $\frac{d}{ds}\left(\frac{1}{\rho}\right)$.

82. These results lead to the magnitudes of the third order, which occur in connection with the arc-rate of change of the circular curvature of an amplitudinal geodesic. We have

$$\frac{1}{\rho} = \sum_i \sum_j L_{ij} x_i' x_j',$$

and therefore

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \sum_i \sum_j \sum_k \frac{\partial L_{ij}}{\partial x_k} x_i' x_j' x_k' + \sum_i \sum_j L_{ij} x_i'' x_j' + \sum_i \sum_j L_{ij} x_j'' x_i'$$
$$= \sum_i \sum_j \sum_k \frac{\partial L_{ij}}{\partial x_k} x_i' x_j' x_k' + 2 \sum_i \sum_j L_{ij} x_i'' x_j'$$
$$= \sum_i \sum_j \sum_k [C_{ijk} - \sum_r \{ij, r\} L_{rk}] x_i' x_j' x_k' + 2 \sum_i \sum_j L_{ij} x_i'' x_j'.$$

On the right-hand side, we take the complete coefficient of $x_\alpha' x_\beta' x_\gamma'$ (for the combination of α, β, γ, and not for the permutations). It is

$= 6C_{\alpha\beta\gamma}$, because there are six such terms owing to the permutations of α, β, γ,

$-2 \sum_r [\{\beta\gamma, r\} L_{r\alpha} + \{\gamma\alpha, r\} L_{r\beta} + \{\alpha\beta, r\} L_{r\gamma}]$, because for any value of k there are two such terms in i and j,

$-4 \sum_i [\{\beta\gamma, i\} L_{i\alpha} + \{\gamma\alpha, i\} L_{i\beta} + \{\alpha\beta, i\} L_{i\gamma}]$, for the like reason,

$= 6C_{\alpha\beta\gamma} - 6 \sum_r [\{\beta\gamma, r\} L_{r\alpha} + \{\gamma\alpha, r\} L_{r\beta} + \{\alpha\beta, r\} L_{r\gamma}].$

We write

$$\frac{d}{ds}\left(\frac{1}{\rho}\right)=\sum_i\sum_j\sum_k c_{ijk}x_i'x_j'x_k',$$

where the summation on the right-hand side is for all values $i, j, k, =1, \ldots, n$, independently of one another. Thus the full coefficient of $x_\alpha' x_\beta' x_\gamma'$ in the triple summation is $6c_{\alpha\beta\gamma}$; and therefore we have

$$c_{\alpha\beta\gamma}=C_{\alpha\beta\gamma}-\sum_r[\{\beta\gamma, r\}L_{r\alpha}+\{\gamma\alpha, r\}L_{r\beta}+\{\alpha\beta, r\}L_{r\gamma}].$$

We retain both sets of quantities $C_{\alpha\beta\gamma}$, $c_{\alpha\beta\gamma}$: the quantities $C_{\alpha\beta\gamma}$ are useful in connection with coordinates in the plenary space, while the quantities $c_{\alpha\beta\gamma}$ are useful in connection with intrinsic magnitudes of the amplitude.

Hence, collecting results, we have

$$C_{ijk}=c_{ijk}+\sum_r[\{jk, r\}L_{ir}+\{ki, r\}L_{jr}+\{ij, r\}L_{kr}],$$

$$\frac{\partial L_{ij}}{\partial x_k}=C_{ijk}-\sum_r\{ij, r\}L_{kr}=c_{ijk}+\sum_r\{jk, r\}L_{ir}+\sum_r\{ik, r\}L_{jr},$$

$$\frac{\partial^3 y_m}{\partial x_i\,\partial x_j\,\partial x_k}=Y_m C_{ijk}+\sum_r\begin{pmatrix}ijk\\ r\end{pmatrix}\frac{\partial y_m}{\partial x_r}.$$

Manifestly,

$$\sum_m Y_m\frac{\partial^3 y_m}{\partial x_i\,\partial x_j\,\partial x_k}=C_{ijk},$$

thus associating the quantities C with the point-coordinates of amplitudinal position in the plenary space: and

$$\sum_m\frac{\partial^3 y_m}{\partial x_i\,\partial x_j\,\partial x_k}\frac{\partial y_m}{\partial x_l}=\sum_r\begin{pmatrix}ijk\\ r\end{pmatrix}A_{rl},$$

or

$$\Omega\begin{pmatrix}ijk\\ r\end{pmatrix}=\sum_l a_{rl}\left[\sum_m\frac{\partial^3 y_m}{\partial x_i\,\partial x_j\,\partial x_k}\frac{\partial y_m}{\partial x_l}\right],$$

thus associating the quantities $\begin{pmatrix}ijk\\ r\end{pmatrix}$ also with these point-coordinates in plenary space. The quantities c are associated with intrinsic properties of the amplitude by the relation

$$\frac{d}{ds}\left(\frac{1}{\rho}\right)=\sum_i\sum_j\sum_k c_{ijk}x_i'x_j'x_k'.$$

We shall find it convenient to write

$$w_i=\tfrac{1}{3}\frac{\partial}{\partial x_i'}\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)\right\},\quad (i=1, \ldots, n),$$

in conformity with the notation already adopted for the symbols

$$u_i=\sum_j A_{ij}x_j',\quad v_i=\sum_j L_{ij}x_j';$$

and, in accordance with the earlier notation (p. 27), we use g_{pr}, where

$$g_{pr} = \sum_k \{kr, p\} x_k'.$$

Differentiating the typical relation

$$u_i = \sum_j A_{ij} x_j'$$

along the amplitudinal geodesic, we have

$$\frac{du_i}{ds} = \sum_j \sum_k \frac{\partial A_{ij}}{\partial x_k} x_k' x_j' + \sum_j A_{ij} x_j''.$$

Now

$$\sum_j \sum_k \frac{\partial A_{ij}}{\partial x_k} x_j' x_k' = \sum_\alpha \sum_\beta \frac{\partial A_{i\alpha}}{\partial x_\beta} x_\alpha' x_\beta'$$
$$= \sum_\alpha \sum_\beta x_\alpha' x_\beta' \left[\sum_p A_{p\alpha} \{i\beta, p\} + \sum_q A_{qi} \{\alpha\beta, q\} \right],$$

and

$$\sum A_{ij} x_j'' = - \sum_q A_{qi} \sum_\alpha \sum_\beta \{\alpha\beta, q\} x_\alpha' x_\beta';$$

hence

$$\frac{du_i}{ds} = \sum_p \sum_\alpha \sum_\beta \{i\beta, p\} A_{p\alpha} x_\alpha' x_\beta'.$$

But

$$\sum_\alpha A_{p\alpha} x_\alpha' = u_p, \quad \sum_\beta \{i\beta, p\} x_\beta' = g_{pi},$$

all the terms in the first summation being independent of β, while those in the second are independent of α; thus

$$\frac{du_i}{ds} = \sum_p g_{pi} u_p$$
$$= g_{1i} u_1 + g_{2i} u_2 + \dots + g_{ni} u_n.$$

Similarly, differentiating the relation

$$v_i = \sum_j L_{ij} x_j'$$

along the amplitudinal geodesic, we have

$$\frac{dv_i}{ds} = \sum_j \sum_k \frac{\partial L_{ij}}{\partial x_k} x_k' x_j' + \sum_j L_{ij} x_j''$$
$$= \sum_j \sum_k c_{ijk} x_k' x_j' + \sum_j \sum_k \left[\sum_r \{jk, r\} L_{ir} + \sum_r \{ik, r\} L_{jr} \right] x_k' x_j'$$
$$- \sum_j \sum_\alpha \sum_\beta L_{ij} \{\alpha\beta, j\} x_\alpha' x_\beta'.$$

But

$$\sum_j \sum_k \sum_r \{jk, r\} L_{ir} x_k' x_j' = \sum_\alpha \sum_\beta \sum_r \{\alpha\beta, r\} L_{ir} x_\alpha' x_\beta'$$
$$= \sum_j \sum_\alpha \sum_\beta \{\alpha\beta, j\} L_{ij} x_\alpha' x_\beta';$$

and therefore

$$\frac{dv_i}{ds} = \sum_j \sum_k c_{ijk} x_j' x_k' + \sum_j \sum_k \sum_r \{ik, r\} L_{jr} x_k' x_j'.$$

Now

$$\sum_j L_{jr} x_j' = v_r, \quad \sum_k \{ik, r\} x_k' = g_{ri},$$

the first summation being independent of k and second being independent of j; hence

$$\frac{dv_i}{ds} = \sum_j \sum_k c_{ijk} x_j' x_k' + \sum_r g_{ri} v_r$$
$$= w_i + g_{1i} v_1 + g_{2i} v_2 + \dots + g_{ni} v_n,$$

with the assigned significance for w_i.

A partial simple verification can be provided at once. We have

$$\sum_i x_i' \frac{dv_i}{ds} = \sum w_i x_i' + \sum v_\mu g_{\mu i} x_i'$$
$$= \frac{d}{ds}\left(\frac{1}{\rho}\right) - \sum_\mu v_\mu x_\mu''$$
$$= \frac{d}{ds}\left(\frac{1}{\rho}\right) - \sum_i v_i x_i'',$$

in accordance with the earlier result

$$\frac{1}{\rho} = \sum_i x_i' v_i.$$

The results are in complete formal agreement with the like results for the general amplitude already (§ 37) obtained.

83. We can obtain, directly, an expression for Y_m'', also in accordance with the general result.

Two relations will be required, in the construction of this required expression. By § 13, we have

$$\Omega \frac{\partial}{\partial x_k}\left(\frac{a_{ij}}{\Omega}\right) = -\sum_\mu [a_{i\mu}\{k\mu, j\} + a_{j\mu}\{k\mu, i\}],$$

and therefore

$$\Omega \frac{d}{ds}\left(\frac{a_{ij}}{\Omega}\right) = -\sum_\mu \sum_k [a_{i\mu}\{k\mu, j\} + a_{j\mu}\{k\mu, i\}] x_k'$$
$$= -\sum_\mu (a_{i\mu} g_{j\mu} + a_{j\mu} g_{i\mu}).$$

For the other relation, we have

$$\frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) = x_1' \frac{\partial^2 y}{\partial x_1 \partial x_r} + x_2' \frac{\partial^2 y}{\partial x_2 \partial x_r} + \dots + x_n' \frac{\partial^2 y}{\partial x_n \partial x_r}$$
$$= Y v_r + g_{1r} \frac{\partial y}{\partial x_1} + g_{2r} \frac{\partial y}{\partial x_2} + \dots + g_{nr} \frac{\partial y}{\partial x_n},$$

on substituting for the second derivatives: that is,

$$\frac{d}{ds}\left(\frac{\partial y}{\partial x_r}\right) = Yv_r + \sum_\mu g_{\mu r}\frac{\partial y}{\partial x_\mu}.$$

We now proceed from the equation (§ 79)

$$Y_m' = -\frac{1}{\Omega}\sum_i\sum_j a_{ij}v_i\frac{\partial y_m}{\partial x_j},$$

which, when differentiated, gives

$$\begin{aligned} Y_m'' = &-\sum_i\sum_j v_i\frac{\partial y_m}{\partial x_j}\frac{d}{ds}\left(\frac{a_{ij}}{\Omega}\right) \\ &-\sum_i\sum_j \frac{a_{ij}}{\Omega}\frac{\partial y_m}{\partial x_j}\left(w_i + \sum_\mu g_{\mu i}v_\mu\right) \\ &-\sum_i\sum_j \frac{a_{ij}}{\Omega}v_i\frac{d}{ds}\left(\frac{\partial y_m}{\partial x_j}\right). \end{aligned}$$

When the first of the foregoing relations is used, the first line in this expression

$$= \frac{1}{\Omega}\sum_i\sum_j\sum_\mu v_i\frac{\partial y_m}{\partial x_j}(a_{i\mu}g_{j\mu} + a_{j\mu}g_{i\mu}):$$

in this form, the terms

$$\begin{aligned} &\frac{1}{\Omega}\sum_i\sum_j\sum_\mu v_i\frac{\partial y_m}{\partial x_j}a_{j\mu}g_{i\mu} \\ &= \frac{1}{\Omega}\sum_\alpha\sum_j\sum_\gamma v_\alpha\frac{\partial y_m}{\partial x_j}a_{j\gamma}g_{\alpha\gamma} \\ &= \frac{1}{\Omega}\sum_\mu\sum_j\sum_i v_\mu\frac{\partial y_m}{\partial x_j}a_{j\mu}g_{\mu i}, \end{aligned}$$

and therefore they cancel the same terms in the second line: while the remaining terms

$$\begin{aligned} &\frac{1}{\Omega}\sum_i\sum_j\sum_\mu v_i\frac{\partial y_m}{\partial x_j}a_{i\mu}g_{j\mu} \\ &= \frac{1}{\Omega}\sum_i\sum_\beta\sum_\gamma v_i\frac{\partial y_m}{\partial x_\beta}a_{i\gamma}g_{\beta\gamma} \\ &= \frac{1}{\Omega}\sum_i\sum_\mu\sum_j v_i\frac{\partial y_m}{\partial x_\mu}a_{ij}g_{\mu j}. \end{aligned}$$

When the second of the foregoing relations is used, the third line in the expression for Y_m''

$$= -\sum_i\sum_j\frac{a_{ij}}{\Omega}Y_m v_i v_j - \sum_i\sum_j\sum_\mu\frac{a_{ij}}{\Omega}v_i\frac{\partial y_m}{\partial x_\mu}g_{\mu j},$$

the triple summation in which cancels the triple summation still remaining from the terms in the first line. Hence, collecting the results, we have

$$Y_m'' = -\sum_i\sum_j\frac{a_{ij}}{\Omega}\frac{\partial y_m}{\partial x_j}w_i - \sum_i\sum_j\frac{a_{ij}}{\Omega}Y_m v_i v_j:$$

or, on using the relation

$$\sum_i \sum_j a_{ij} v_i v_j = \Omega \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right),$$

there results the equation

$$Y_m'' + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) Y_m = -\frac{1}{\Omega} \sum_i \sum_j a_{ij} w_i \frac{\partial y_m}{\partial x_j},$$

in accordance with the earlier relation (§ 38).

As an immediate corollary, we have

$$\begin{aligned}\sum_m Y_m'' \frac{\partial y_m}{\partial x_r} &= -\frac{1}{\Omega} \sum_m \sum_i \sum_j a_{ij} w_i \frac{\partial y_m}{\partial x_j} \frac{\partial y_m}{\partial x_r} \\ &= -\frac{1}{\Omega} \sum_i \sum_j A_{jr} a_{ij} w_i \\ &= -w_r,\end{aligned}$$

by the customary theorems for the minors of Ω. This result can also be established otherwise, as follows. We have had

$$\frac{d}{ds}\left(\frac{\partial y_m}{\partial x_r}\right) = -Y_m v_r + \sum_\mu g_{\mu r} \frac{\partial y_m}{\partial x_\mu},$$

$$Y_m' = -\frac{1}{\Omega} \sum_i \sum_j a_{ij} v_i \frac{\partial y_m}{\partial x_j},$$

and therefore

$$\begin{aligned}\sum_m Y_m' \frac{d}{ds}\left(\frac{\partial y_m}{\partial x_r}\right) &= -\frac{1}{\Omega} \sum_i \sum_j \sum_\mu a_{ij} v_i g_{\mu r} \left(\sum_m \frac{\partial y_m}{\partial x_\mu} \frac{\partial y_m}{\partial x_j}\right) \\ &= -\frac{1}{\Omega} \sum_i \sum_j \sum_\mu a_{ij} A_{\mu j} v_i g_{\mu r} \\ &= -\sum_\mu v_\mu g_{\mu r}.\end{aligned}$$

An earlier result gave

$$\sum_m Y_m' \frac{\partial y_m}{\partial x_r} = -v_r,$$

so that

$$\begin{aligned}\sum_m Y_m'' \frac{\partial y_m}{\partial x_r} + \sum_m Y_m' \frac{d}{ds}\left(\frac{\partial y_m}{\partial x_r}\right) &= -\frac{dv_r}{ds} \\ &= -w_r - \sum_\mu v_\mu g_{\mu r};\end{aligned}$$

and therefore, as above,

$$\sum_m Y_m'' \frac{\partial y_m}{\partial x_r} = -w_r.$$

Also, squaring the equation

$$Y_m'' + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) Y_m = -\frac{1}{\Omega} \sum_i \sum_j a_{ij} w_i \frac{\partial y_m}{\partial x_j},$$

adding for all the equations, and using the relation

$$\sum_m Y_m Y_m'' = -\sum_m Y_m'^2 = -\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right),$$

we have the former result (§ 46)

$$\sum Y_m''^2 - \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)^2 = \frac{1}{\Omega}\sum_i \sum_j a_{ij} w_i w_j.$$

Moreover,

$$-\left(\frac{\rho'}{\rho_3}+\frac{\sigma'}{\sigma^3}\right) = \sum_m Y_m' Y_m'' = -\frac{1}{\Omega}\sum_m \sum_i \sum_j a_{ij} v_i \frac{\partial y_m}{\partial x_j} Y_m''$$

$$= \frac{1}{\Omega}\sum_i \sum_j a_{ij} v_i w_j,$$

again a result already established.

Trinormal of a geodesic : the tilt.

84. The direction-cosines of the trinormal and the magnitude of the tilt can now be obtained. We have

$$\frac{l_4}{\tau} - \frac{Y}{\sigma} = \frac{dl_3}{ds}$$

$$= y' \frac{d}{ds}\left(\frac{\sigma}{\rho}\right) + \frac{\sigma}{\rho^2} Y + \sigma' Y' + \sigma Y'';$$

and therefore, when substitution takes place for Y' and Y'',

$$\frac{l_4}{\tau} = y' \frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - \frac{1}{\Omega}\sum_i \sum_j \left\{(\sigma' v_i + \sigma w_i) a_{ij} \frac{\partial y}{\partial x_j}\right\}$$

$$= \sum_j \left[x_j' \frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - \frac{1}{\Omega}\sum_i \{a_{ij}(\sigma' v_i + \sigma w_i)\}\right] \frac{\partial y}{\partial x_j},$$

being the equations for the tilt and the direction of the binormal of a geodesic.

Squaring and adding for all the $n+1$ equations for the range of the plenary space, we find

$$\frac{1}{\tau^2} = \left\{\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}^2 + \frac{1}{\Omega}\sum_i \sum_j (\sigma' v_i + \sigma w_i)(\sigma' v_j + \sigma w_j) a_{ij}$$

$$-2\left(\frac{\sigma'}{\rho} - \frac{\sigma\rho'}{\rho^2}\right)\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)$$

$$= -\left(\frac{\sigma'^2}{\sigma^2} + \frac{\sigma^2\rho'^2}{\rho^4}\right) + \frac{\sigma^2}{\Omega}\sum_i \sum_j a_{ij} w_i w_j;$$

or

$$\Omega\left(\frac{1}{\sigma^2\tau^2} + \frac{\sigma'^2}{\sigma^4} + \frac{\rho'^2}{\rho^4}\right) = \sum_i \sum_j a_{ij} w_i w_j,$$

a relation giving the magnitude of the tilt.

Taken in conjunction with the former value of Y''^2, we have

$$\sum Y'''^2 = \frac{1}{\sigma^2\tau^2} + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^2 + \frac{\rho'^2}{\rho^4} + \frac{\sigma'^2}{\sigma^4},$$

a known inference (§ 8) from the Frenet equations applied to a geodesic of a configuration.

Again, when the equation

$$l_4 = y'\left\{\tau \frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\} - \sum_i \sum_j \left\{(\sigma' v_i + \sigma w_i)\tau \frac{a_{ij}}{\Omega}\right\} \frac{\partial y}{\partial x_j}$$

is differentiated along the geodesic, and the relation

$$\frac{d}{ds}\left(\frac{\partial y}{\partial x_j}\right) = Y v_j + g_{1j}\frac{\partial y}{\partial x_1} + g_{2j}\frac{\partial y}{\partial x_2} + \dots + g_{nj}\frac{\partial y}{\partial x_n}$$

is used, the coefficient of Y on the right-hand side of the differentiated equation becomes

$$= \frac{\tau}{\rho}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - \sum_i \sum_j \left\{(\sigma' v_i + \sigma w_i)\tau \frac{a_{ij}}{\Omega} v_j\right\},$$

a quantity which vanishes on the insertion of the values of the two magnitudes

$$\sum_i \sum_j a_{ij} v_i v_j, \quad \sum_i \sum_j a_{ij} w_i v_j$$

already (§ 40) obtained. Thus the equation is

$$\frac{l_5}{\kappa} - \frac{l_3}{\tau} = y'\frac{d}{ds}\left\{\tau \frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\} + \sum \bar{D}_\mu \frac{\partial y}{\partial x_\mu},$$

where

$$-\bar{D}_\mu = \frac{d}{ds}\left\{\sum_i (\sigma' v_i + \sigma w_i)\tau\frac{a_{i\mu}}{\Omega}\right\} + \sum_i \sum_j \left\{(\sigma' v_i + \sigma w_i)\frac{\tau}{\Omega} a_{ij} g_{\mu j}\right\}.$$

Also (§ 80)

$$\frac{l_3}{\sigma} = \sum_j \left\{\frac{1}{\rho} x_j' - \frac{1}{\Omega}\left(\sum_i a_{ij} v_i\right)\right\}\frac{\partial y}{\partial x_j} = \frac{1}{\Omega}\sum_i \sum_j \left\{a_{ij}\left(\frac{u_i}{\rho} - v_i\right)\right\}\frac{\partial y}{\partial x_j}:$$

and thus we have equations for determining the direction-cosines of the quartinormal and the magnitude of the coil.

These results express l_3, l_4, l_5, the typical direction-cosines of the binormal, the trinormal, and the quartinormal, as linear functions of the quantities $\frac{\partial y}{\partial x_i}$: and thus there is a verification that these successive principal lines of an amplitudinal geodesic lie in the n-fold homaloid that touches the amplitude.

As all the principal lines (other than the prime normal) in the orthogonal frame of a geodesic, which pertains to a primary amplitude, lie in the n-fold

homaloid touching the amplitude, their respective typical direction-cosines are of the form

$$l_p = C_{p1}\frac{\partial y}{\partial x_1} + C_{p2}\frac{\partial y}{\partial x_2} + \dots + C_{pn}\frac{\partial y}{\partial x_n},$$

except for $p=2$ when the typical direction-cosine is Y where

$$Y = \sum_i \sum_j \eta_{ij} x_i' x_j'.$$

Also, when $p=1$, the value of C_{1j} is x_j'.

Certain relations, of a difference-differential type, can be established among these coefficients. When differentiation along a geodesic is effected, we have

$$\frac{d}{ds}\left(\frac{\partial y}{\partial x_m}\right) = Y v_m + \sum_\lambda g_{\lambda m}\frac{\partial y}{\partial x_\lambda},$$

from the relation in § 83, where (p. 27)

$$g_{qr} = \sum_k \{kr, q\} x_k'.$$

Accordingly, when the equation for l_p is differentiated along the amplitudinal geodesic, we have

$$\begin{aligned}\frac{l_{p+1}}{\rho_p} - \frac{l_{p-1}}{\rho_{p-1}} &= \sum_\mu \frac{\partial y}{\partial x_\mu}\frac{dC_{p\mu}}{ds} + \sum_\mu C_{p\mu}\frac{d}{ds}\left(\frac{\partial y}{\partial x_\mu}\right)\\ &= Y\sum_\mu C_{p\mu}v_\mu + \sum_\mu \frac{\partial y}{\partial x_\mu}\frac{dC_{p\mu}}{ds} + \sum_\mu\sum_\lambda C_{p\mu}g_{\lambda\mu}\frac{\partial y}{\partial x_\lambda}\\ &= Y\sum_\mu C_{p\mu}v_\mu + \sum_\mu \frac{\partial y}{\partial x_\mu}\left(\frac{dC_{p\mu}}{ds} + \sum_\lambda C_{p\lambda}g_{\mu\lambda}\right).\end{aligned}$$

We shall assume that $p>2$; for $p=1$, the derived formula does not hold, and for $p=2$, the original formula does not apply.

When $p=3$, so that $\rho_p=\tau$, $\rho_{p-1}=\sigma$, the left-hand side is

$$\frac{l_4}{\tau} - \frac{Y}{\sigma};$$

and the equation becomes

$$\frac{l_4}{\tau} - Y\left(\frac{1}{\sigma} + \sum C_{3\mu}v_\mu\right) = \sum_\mu \frac{\partial y}{\partial x_\mu}\left(\frac{dC_{3\mu}}{ds} + \sum_\lambda C_{3\lambda}g_{\mu\lambda}\right).$$

When we multiply this relation by Y, and add for all the dimensions, we have

$$\frac{1}{\sigma} + \sum C_{3\mu}v_\mu = 0,$$

because

$$\sum l_4 Y = 0, \quad \sum Y\frac{\partial y}{\partial x_\mu} = 0, \quad (\mu = 1, \dots, n).$$

This result can be verified independently; for the expression for l_3 was obtained (§ 80) in the form

$$\frac{l_3}{\sigma}=\sum_\mu \frac{\partial y}{\partial x_\mu}\left\{\frac{x_\mu'}{\rho}-\frac{1}{\Omega}\left(\sum_i a_{i\mu}v_i\right)\right\},$$

so that

$$\frac{1}{\sigma}C_{3\mu}=\frac{x_\mu'}{\rho}-\frac{1}{\Omega}\left(\sum_i a_{i\mu}v_i\right),$$

and the result follows at once by means of the relation

$$\Omega\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)=\sum_i\sum_\mu a_{i\mu}v_iv_\mu.$$

Then

$$\frac{l_4}{\tau}=\sum_\mu \frac{\partial y}{\partial x_\mu}\left(\frac{dC_{3\mu}}{ds}+\sum_\lambda C_{3\lambda}g_{\mu\lambda}\right);$$

and therefore we have the difference-differential equation

$$\frac{1}{\tau}C_{4\mu}=\frac{dC_{3\mu}}{ds}+\sum_\lambda C_{3\lambda}g_{\mu\lambda}.$$

When $p>3$, there is no term on the left-hand side involving Y, and we have (for all such values of p)

$$\sum Yl_{p+1}=0,\quad \sum Yl_{p-1}=0.$$

Then, when we multiply the relation by Y, and add for all the dimensions, we have

$$\sum_\mu C_{p\mu}v_\mu=0,$$

this holding for all the values $p=4, \ldots, n$; and the equation becomes

$$\frac{l_{p+1}}{\rho_p}-\frac{l_{p-1}}{\rho_{p-1}}=\sum_\mu \frac{\partial y}{\partial x_\mu}\left(\frac{dC_{p\mu}}{ds}+\sum_\lambda C_{p\lambda}g_{\mu\lambda}\right).$$

Now let the postulated values of l_{p+1} and l_{p-1} be inserted; so that we have a relation

$$\sum_\mu\left\{\frac{C_{p+1,\,\mu}}{\rho_p}-\frac{C_{p-1,\,\mu}}{\rho_{p-1}}-\left(\frac{dC_{p\mu}}{ds}+\sum_\lambda C_{p\lambda}g_{\mu\lambda}\right)\right\}\frac{\partial y}{\partial x_\mu}=0.$$

This relation holds for the $n+1$ equations, of which it is typical. Let it be multiplied by $\dfrac{\partial y}{\partial x_\lambda}$, where λ has any one of the values $1, \ldots, n$. The same equation now holds with the quantities $A_{\lambda\mu}$ instead of the quantities $\dfrac{\partial y}{\partial x_\mu}$ as coefficients; and the determinant of the quantities $A_{\lambda\mu}$ is not zero, being equal to Ω. Hence we have

$$\frac{C_{p+1,\,\mu}}{\rho_p}-\frac{C_{p-1,\,\mu}}{\rho_{p-1}}=\frac{dC_{p\mu}}{ds}+\left(\sum_\lambda C_{p\lambda}g_{\mu\lambda}\right),$$

for each of the values $\mu=1, \ldots, n$, and for all the values $p=4, \ldots, n$.

Thus we have

$$\frac{C_{5\mu}}{\kappa}-\frac{C_{3\mu}}{\tau}=\frac{dC_{4\mu}}{ds}+\left(\sum_\lambda C_{4\lambda}g_{\mu\lambda}\right),$$

$$\frac{C_{6\mu}}{\rho_5}-\frac{C_{4\mu}}{\kappa}=\frac{dC_{5\mu}}{ds}+\left(\sum_\lambda C_{5\lambda}g_{\mu\lambda}\right),$$

as far as

$$\frac{C_{n\mu}}{\rho_{n-1}}-\frac{C_{n-2,\,\mu}}{\rho_{n-2}}=\frac{dC_{n-1,\,\mu}}{ds}+\left(\sum_\lambda C_{n-1,\,\lambda}g_{\mu\lambda}\right),$$

together with an equation of verification, in the form

$$-\frac{C_{n-1,\,\mu}}{\rho_{n-1}}=\frac{dC_{n\mu}}{ds}+\left(\sum_\lambda C_{n\lambda}g_{\mu\lambda}\right).$$

Ex. The relation

$$\sum_\mu C_{p\mu}v_\mu=0,$$

is to hold for all the values of p greater than 3; verify it for $p=4$, $p=5$, by means of the known expressions for l_4, l_5.

85. The values of the first derivatives of the direction-cosines of the normal to the amplitude with respect to the amplitudinal parameters have been obtained; those of the second derivatives can also be constructed simply.

Direct differentiation of the relation

$$\frac{\partial Y}{\partial x_k}=-\sum_i\sum_j\frac{a_{ij}}{\Omega}L_{ik}\frac{\partial y}{\partial x_j}$$

with respect to x_l gives

$$\frac{\partial^2 Y}{\partial x_k\partial x_l}=-\sum_i\sum_j\frac{a_{ij}}{\Omega}L_{ik}\frac{\partial^2 y}{\partial x_j\partial x_l}-\sum_i\sum_j\frac{a_{ij}}{\Omega}\frac{\partial y}{\partial x_j}\frac{\partial L_{ik}}{\partial x_l}-\sum_i\sum_j L_{ik}\frac{\partial y}{\partial x_j}\frac{\partial}{\partial x_l}\left(\frac{a_{ij}}{\Omega}\right).$$

When the value of $\dfrac{\partial^2 y}{\partial x_j\partial x_l}$ is substituted in the first summation, that summation (with its sign) becomes

$$=-Y\sum_i\sum_j\frac{a_{ij}}{\Omega}L_{ik}L_{jl}-\sum_i\sum_j\frac{a_{ij}}{\Omega}L_{ik}\sum_p\frac{\partial y}{\partial x_p}\{jl,\,p\}.$$

When the value of $\dfrac{\partial L_{ik}}{\partial x_l}$ is substituted in the second summation, that summation (with its sign) becomes

$$=-\sum_i\sum_j\frac{a_{ij}}{\Omega}\frac{\partial y}{\partial x_j}\left[C_{ikl}-\sum_r\{ik,\,r\}L_{rl}\right].$$

When the value of $\dfrac{\partial}{\partial x_l}\left(\dfrac{a_{ij}}{\Omega}\right)$ is substituted in the third summation, that summation (with its sign) becomes

$$=\frac{1}{\Omega}\sum_i\sum_j L_{ik}\frac{\partial y}{\partial x_j}\sum_\mu\left[a_{\mu i}\{l\mu,\,j\}+a_{\mu j}\{l\mu,\,i\}\right].$$

Thus, in $\dfrac{\partial^2 Y}{\partial x_k \partial x_l}$, the coefficient of Y is

$$= -\sum_i \sum_j \frac{a_{ij}}{\Omega} L_{ik} L_{jl}.$$

The total coefficient of $\dfrac{\partial y}{\partial x_p}$ is

$$= -\sum_i \sum_j \frac{a_{ij}}{\Omega} L_{ik} \{jl, p\}$$

$$- \sum_i \frac{a_{ip}}{\Omega} [C_{ikl} - \sum_r \{ik, r\} L_{rl}]$$

$$+ \frac{1}{\Omega} \sum_i L_{ik} \sum_\mu [a_{\mu i} \{l\mu, p\} + a_{\mu p} \{l\mu, i\}].$$

As

$$\sum_\mu a_{\mu i} \{l\mu, p\} = \sum_j a_{ji} \{lj, p\},$$

the prior term in the third line of this coefficient cancels the whole of the first line. Also, the later term in that third line

$$= \frac{1}{\Omega} \sum_\mu \sum_i L_{ik} a_{\mu p} \{l\mu, i\}$$

$$= \frac{1}{\Omega} \sum_\mu \sum_t L_{tk} a_{\mu p} \{l\mu, t\}$$

$$= \frac{1}{\Omega} \sum_i \sum_t a_{ip} \{li, t\} L_{tk}$$

$$= \frac{1}{\Omega} \sum_i \sum_r a_{ip} \{li, r\} L_{rk},$$

and therefore can be associated with the terms in the second line; thus the total coefficient of $\dfrac{\partial y}{\partial x_p}$ is

$$= -\sum_i \frac{a_{ip}}{\Omega} \{C_{ikl} - \sum_r [\{ik, r\} L_{rl} + \{il, r\} L_{rk}]\}$$

$$= -\sum_i \frac{a_{ip}}{\Omega} [c_{ikl} + \sum_r \{kl, r\} L_{ri}] = R_p,$$

using R_p to denote the expression. Hence, finally,

$$\frac{\partial^2 Y}{\partial x_k \partial x_l} = -Y \sum_i \sum_j \frac{a_{ij}}{\Omega} L_{ik} L_{jl} + \sum_p \frac{\partial y}{\partial x_p} R_p.$$

Evidently we have

$$\sum Y \frac{\partial^2 Y}{\partial x_k \partial x_l} = -\sum_i \sum_j \frac{a_{ij}}{\Omega} L_{ik} L_{jl};$$

a result easily verified, because

$$\sum Y \frac{\partial Y}{\partial x_k} = 0,$$

and therefore

$$\begin{aligned}\sum Y \frac{\partial^2 Y}{\partial x_k \partial x_l} &= -\sum \frac{\partial Y}{\partial x_l}\frac{\partial Y}{\partial x_k} \\ &= -\sum_m \left(\sum_i \sum_j \frac{a_{ij}}{\Omega} L_{ik} \frac{\partial y_m}{\partial x_j}\right)\left(\sum_\alpha \sum_\beta \frac{a_{\alpha\beta}}{\Omega} L_{\alpha l} \frac{\partial y_m}{\partial x_\beta}\right) \\ &= -\sum_i \sum_j \frac{a_{ij}}{\Omega} L_{ik} L_{jl}.\end{aligned}$$

Again, we have

$$\sum \frac{\partial y}{\partial x_i} \frac{\partial^2 y}{\partial x_k \partial x_l} = \sum_i A_{pi} R_p = -[c_{ikl} + \sum_r \{kl, r\} L_{ri}];$$

and thence, by differentiating the relation

$$\sum \frac{\partial y_m}{\partial x_i} \frac{\partial Y_m}{\partial x_k} = -L_{ik},$$

we easily find (or otherwise verify) the result

$$\sum \frac{\partial^2 y}{\partial x_\alpha \partial x_\beta} \frac{\partial Y}{\partial x_k} = -\sum_i \{\alpha\beta, i\} L_{ik}.$$

Two measures of superficial curvature of a primary amplitude.

86. The maximum and the minimum values of geodesic circular curvature in a primary amplitude, when account is taken of all directions possible within the amplitude, have been discussed (§ 77). But account must be taken of the maximum and the minimum values arising from less extensive ranges of directions; in particular, when a direction lies in a superficial orientation within the amplitude, and when a direction lies within a regional orientation within the amplitude.

When a direction lies within a superficial orientation, the range may be defined by two leading lines with direction-variables $p_1', \ldots, p_n'$, and $q_1', \ldots, q_n'$, so that the direction-variables are given by

$$x_i' = \lambda p_i' + \mu q_i', \quad (i = 1, \ldots, n),$$

where λ and μ are parametric. The permanent relation is

$$\lambda^2 \sum_i \sum_j A_{ij} p_i' p_j' + 2\lambda\mu \sum_i \sum_j A_{ij} p_i' q_j' + A^2 \sum_i \sum_j A_{ij} q_i' q_j' = 1,$$

that is,

$$\lambda^2 + \mu^2 + 2\lambda\mu \cos\epsilon = 1.$$

The circular curvature of the geodesic in the direction $x_1', \ldots, x_n'$, is given by

$$\begin{aligned}\frac{1}{\rho} &= \sum_i \sum_j L_{ij} x_i' x_j' \\ &= \lambda^2 \sum_i \sum_j L_{ij} p_i' p_j' + 2\lambda\mu \sum_i \sum_j L_{ij} p_i' q_j' + \mu^2 \sum_i \sum_j L_{ij} q_i' q_j' \\ &= \lambda^2 V_{11} + 2\lambda\mu V_{12} + \mu^2 V_{22},\end{aligned}$$

with obvious symbolical significance for V_{11}, V_{12}, V_{22}. The parametric quantities in $1/\rho$ now are λ and μ; and thus the maximum and the minimum of the circular curvature of geodesics in directions within the superficial orientation are given by the equation

$$\begin{vmatrix} V_{11}-\dfrac{1}{\rho}, & V_{12}-\dfrac{\cos\epsilon}{\rho} \\ V_{12}-\dfrac{\cos\epsilon}{\rho}, & V_{22}-\dfrac{1}{\rho} \end{vmatrix}=0.$$

There are two such directions within the orientation, one providing a maximum circular curvature, the other providing a minimum circular curvature; it is easy to prove that these directions are at right angles.

The equation for the magnitudes of the two curvatures is

$$\frac{\sin^2\epsilon}{\rho^2}-\frac{1}{\rho}(V_{11}+V_{22}-2V_{12}\cos\epsilon)+V_{11}V_{22}-V_{12}^2=0.$$

Now

$$\begin{aligned} V_{11}V_{22}-V_{12}^2 &= \left(\sum_i\sum_j L_{ij}p_i'p_j'\right)\left(\sum_i\sum_j L_{ij}q_i'q_j'\right)-\left[\sum_i\sum_j L_{ij}p_i'q_j'\right]^2 \\ &= \sum_i\sum_j\sum_k\sum_l (L_{ik}L_{jl}-L_{jk}L_{il})(p_i'q_j'-p_j'q_i')(p_k'q_l'-p_l'q_k') \\ &= \sum_i\sum_j\sum_k\sum_l (ij,\,kl)(p_i'q_j'-p_j'q_i')(p_k'q_l'-p_l'q_k'), \end{aligned}$$

because of the value (§ 78) of the Riemann four-index symbol when the amplitude is primary. Also

$$\sin^2\epsilon=\left(\sum_i\sum_j A_{ij}p_i'p_j'\right)\left(\sum_i\sum_j A_{ij}q_i'q_j'\right)-\left[\sum_i\sum_j A_{ij}p_i'q_j'\right]^2.$$

If therefore we denote by α and β the two roots of the foregoing quadratic in ρ, we have

$$\begin{aligned} \frac{1}{\alpha\beta} &= \frac{V_{11}V_{22}-V_{12}^2}{\sin^2\epsilon} \\ &= K, \end{aligned}$$

where K denotes the Riemann measure of superficial curvature (§ 65) of the amplitude, estimated in the orientation prescribed by the two directions $p_1', \dots, p_n'$; $q_1', \dots, q_n'$.

Thus for a primary amplitude, the Riemann measure K, already entitled generally the sphericity for any amplitude not restricted to be primary, is equal to the product of the two principal circular curvatures of amplitudinal geodesics originating in direction in the assigned orientation. Hence a further significance is assigned to K, which is special to a primary amplitude; and the simplest instance occurs when the amplitude is a surface in homaloidal triple space, the measure K then becoming the Gauss measure or the specific curvature.

The foregoing equation shews that the amplitude possesses another measure of superficial curvature in the assigned orientation. Denoting it by M, we take

$$\begin{aligned} M &= \frac{1}{\alpha} + \frac{1}{\beta} \\ &= \frac{1}{\sin^2 \epsilon} (V_{11} + V_{22} - V_{12} \cos \epsilon). \end{aligned}$$

Now we have

$$\begin{aligned} V_{11} &+ V_{22} - 2 V_{12} \cos \epsilon \\ &= \Big(\sum_i \sum_j L_{ij} p_i' p_j' \Big) \Big(\sum_i \sum_j A_{ij} q_i' q_j' \Big) + \Big(\sum_i \sum_j L_{ij} q_i' q_j' \Big) \Big(\sum_i \sum_j A_{ij} p_i' p_j' \Big) \\ &\qquad\qquad - 2 \Big\{ \sum_i \sum_j L_{ij} p_i' q_j' \Big\} \Big\{ \sum_i \sum_j A_{ij} p_i' q_j' \Big\} \\ &= \sum_i \sum_j \sum_k \sum_l (L_{ik} A_{jl} - L_{jk} A_{il} - L_{il} A_{jk} + L_{jl} A_{ik}) (p_i' q_j' - p_j' q_i') (p_k' q_l' - p_l' q_k'); \end{aligned}$$

and there is the same expression for $\sin^2 \epsilon$ as before. Thus there is an expression for M, of the same character as the expression for K.

The magnitudes K and M are the two superficial measures of superficial curvature of the primary amplitude in the assigned orientation. As for K, so for M, the simplest instance arises when the amplitude is a surface in homaloidal triple space; the measure M then becomes the sum of the two principal curvatures and has been (inaccurately) called the mean curvature.

Ex. Shew that the values of λ and μ for the value α of ρ are given by

$$\frac{\lambda_1^2}{V_{22} - \dfrac{1}{\alpha}} = \frac{\lambda_1 \mu_1}{-\left(V_{12} - \dfrac{\cos \epsilon}{\alpha} \right)} = \frac{\mu_1^2}{V_{11} - \dfrac{1}{\alpha}} = \frac{1}{\left(\dfrac{1}{\beta} - \dfrac{1}{\alpha} \right) \sin^2 \epsilon};$$

and that those for the value β are deduced by interchanging α and β.

These measures K and M for a primary amplitude are homogeneous functions of the surface-variables $p_i' q_j' - q_i' p_j'$, specifying the orientation; and they are of net order zero in those variables, because they are the quotients of homogeneous quadratic functions in the variables. Thus each has its own set of principal values, obtained by assigning the conditions for a maximum or a minimum; and it can be proved * that the principal values of each of the measures K and M is expressible in terms of the principal circular curvatures obtained in § 78. There are $\frac{1}{2} n (n - 1)$ principal values of each of the two measures; these principal values of K are

$$\frac{1}{\rho_i \rho_j},$$

and those of M are

$$\frac{1}{\rho_i} + \frac{1}{\rho_j},$$

* *G.F.D.*, vol. ii, §§ 432-437.

for all the combinations $i, j, = 1, \ldots, n$. Moreover, the orientations for which these are the principal values are correspondingly the same for the two measures: and, in each instance, they are composed by taking, as leading lines of the orientation, the corresponding directions of principal circular curvature.

Measures of regional curvature.

87. Similarly when a direction lies within a regional orientation, the region can be defined by taking three non-complanar directions as leading lines with direction-variables $p_1', \ldots, p_n'$; $q_1', \ldots, q_n'$; $r_1', \ldots, r_n'$: so that, for any included direction, the direction-variables are

$$x_i' = \lambda p_i' + \mu q_i' + \nu r_i', \quad (i = 1, \ldots, n),$$

with λ, μ, ν, as parameters. The permanent relation becomes

$$\lambda^2 + \mu^2 + \nu^2 + 2\mu\nu\cos\gamma + 2\nu\lambda\cos\delta + 2\lambda\mu\cos\epsilon = 1,$$

where γ, δ, ϵ are the inclinations of pairs of leading lines. The circular curvature of the amplitudinal geodesic in the direction $x_1', \ldots, x_n'$, is given by

$$\begin{aligned} \frac{1}{\rho} &= \sum_i \sum_j L_{ij} x_i' x_j' \\ &= \lambda^2 \sum_i \sum_j L_{ij} p_i' p_j' + 2\mu\nu \sum_i \sum_j L_{ij} q_i' r_j' \\ &\quad + \mu^2 \sum_i \sum_j L_{ij} q_i' q_j' + 2\nu\lambda \sum_i \sum_j L_{ij} r_i' p_j' \\ &\quad + \nu^2 \sum_i \sum_j L_{ij} r_i' r_j' + 2\lambda\mu \sum_i \sum_j L_{ij} p_i' q_j' \\ &= \lambda^2 V_{11} + \mu^2 V_{22} + \nu^2 V_{33} + 2\mu\nu V_{23} + 2\nu\lambda V_{31} + 2\lambda\mu V_{12}, \end{aligned}$$

with an obvious significance for V_{11}, V_{22}, V_{33}, V_{23}, V_{31}, V_{12}. The modifiable quantities in this value of $1/\rho$ are the parameters λ, μ, ν, subjected to the foregoing permanent relation; and therefore the maximum and the minimum values of the circular curvature of amplitudinal geodesics, whose directions lie within the regional orientation, are given by the cubic equation

$$\begin{vmatrix} V_{11} - \dfrac{1}{\rho}, & V_{12} - \dfrac{\cos\epsilon}{\rho}, & V_{13} - \dfrac{\cos\delta}{\rho} \\ V_{21} - \dfrac{\cos\epsilon}{\rho}, & V_{22} - \dfrac{1}{\rho}, & V_{23} - \dfrac{\cos\gamma}{\rho} \\ V_{31} - \dfrac{\cos\delta}{\rho}, & V_{32} - \dfrac{\cos\gamma}{\rho}, & V_{33} - \dfrac{1}{\rho} \end{vmatrix} = 0.$$

There are thus three principal measures for the regional orientation. In this equation, the coefficient of $\dfrac{1}{\rho^3}$ is

$$= 1 - \cos^2\gamma - \cos^2\delta - \cos^2\epsilon + 2\cos\gamma\cos\delta\cos\epsilon = S,$$

the customary expression characteristic of the solid angle formed by the three leading lines; and the three principal measures R_1, R_2, R_3, for the regional orientation under consideration are

$$\begin{aligned}
SR_3 &= V_{11}V_{22}V_{33}+2V_{23}V_{31}V_{12}-V_{11}V_{23}^2-V_{22}V_{31}^2-V_{33}V_{12}^2,\\
SR_2 &= V_{22}V_{33}-V_{23}^2+V_{33}V_{11}-V_{13}^2+V_{11}V_{22}-V_{12}^2\\
&\qquad +2(V_{13}V_{12}-V_{11}V_{23})\cos\gamma+2(V_{12}V_{23}-V_{22}V_{13})\cos\delta\\
&\qquad\qquad +2(V_{23}V_{13}-V_{33}V_{12})\cos\epsilon,\\
SR_1 &= \quad V_{11}\sin^2\gamma-2V_{23}(\cos\gamma-\cos\delta\cos\epsilon)\\
&\quad + V_{22}\sin^2\delta-2V_{13}(\cos\delta-\cos\epsilon\cos\gamma)\\
&\quad + V_{33}\sin^2\epsilon-2V_{12}(\cos\epsilon-\cos\gamma\cos\delta).
\end{aligned}$$

Of these measures, R_1 is linear in quality, R_2 is superficial, and R_3 is volumetric.

These regional measures can be expressed as homogeneous functions of the regional variables of the determinantal form $(p_i'q_j'r_k')$ which specify the orientation of the region; and they are of net order zero in those variables, because each of the quantities SR_3, SR_2, SR_1, is a homogeneous quadratic function of the variables as is also the magnitude S. Hence each of the three measures has its own set of principal values, obtained by assigning the conditions for a maximum and a minimum; and as for the measures of superficial curvature K and M of a primary amplitude, so for the measures R_3, R_2, R_1, of regional curvature of the amplitude, it can be proved * that each set of principal values can be expressed in terms of the principal circular curvatures, and that the principal regional orientations are compounded of the principal directions of circular curvatures.

* *G.F.D.*, vol. ii, §§ 438-440.

SECTION II : SURFACES

CHAPTER VIII

Free Surfaces : Introductory

88. Instead of continuing the investigation of further detailed properties of a general amplitude, which are not associated solely or even mainly with the curvatures of its organic geodesics, it is desirable (if only because of the less elaborate formulæ) to discuss in detail some characteristic properties of such specifically restricted amplitudes as surfaces, regions, domains. These restricted amplitudes may exist in some more extensive n-fold curved amplitude : in that event, we must assume n to be less than the dimension-number of the plenary homaloidal space of the less extensive amplitude. Also, then, the properties of the enclosed amplitude will be affected by those of the containing amplitude ; and thus we are compelled, in fact, to consider the geometry of amplitudes which are contained within other more extensive amplitudes.

For the sake of simplicity in the arrangement of issues to be considered, it is convenient to begin with the discussion of amplitudes, of successive specific types, each existing freely in its own plenary homaloidal space : that is, we implicitly exclude, from the beginning of such discussion, the existence of any single amplitude which, itself existing in that plenary homaloidal space, can be made to yield the whole of the specific amplitude by relations solely among its own parameters. On the other hand, in order to complete the consideration of the specific amplitude, it is necessary to consider all the classes of sub-amplitudes which it may contain. Thus in due course, we shall consider free surfaces, as well as curves on a free surface : we shall consider free regions, as well as surfaces and curves enclosed in a free region : and we shall consider free domains, as well as regions, surfaces, and curves, existing within a domain. And it may well happen that some geometrical characteristic property of a sub-amplitude is to be regarded as a characteristic property of the containing amplitude : for instance, the Riemann sphericity of a surface, geodesic to an amplitude at a point O, has been postulated (§ 65) as the sphericity of the amplitude at O in the orientation there defined by that geodesic surface.

Accordingly, we now proceed to consider the intrinsic geometry of surfaces existing freely in a plenary homaloidal space of any number of dimensions ; and, later in this connection, it will be necessary to consider the geometry of curves lying on the free surface.

We shall denote by N the number of dimensions of the plenary homaloidal space in which the surface exists freely; and the space-coordinates of a point on the surface will be represented by $y_1, y_2, y_3, \ldots$, as before. Many results, applicable to a surface, can be deduced from those of the n-fold amplitude by merely taking $n=2$. It is, however, not undesirable to change the notation for the parameters of the surface and for its prime magnitudes, as well as the notation for some of the derived quantities. We shall take p, q, as parameters instead of x_1, x_2; we shall write

$$A_{11},\ A_{12},\ A_{22},\ = A,\ H,\ B;$$

we shall use V^2 to denote the positive quantity $AB - H^2$; and we shall dispense with the notation of the minors of V^2 which, for surfaces, takes the place of the former magnitude Ω. The arc-element ds is given by

$$ds^2 = A\,dp^2 + 2H\,dp\,dq + B\,dq^2,$$

where

$$A = \sum_m \left(\frac{\partial y_m}{\partial p}\right)^2, \quad H = \sum_m \frac{\partial y_m}{\partial p}\frac{\partial y_m}{\partial q}, \quad B = \sum_m \left(\frac{\partial y_m}{\partial q}\right)^2.$$

Frequently, for parametric differentiation of combined order higher than two, with respect to p and q, a double-suffix notation will be used for the sake of brevity; thus we shall write, for any quantity θ,

$$\theta_{ij} = \frac{\partial^{i+j}\theta}{\partial p^i\,\partial q^j},$$

for all positive integer values of i and j, zero included.

When the values of A, H, B, have been explicitly obtained, or when they are arbitrarily assigned, as functions of p and q, no indication of the value of N survives in the first alternative or is forthcoming in the second alternative. Consequently when the arc-element of a surface occurs in this parametric form, there can at once arise a question as to the dimensionality of the plenary homaloidal space in which the surface can exist freely *. The question has already been mentioned in § 11: it will be discussed in some detail for its simplest occurrence here.

In the first place, it may be pointed out that surfaces conceivably can exist in a plenary homaloidal space of an unlimited number of dimensions. Thus consider the superficial configuration, the points of which have their space-coordinates $y_1, y_2, y_3, \ldots$, given by relations

$$y_r = \frac{1}{r\,!}\left(\frac{p^r}{a^{r-1}} + \frac{q^r}{c^{r-1}}\right),$$

* For the present purpose, as already indicated (p. 229) in general, we omit from consideration the possible existence of a plenary curved amplitude containing the surface: there would still remain a question as to the dimensionality of the plenary homaloidal space for the supposed amplitude.

for all positive integer values of $r=1, 2, 3, \ldots, \infty$. We have

$$A=\sum_{s=0}\frac{1}{s!\,s!}\left(\frac{p}{a}\right)^{2s}=J_0\left(2i\frac{p}{a}\right),$$

$$H=\sum_{s=0}\frac{1}{s!\,s!}\left(\frac{p}{a}\right)^{s}\left(\frac{q}{c}\right)^{s}=J_0\left\{2i\left(\frac{pq}{ac}\right)^{\frac{1}{2}}\right\},$$

$$B=\sum_{s=0}\frac{1}{s!\,s!}\left(\frac{q}{c}\right)^{2s}=J_0\left(2i\frac{q}{c}\right),$$

with the customary notation for the Bessel Function of order zero. Such values of A, H, B, are finite for all finite values of p, q; when they occur in the expression for the element of arc on the surface, there is nothing to indicate their source in a plenary space of unlimited dimensionality. Of course, a particular method of derivation does not preclude the ultimate expression for an arc-element from being characteristic of a surface in some less extensive plenary space. But even in that case, and certainly in general, there remains the question: what is the dimensionality of the most restricted plenary homaloidal space containing the surface?

Plenary homaloidal space of a surface.

89. To obtain the integer denoting this dimensionality, it would suffice to determine the smallest number of point-variables $y_1, y_2, y_3, \ldots$, in plenary homaloidal space which allow the simultaneous existence of the three equations

$$A=\sum\left(\frac{\partial y}{\partial p}\right)^2,\quad H=\sum\frac{\partial y}{\partial p}\frac{\partial y}{\partial q},\quad B=\sum\left(\frac{\partial y}{\partial q}\right)^2.$$

It is sometimes assumed, presumably as obvious, that, because there are three equations, they can determine three unknown quantities, and we therefore may postulate three variables y_1, y_2, y_3; these three variables would then be determinate, save as to arbitrary elements from integration, by the three equations. The implicit argument could be valid if the three equations were algebraical, so far as concerns the occurrence of y_1, y_2, y_3. But they are partial differential equations; and no such existence-theorem, suited to their present form, can be cited for the existence of integrals, as could be cited for the existence of roots were three purely algebraical equations propounded.

In the next place, a completely equivalent set of partial differential equations can be taken in the form

$$\frac{\partial y_1}{\partial p}=P_1,\quad \frac{\partial y_2}{\partial p}=P_2,\quad \frac{\partial y_3}{\partial p}=P_3,$$

$$\frac{\partial y_1}{\partial q}=Q_1,\quad \frac{\partial y_2}{\partial q}=Q_2,\quad \frac{\partial y_3}{\partial q}=Q_3,$$

$$P_1^2+P_2^2+P_3^2=A,$$

$$P_1Q_1+P_2Q_2+P_3Q_3=H,$$

$$Q_1^2+Q_2^2+Q_3^2=B,$$

nine differential equations in nine variables y_1, P_1, Q_1; y_2, P_2, Q_2; y_3, P_3, Q_3; but, in addition, there are three necessary equations

$$\frac{\partial P_1}{\partial q}=\frac{\partial Q_1}{\partial p}, \quad \frac{\partial P_2}{\partial q}=\frac{\partial Q_2}{\partial p}, \quad \frac{\partial P_3}{\partial q}=\frac{\partial Q_3}{\partial p},$$

so that an argument from the number of equations alone ceases to be valid. And even to a set of equations of this form, there is no existence-theorem which is applicable.

But further, in the absence alike of an applicable existence-theorem and of any set of direct solutions of the three equations, we can proceed by way of indirect inference. Let it be supposed that, though unobtained, such direct solutions exist; and let x, y, z, denote such a set, so that x, y, z, are functions of p and q. Then the surface, thus determined, must exist in a triple homaloidal space where x, y, z, denote point-coordinates, and the customary Gauss theory of surfaces will apply, valid for a plenary triple space, but requiring modification for any more extensive plenary space. In that theory, it is necessary to consider the quadratic form associated with the circular curvature of a superficial geodesic; and the coefficients in that quadratic form satisfy three distinct equations, such coefficients being constructed from the values of x, y, z, and therefore (by this mode of derivation) implicitly depending upon the original magnitudes A, H, B. Of the three equations indicated, two are the customary Mainardi-Codazzi relations *; the third is the Gauss characteristic equation, so often connected with the Gauss measure of curvature of the surface. Into all these equations, A, H, B, and their derivatives enter: and so it appears that the quantities A, H, B, for a Gauss surface, are subject to certain relations.

To render this statement more precise, we refer the obtained surface to its curves of curvature as the parametric curves. The arc-element is then given by a relation

$$ds^2=E\,dp^2+G\,dq^2;$$

the circular curvature of a superficial geodesic is given by a relation

$$\frac{ds^2}{\rho}=L\,dp^2+N\,dq^2;$$

and the Gauss characteristic equation, associated with his measure of curvature of a surface in triple homaloidal space, is

$$LN=-\tfrac{1}{2}(E_{22}+G_{11})+\frac{1}{4E}(E_1G_1+E_2{}^2)+\frac{1}{4G}(E_2G_2+G_1{}^2).$$

* They will be found in any treatise on the differential geometry of surfaces; references to the memoir (1856) of Mainardi and the memoir (1868) of Codazzi will be found in my *Lectures on Differential Geometry*, § 35. The Gauss characteristic equation occurs there also, § 34; it was first obtained in Gauss's classical memoir, *Disquisitiones generales circa superficies curvas.*

When the arc-element is thus postulated, E and G can be regarded as known functions. Further, the Gauss measure of curvature K, which is an absolute invariant belonging to the arc-element only and which can be expressed solely in terms of the coefficients in the arc-relation, is now given by

$$LN = KEG.$$

Accordingly, when the foregoing value of LN is denoted by u, we have

$$LN = u, \quad KEG = u.$$

It will now be proved that the known necessary relations to be satisfied by L, N, E, G, require two partial differential equations to be satisfied by K, the absolute invariant; as K is expressible in terms solely of unconditioned original quantities, and so itself is an unconditioned quantity, the result leads to a contradiction.

The known necessary relations, being the Mainardi-Codazzi relations, now are

$$L_2 = \tfrac{1}{2}E_2\left(\frac{L}{E} + \frac{N}{G}\right), \quad N_1 = \tfrac{1}{2}G_1\left(\frac{L}{E} + \frac{N}{G}\right).$$

Let $L^2 = \theta$, $N^2 = \phi$, so that $\theta\phi = u^2$; then

$$\theta_2 = E_2\left(\frac{\theta}{E} + \frac{u}{G}\right), \qquad \phi_1 = G_1\left(\frac{u}{E} + \frac{\phi}{G}\right).$$

In the second of these transformed equations, substitute u^2/θ for ϕ; we find, after a slight reduction,

$$\theta_1 = \theta\left(2\frac{u_1}{u} - \frac{G_1}{G}\right) - \frac{G_1}{Eu}\theta^2.$$

Hence

$$\frac{\partial}{\partial p}\left\{E_2\left(\frac{\theta}{E} + \frac{u}{G}\right)\right\} = \frac{\partial}{\partial q}\left\{\theta\left(2\frac{u_1}{u} - \frac{G_1}{G}\right) - \frac{G_1}{Eu}\theta^2\right\},$$

which, on evaluation and after substitution for θ_1 and θ_2 where they occur, becomes

$$\frac{\theta^2}{u^2}\frac{G_{12}u - G_1 u_2}{E} - 2\theta\left\{\frac{u_{12}}{u} - \frac{u_1 u_2}{u^2} - Q\right\} + \frac{E_{12}u - E_2 u_1}{G} = 0,$$

where

$$2Q = \frac{E_{12}}{E} + \frac{G_{12}}{G} - \left(\frac{E_1 E_2}{E^2} - 2\frac{G_1 E_2}{EG} + \frac{G_1 G_2}{G^2}\right).$$

Had we proceeded similarly from the first equation by substituting u^2/ϕ for θ, we should have been led to the equation

$$\phi_2 = \phi\left(2\frac{u_2}{u} - \frac{E_2}{E}\right) - \frac{E_2}{Gu}\phi^2,$$

and in due course to the equation

$$\frac{\phi^2}{u^2}\frac{E_{12}u - E_2 u_1}{G} - 2\phi\left\{\frac{u_{12}}{u} - \frac{u_1 u_2}{u^2} - Q\right\} + \frac{G_{12}u - G_1 u_2}{E} = 0.$$

The two relations, which involve u_{12}, are transformed into one another by the equation

$$\theta\phi = u^2;$$

and therefore they are equivalent to only a single relation. Thus θ is a non-rational function of u_{12}, u_1, u_2, u, being the root of a quadratic equation. When this non-rational function is substituted in the two equations

$$\theta_2 = E_2\left(\frac{\theta}{E} + \frac{u}{G}\right), \quad \theta_1 = \theta\left(2\frac{u_1}{u} - \frac{G_1}{G}\right) - \frac{G_1}{Eu}\theta^2,$$

in turn, there result two partial differential equations of the third order in u. But $u = KEG$; and therefore, on substituting, we obtain two partial differential equations of the third order in K, the coefficients in which are derivatives of E and G, of the third order and of lower orders. Now as

$$K = \frac{1}{EG}u,$$

where u contains second-order derivatives of E and G, the quantity K itself involves derivatives of E and G of the second order; and therefore the resulting equations ultimately involve fifth-order derivatives of E and G. Moreover, the two equations are distinct from one another, for one of them involves the magnitude $\dfrac{\partial^3 u}{\partial p\, \partial q^2}$ $\left(\text{but not } \dfrac{\partial^3 u}{\partial p^2 \partial q}\right)$ and the other involves the magnitude $\dfrac{\partial^3 u}{\partial p^2 \partial q}$ $\left(\text{but not } \dfrac{\partial^3 u}{\partial p\, \partial q^2}\right)$: we therefore, in the end, have two distinct partial differential equations of the third order as consequences of the initial hypothesis of a plenary triple space.

Reverting to the initial form

$$ds^2 = E\, dp^2 + G\, dq^2$$

for the arc-element, we note that there are two analytical limitations: one is descriptive, appertaining solely to the form: the other is organic, appertaining to an intrinsic property. The descriptive limitation is that the parametric curves are orthogonal: but this limitation is only analytically formal, because, by a mere transformation of independent variables, the quantity

$$A\, dx_1^2 + 2H\, dx_1 dx_2 + B\, dx_2^2,$$

where A, H, B, are arbitrary functions of x_1 and x_2, can be changed to the assumed type in an unrestricted number of ways. Thus the limitation is only a simplification in the calculations and does not necessitate any pervading condition.

The other limitation is of an intrinsic character. We have assumed, in the calculations, that the circular curvature of a geodesic is given by

$$\frac{1}{\rho} = L\, dp^2 + N\, dq^2:$$

in other words, we have assumed that, from among the aggregate of orthogonal

parametric curves, a particular and unique selection has been made by taking the curves of curvature. Such a selection, after the postulation of a general form $E\,dp^2 + G\,dq^2$, implies a single restrictive (and pervading) condition whatever shape it may assume; but the single condition of such a character does not exact, as its equivalent, more than a single equation. There must, however, persist some such single equation, or an equivalent; with the customary notation in the Gauss theory, the equation is $M=0$, and it has been used in the form of the Mainardi-Codazzi relations as stated.

The foregoing analysis has resulted in the provision of two distinct partial differential equations in the quantity u. The implicit assumption, which has been indicated, requires that some one condition shall be satisfied. There are two equations in virtue of which it can be satisfied: after it has been satisfied, there survives one further relation, independent of that condition. The only intrinsic magnitude of the surface, now outstanding in occurrence without presumable conditions, is the Gauss measure K which is a definite function, invariantive in its nature whatever be the variables of reference, and derived from the arbitrarily assumed magnitudes E and G. In deriving it, differential coefficients have to be taken with respect to the variables used, in this instance the parameters of the curves of curvature; but the one condition thus imposed has already been taken into account. It therefore is to be inferred that K must itself satisfy an equation, to be satisfied in virtue of the two constructed equations; or (if we take account of the condition required to justify the implicit assumption indicated) the surviving relation must be the equivalent of the equation to be satisfied by K: that is, there must be such a further relation to be satisfied by the two arbitrary magnitudes E and G, because K is expressible in terms of the primary magnitudes alone.

Now equations of the character obtained, in the form of particular explicit differential equations, involving fifth-order derivatives of quantities E and G, cannot be satisfied by assuming E and G to be arbitrary functions of the two variables free from all limitations. Consequently the initial hypothesis, as it has led to an untenable inference for which it is argumentatively essential, cannot be maintained: we cannot declare that a surface, which has its arc-element represented by the equation

$$ds^2 = A\,du^2 + 2H\,du\,dv + B\,dv^2$$

where each of the coefficients A, H, B, is a function of u and v, arbitrarily assumed and subject to no limitations, necessarily exists in a triple plenary homaloidal space.

Moreover, the preceding analysis provides no indication of any upper limit to the number of dimensions of the plenary homaloidal space for the surface. We have adduced (§ 88) one instance of a surface which, when regard is paid to the analytical source of its arc-element, may be deemed to require an unlimited number of dimensions for its plenary homaloidal space.

90. After this discussion, it will be assumed that the plenary homaloidal space, in which the surface exists freely, has a merely general number of dimensions. This number will be assumed to be greater than three: investigations of surfaces, in triple homaloidal space, belong to the well-established Gauss theory. The number will usually be assumed to be greater than four: the known results * for surfaces, in quadruple homaloidal space, can sometimes be used as simple verifications of results appertaining to a quite general plenary space.

The primitive magnitudes A, H, B, occur in the expression

$$ds^2 = A\,dp^2 + 2H\,dp\,dq + B\,dq^2$$

for the arc-element of the surface. When they are constructed from the space-coordinates of a point on the surface, they involve only the first parametric derivatives of the coordinates.

The first parametric derivatives of A, H, B, therefore involve the second derivatives of these coordinates; and thus these second derivatives occur implicitly in one mode of construction of the Christoffel symbols $[ab, c]$, $\{ab, c\}$, first defined in connection with the derivatives of the primary magnitudes. As already indicated (§ 12), for a surface we shall write

$$\Gamma_{ij} = \{ij, 1\}, \quad \Delta_{ij} = \{ij, 2\},$$

for the combinations $i, j, = 1, 2$, with the convention $x_1 = p$, $x_2 = q$. We thus have

$$\left.\begin{aligned}
\sum \frac{\partial y}{\partial p}\frac{\partial^2 y}{\partial p^2} &= \tfrac{1}{2}\frac{\partial A}{\partial p} &&= A\Gamma_{11} + H\Delta_{11}\\
\sum \frac{\partial y}{\partial p}\frac{\partial^2 y}{\partial p\,\partial q} &= \tfrac{1}{2}\frac{\partial A}{\partial q} &&= A\Gamma_{12} + H\Delta_{12}\\
\sum \frac{\partial y}{\partial p}\frac{\partial^2 y}{\partial q^2} &= \frac{\partial H}{\partial q} - \tfrac{1}{2}\frac{\partial B}{\partial p} &&= A\Gamma_{22} + H\Delta_{22}
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\sum \frac{\partial y}{\partial q}\frac{\partial^2 y}{\partial p^2} &= \frac{\partial H}{\partial p} - \tfrac{1}{2}\frac{\partial A}{\partial q} &&= H\Gamma_{11} + B\Delta_{11}\\
\sum \frac{\partial y}{\partial q}\frac{\partial^2 y}{\partial p\,\partial q} &= \tfrac{1}{2}\frac{\partial B}{\partial p} &&= H\Gamma_{12} + B\Delta_{12}\\
\sum \frac{\partial y}{\partial q}\frac{\partial^2 y}{\partial q^2} &= \tfrac{1}{2}\frac{\partial B}{\partial q} &&= H\Gamma_{22} + B\Delta_{22}
\end{aligned}\right\}.$$

Also we use V^2 (instead of Ω) to denote the determinant

$$V^2 = \begin{vmatrix} A, & H \\ H, & B \end{vmatrix} = AB - H^2,$$

which is the discriminant of the quadratic differential form ds^2. Thus V^2 is a

* *G.F.D.*, vol. i, chaps. xii-xiv.

relative invariant when the parameters p and q of the form are subjected to transformations: in fact, when p_0 and q_0 denote the transformed parameters

$$A_0B_0 - H_0^2 = \left\{\frac{\partial(p, q)}{\partial(p_0, q_0)}\right\}^2 (AB - H^2).$$

Further, we have

$$\frac{1}{V}\frac{\partial V}{\partial p} = \Gamma_{11} + \Delta_{12}, \quad \frac{1}{V}\frac{\partial V}{\partial q} = \Gamma_{12} + \Delta_{22}.$$

As the quantities Γ_{ij} and Δ_{ij} are expressible in terms of the first derivatives of A, H, B, it is to be expected that relations will exist among some of the first derivatives of Γ_{ij} and Δ_{ij} because of the identities

$$\frac{\partial}{\partial q}\left(\frac{\partial \Theta}{\partial p}\right) = \frac{\partial}{\partial p}\left(\frac{\partial \Theta}{\partial q}\right),$$

for $\Theta = A, H, B$. The simplest expression of such relations is obtained by the use of a magnitude, to be denoted by K, and analytically defined at this stage by the equation

$$V^2K = -\tfrac{1}{2}\left(\frac{\partial^2 A}{\partial q^2} - 2\frac{\partial^2 H}{\partial p\,\partial q} + \frac{\partial^2 B}{\partial p^2}\right) \\ + (A, H, B \between \Gamma_{12} \between \Delta_{12})^2 - (A, H, B \between \Gamma_{11}, \Delta_{11} \between \Gamma_{22}, \Delta_{22}).$$

Moreover, we verify (by direct substitution) that

$$\tfrac{1}{2}\left(\frac{\partial^2 A}{\partial q^2} - 2\frac{\partial^2 H}{\partial p\,\partial q} + \frac{\partial^2 B}{\partial p^2}\right) = \sum\left(\frac{\partial^2 y}{\partial p\,\partial q}\right)^2 - \sum\left(\frac{\partial^2 y}{\partial p^2}\frac{\partial^2 y}{\partial q^2}\right).$$

The two relations, when the plenary space is triple, constitute the Gauss equation for the measure of specific curvature.

It proves convenient, for ulterior purposes, to possess the actual values of the derivatives of Γ_{ij} and Δ_{ij}: the indicated relations will be deduced from these actual values, which are as follows:

$$\left.\begin{aligned} V^2\frac{\partial \Gamma_{11}}{\partial p} &= \tfrac{1}{2}B\frac{\partial^2 A}{\partial p^2} - H\left(\frac{\partial^2 H}{\partial p^2} - \tfrac{1}{2}\frac{\partial^2 A}{\partial p\,\partial q}\right) \\ &\qquad - V^2(\Gamma_{11}\Gamma_{11} + \Delta_{11}\Gamma_{12}) - B(11 \between 11) + H(11 \between 12) \\ V^2\frac{\partial \Gamma_{11}}{\partial q} &= \tfrac{1}{2}B\frac{\partial^2 A}{\partial p\,\partial q} - H\left(\frac{\partial^2 H}{\partial p\,\partial q} - \tfrac{1}{2}\frac{\partial^2 A}{\partial q^2}\right) \\ &\qquad - V^2(\Gamma_{11}\Gamma_{12} + \Delta_{11}\Gamma_{22}) - B(11 \between 12) + H(11 \between 22) \end{aligned}\right\};$$

$$\left.\begin{aligned} V^2\frac{\partial \Gamma_{12}}{\partial p} &= \tfrac{1}{2}B\frac{\partial^2 A}{\partial p\,\partial q} - \tfrac{1}{2}H\frac{\partial^2 B}{\partial p^2} \\ &\qquad - V^2(\Gamma_{12}\Gamma_{11} + \Delta_{12}\Gamma_{12}) - B(12 \between 11) + H(12 \between 12) \\ V^2\frac{\partial \Gamma_{12}}{\partial q} &= \tfrac{1}{2}B\frac{\partial^2 A}{\partial q^2} - \tfrac{1}{2}H\frac{\partial^2 B}{\partial p\,\partial q} \\ &\qquad - V^2(\Gamma_{12}\Gamma_{12} + \Delta_{12}\Gamma_{22}) - B(12 \between 12) + H(12 \between 22) \end{aligned}\right\};$$

$$\left.\begin{aligned}
V^2\frac{\partial \Gamma_{22}}{\partial p} &= B\left(\frac{\partial^2 H}{\partial p\,\partial q}-\tfrac{1}{2}\frac{\partial^2 B}{\partial p^2}\right)-\tfrac{1}{2}H\frac{\partial^2 B}{\partial p\partial q}\\
&\qquad - V^2(\Gamma_{22}\Gamma_{11}+\Delta_{22}\Gamma_{12})-B(22\between 11)+H(22\between 12)\\
V^2\frac{\partial \Gamma_{22}}{\partial q} &= B\left(\frac{\partial^2 H}{\partial q^2}-\tfrac{1}{2}\frac{\partial^2 B}{\partial p\,\partial q}\right)-H\left(\tfrac{1}{2}\frac{\partial^2 B}{\partial q^2}\right)\\
&\qquad - V^2(\Gamma_{22}\Gamma_{12}+\Delta_{22}\Gamma_{22})-B(22\between 12)+H(22\between 22)
\end{aligned}\right\};$$

$$\left.\begin{aligned}
V^2\frac{\partial \Delta_{11}}{\partial p} &= -\tfrac{1}{2}H\frac{\partial^2 A}{\partial p^2}+A\left(\frac{\partial^2 H}{\partial p^2}-\tfrac{1}{2}\frac{\partial^2 A}{\partial p\,\partial q}\right)\\
&\qquad - V^2(\Gamma_{11}\Delta_{11}+\Delta_{11}\Delta_{12})+H(11\between 11)-A(11\between 12)\\
V^2\frac{\partial \Delta_{11}}{\partial q} &= -\tfrac{1}{2}H\frac{\partial^2 A}{\partial p\,\partial q}+A\left(\frac{\partial^2 H}{\partial p\,\partial q}-\tfrac{1}{2}\frac{\partial^2 A}{\partial q^2}\right)\\
&\qquad - V^2(\Gamma_{11}\Delta_{12}+\Delta_{11}\Delta_{22})+H(11\between 12)-A(11\between 22)
\end{aligned}\right\};$$

$$\left.\begin{aligned}
V^2\frac{\partial \Delta_{12}}{\partial p} &= -\tfrac{1}{2}H\frac{\partial^2 A}{\partial p\,\partial q}+\tfrac{1}{2}A\frac{\partial^2 B}{\partial p^2}\\
&\qquad - V^2(\Gamma_{12}\Delta_{11}+\Delta_{12}\Delta_{12})+H(12\between 11)-A(12\between 12)\\
V^2\frac{\partial \Delta_{12}}{\partial q} &= -\tfrac{1}{2}H\frac{\partial^2 A}{\partial q^2}+\tfrac{1}{2}A\frac{\partial^2 B}{\partial p\,\partial q}\\
&\qquad - V^2(\Gamma_{12}\Delta_{12}+\Delta_{12}\Delta_{22})+H(12\between 12)-A(12\between 22)
\end{aligned}\right\};$$

$$\left.\begin{aligned}
V^2\frac{\partial \Delta_{22}}{\partial p} &= -H\left(\frac{\partial^2 H}{\partial p\,\partial q}-\tfrac{1}{2}\frac{\partial^2 B}{\partial p^2}\right)+\tfrac{1}{2}A\frac{\partial^2 B}{\partial p\,\partial q}\\
&\qquad - V^2(\Gamma_{22}\Delta_{11}+\Delta_{22}\Delta_{12})+H(22\between 11)-A(22\between 12)\\
V^2\frac{\partial \Delta_{22}}{\partial q} &= -H\left(\frac{\partial^2 H}{\partial q^2}-\tfrac{1}{2}\frac{\partial^2 B}{\partial p\,\partial q}\right)+\tfrac{1}{2}A\frac{\partial^2 B}{\partial q^2}\\
&\qquad - V^2(\Gamma_{22}\Delta_{12}+\Delta_{22}\Delta_{22})+H(22\between 12)-A(22\between 22)
\end{aligned}\right\};$$

where symbols, such as $(11\between 11)$ throughout, have the significance

$$\begin{aligned}
(ij\between kl) &= (A,\ H,\ B\between \Gamma_{ij},\ \Delta_{ij}\between \Gamma_{kl},\ \Delta_{kl})\\
&= A\Gamma_{ij}\Gamma_{kl}+H(\Gamma_{ij}\Delta_{kl}+\Delta_{ij}\Gamma_{kl})+B\Delta_{ij}\Delta_{kl},
\end{aligned}$$

for all the indices $i, j, k, l, =1, 2$.

From these values, we at once derive the four differential relations between first derivatives of Γ_{ij} and Δ_{ij} in the forms

$$\left.\begin{aligned}
\left(\frac{\partial \Gamma_{11}}{\partial q}+\Gamma_{22}\Delta_{11}\right)-\left(\frac{\partial \Gamma_{12}}{\partial p}+\Gamma_{12}\Delta_{12}\right) &= -HK\\
\left(\frac{\partial \Delta_{22}}{\partial p}+\Delta_{11}\Gamma_{22}\right)-\left(\frac{\partial \Delta_{12}}{\partial q}+\Delta_{12}\Gamma_{12}\right) &= -HK
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\left(\frac{\partial \Gamma_{22}}{\partial p}+\Gamma_{11}\Gamma_{22}+\Gamma_{12}\Delta_{22}\right)-\left(\frac{\partial \Gamma_{12}}{\partial q}+\Gamma_{12}{}^2+\Gamma_{22}\Delta_{12}\right) &= \ \ BK\\
\left(\frac{\partial \Delta_{11}}{\partial q}+\Delta_{12}\Gamma_{11}+\Delta_{22}\Delta_{11}\right)-\left(\frac{\partial \Delta_{12}}{\partial p}+\Delta_{11}\Gamma_{12}+\Delta_{12}{}^2\right) &= \ \ AK
\end{aligned}\right\}$$

It has already been pointed out (§ 16) that there is only a single Riemann four-index symbol appertaining to a surface; and it was given in the form

$$(12, 12) = -\tfrac{1}{2}\left(\frac{\partial^2 A}{\partial q^2} - 2\frac{\partial^2 H}{\partial p\,\partial q} + \frac{\partial^2 B}{\partial p^2}\right)$$
$$+ (A, H, B\between\Gamma_{12}, \Delta_{12})^2 - (A, H, B\between\Gamma_{11}, \Delta_{11}\between\Gamma_{22}, \Delta_{22}),$$

that is, we have

$$(12, 12) = V^2 K.$$

The Gauss characteristic equation for a surface in triple homaloidal space, combined with the equation determining the principal radii of curvature at any point, yields an interpretation of K: it is the product of the two principal curvatures of such a surface; and frequently it is called the Gauss measure of curvature (or the specific curvature) of the surface, existing in triple homaloidal space.

But the quantity K exists for any surface, whatever be the dimensions of the plenary homaloidal surface: its significance will be obtained later (§ 112).

It is to be noted that the four relations are the aggregate of the relations arising from the general relation

$$\frac{\partial}{\partial x_j}\{ki, l\} - \frac{\partial}{\partial x_i}\{kj, l\} + \sum_p [\{ki, p\}\{jp, l\} - \{kj, p\}\{ip, l\}] = \frac{1}{\Omega}\sum_\mu a_{\mu l}(ij, k\mu)$$

of § 22, when this is applied to the surface. For we then have

$$a_{11} = B, \quad a_{12} = -H, \quad a_{22} = A, \quad \Omega = V^2;$$

the only possible independent forms arise from $i=1$, $j=2$; the first of the four relations arises for $k=1$, $l=1$; the second for $k=2$, $l=2$; the third for $k=2$, $l=1$; and the fourth for $k=1$, $l=2$; always in connection with the relation

$$-(12, 21) = (12, 12) = V^2 K.$$

Ex. Verify, by direct calculation from the equations

$$A = \left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2, \quad H = \frac{\partial x}{\partial p}\frac{\partial x}{\partial q} + \frac{\partial y}{\partial p}\frac{\partial y}{\partial q}, \quad B = \left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2,$$

that, when the surface is actually a plane,

$$K = 0;$$

and prove that the same result is true for developable surfaces which exist in triple homaloidal space.

Tangent plane of a surface.

91. The equations of the tangent line to any curve on the surface at the point $y_1, y_2, \ldots$ are

$$\frac{\bar{y}_1 - y_1}{y_1'} = \frac{\bar{y}_2 - y_2}{y_2'} = \ldots = D,$$

so that the coordinates of any point on the line are represented by the typical equation

$$\begin{aligned} \bar{y}_m - y_m &= y_m{}'D \\ &= \left(\frac{\partial y_m}{\partial p} p' + \frac{\partial y_m}{\partial q} q'\right) D \\ &= \lambda \frac{\partial y_m}{\partial p} + \mu \frac{\partial y_m}{\partial q}, \end{aligned}$$

where λ and μ are parametric along the line. Hence every point on every line touching the surface satisfies equations

$$\left\| \begin{matrix} \bar{y}_m - y_m \\ \dfrac{\partial y_m}{\partial p} \\ \dfrac{\partial y_m}{\partial q} \end{matrix} \right\| = 0,$$

which will be written

$$\left\| \bar{y} - y, \quad \frac{\partial y}{\partial p}, \quad \frac{\partial y}{\partial q} \right\| = 0.$$

Accordingly, these are the equations of the tangent plane of the surface containing all such lines, tangents to all the curves on the surface at the specified initial point. When the plenary space is of N dimensions, there are $N-2$ independent equations in the full set for a tangent plane; but the simpler analytical expression of the plane is given by the parametric form

$$\bar{y} - y = \lambda \frac{\partial y}{\partial p} + \mu \frac{\partial y}{\partial q}.$$

It is convenient to use direction-variables p', q', to determine a direction in the surface; and then the typical equation of the line in that direction touching the surface is

$$\bar{y} - y = \frac{\partial y}{\partial p} p' + \frac{\partial y}{\partial q} q'.$$

We shall require a knowledge of the direction, which lies in the tangent plane and is perpendicular to the forègoing direction. If such direction is represented typically by

$$\lambda \frac{\partial y}{\partial p} + \mu \frac{\partial y}{\partial q},$$

the condition of perpendicularity is

$$\sum \left\{ \left(\lambda \frac{\partial y}{\partial p} + \mu \frac{\partial y}{\partial q}\right) \left(\frac{\partial y}{\partial p} p' + \frac{\partial y}{\partial q} q'\right) \right\} = 0,$$

that is,

$$\lambda (Ap' + Hq') + \mu (Hp' + Bq') = 0.$$

Also, when the foregoing typical magnitude is actually a spatial direction-cosine, there is a necessary relation

$$\sum\left(\lambda\frac{\partial y}{\partial p}+\mu\frac{\partial y}{\partial q}\right)^2=1,$$

that is,

$$A\lambda^2+2H\lambda\mu+B\mu^2=1.$$

Consequently,

$$\frac{\lambda}{-(Hp'+Bq')}=\frac{\mu}{(Ap'+Hq')}=\frac{1}{V},$$

where the choice of sign for V is a selection of positive direction of the line. Accordingly, the typical direction-cosine of a line, lying in the tangent plane of the surface and drawn at right angles to the direction p', q', in the tangent plane is

$$\frac{1}{V}\left\{(Ap'+Hq')\frac{\partial y}{\partial q}-(Hp'+Bq')\frac{\partial y}{\partial p}\right\}.$$

This result will be used in determining the torsion of a superficial geodesic; for it will appear that the special form of the general property of § 33 for any amplitude now gives the property that the binormal of a superficial geodesic lies in the tangent plane of the surface *.

The envelope of the tangent plane of a surface, when the plane is represented by the typical equation

$$\bar{y}-y=\lambda\frac{\partial y}{\partial p}+\mu\frac{\partial y}{\partial q},$$

is obtained (as to its equations) by associating the further $2N$ typical relations

$$-\frac{\partial y}{\partial p}=\frac{\partial\lambda}{\partial p}\frac{\partial y}{\partial p}+\frac{\partial\mu}{\partial p}\frac{\partial y}{\partial q}+\lambda\frac{\partial^2 y}{\partial p^2}+\mu\frac{\partial^2 y}{\partial p\,\partial q},$$

$$-\frac{\partial y}{\partial q}=\frac{\partial\lambda}{\partial q}\frac{\partial y}{\partial p}+\frac{\partial\mu}{\partial q}\frac{\partial y}{\partial q}+\lambda\frac{\partial^2 y}{\partial p\,\partial q}+\mu\frac{\partial^2 y}{\partial q^2},$$

with the N equations. Eliminating the p-derivatives of λ and μ from the whole set of equations, we have $N-2$ independent equations of the type

$$\lambda\begin{vmatrix}\dfrac{\partial^2 y_1}{\partial p^2}, & \dfrac{\partial y_1}{\partial p}, & \dfrac{\partial y_1}{\partial q}\\[2ex] \dfrac{\partial^2 y_2}{\partial p^2}, & \dfrac{\partial y_2}{\partial p}, & \dfrac{\partial y_2}{\partial q}\\[2ex] \dfrac{\partial^2 y_m}{\partial p^2}, & \dfrac{\partial y_m}{\partial p}, & \dfrac{\partial y_m}{\partial q}\end{vmatrix}+\mu\begin{vmatrix}\dfrac{\partial^2 y_1}{\partial p\,\partial q}, & \dfrac{\partial y_1}{\partial p}, & \dfrac{\partial y_1}{\partial q}\\[2ex] \dfrac{\partial^2 y_2}{\partial p\,\partial q}, & \dfrac{\partial y_2}{\partial p}, & \dfrac{\partial y_2}{\partial q}\\[2ex] \dfrac{\partial^2 y_m}{\partial p\,\partial q}, & \dfrac{\partial y_m}{\partial p}, & \dfrac{\partial y_m}{\partial q}\end{vmatrix}=0,$$

for $m=3, 4, \ldots, N$; and these can be represented in the forms

$$\lambda E_m+\mu F_m=0.$$

* The result is immediate for a surface in triple homaloidal space, because the unique normal to the surface is the prime normal of a geodesic on the surface.

Similarly eliminating the q-derivatives of λ and μ from the set of equations typified by the second relation, we are led to a like equation

$$\lambda F_m + \mu G_m = 0,$$

also holding for $m=3, 4, \ldots, N$.

When all the quantities

$$E_i G_j - F_i F_j$$

for $i, j=3, \ldots, N$, taken independently of one another vanish, while not all the quantities E, F, G, vanish, the two sets of $N-2$ new equations are satisfied by

$$\lambda = \mu k,$$

where k is a determinate finite quantity. The envelope of the tangent plane then is such that, along it, the equations of the line

$$\frac{\bar{y}_1 - y_1}{k\dfrac{\partial y_1}{\partial p} + \dfrac{\partial y_1}{\partial q}} = \frac{\bar{y}_2 - y_2}{k\dfrac{\partial y_2}{\partial p} + \dfrac{\partial y_2}{\partial q}} = \ldots$$

are satisfied; that is, the envelope can be made to consist of a succession of lines, or the surface is a ruled surface.

When all the quantities E, F, G, vanish, the envelope is a developable surface; and it is easy to verify that the relation

$$K=0$$

then is satisfied.

When some at least of the quantities

$$E_i G_j - F_i F_j$$

differ from zero, the two sets of new equations are satisfied only if

$$\lambda=0, \quad \mu=0:$$

that is, for the point on the envelope, we have

$$\bar{y} - y = 0,$$

or the surface itself is then the envelope.

Orthogonal homaloid of a surface.

92. Every direction in the tangent plane of the surface can be represented typically by the expression

$$\alpha \frac{\partial y}{\partial p} + \beta \frac{\partial y}{\partial q}$$

with parametric values of α and β. Hence the line joining the point $y_1, y_2, \ldots$ to the point $\bar{y}_1, \bar{y}_2, \ldots$ is at right angles to every direction in the tangent plane, that is, the line so drawn is orthogonal to the tangent plane, when the equation

$$\alpha \sum (\bar{y} - y) \frac{\partial y}{\partial p} + \beta \sum (\bar{y} - y) \frac{\partial y}{\partial q} = 0$$

is satisfied for all values of α and β. Accordingly, the locus of these orthogonal lines is given by the two equations

$$\sum(\bar{y}-y)\frac{\partial y}{\partial p}=0, \quad \sum(\bar{y}-y)\frac{\partial y}{\partial q}=0;$$

and these equations represent a homaloid, manifestly of $N-2$ dimensions. It is called the *orthogonal homaloid* of the surface.

When the plenary homaloidal space of the surface is triple, this orthogonal homaloid is a line; it is the unique normal to the surface.

When that plenary space is quadruple, the orthogonal homaloid in question is a plane; it is the orthogonal plane of the surface.

Whatever be the range of the plenary space, the orthogonal homaloid of the surface is independent of any particular direction through the point at which it is orthogonal to the surface.

The envelope of the orthogonal homaloid (if any) can be regarded in two aspects.

We may seek the complete envelope of the orthogonal homaloid for the whole surface. For that purpose, with the two equations

$$\sum(\bar{y}-y)\frac{\partial y}{\partial p}=0, \quad \sum(\bar{y}-y)\frac{\partial y}{\partial q}=0,$$

of the orthogonal, we associate the derived equations

$$\sum(\bar{y}-y)\frac{\partial^2 y}{\partial p^2}=\sum\left(\frac{\partial y}{\partial p}\right)^2=A,$$

$$\sum(\bar{y}-y)\frac{\partial^2 y}{\partial p\,\partial q}=\sum\left(\frac{\partial y}{\partial p}\frac{\partial y}{\partial q}\right)=H,$$

$$\sum(\bar{y}-y)\frac{\partial^2 y}{\partial q^2}=\sum\left(\frac{\partial y}{\partial q}\right)^2=B,$$

manifestly linearly independent of the two original equations. Thus, in general, the envelope is a configuration of $N-3$ dimensions: because the elimination, of the two parameters p and q among the five equations, leaves three equations involving space-coordinates alone.

But the result is not significant unless $N \geqslant 5$. If $N=5$, there are five coordinates $\bar{y}$, each potentially expressible by these equations in terms of p and q; the locus of the points then is a surface in a plenary quintuple homaloidal space. This result assumes that the determinant J of the coefficients of the five magnitudes $\bar{y}_m - y_m$ does not vanish. Now J has constituents

$$\frac{\partial y_m}{\partial p}, \quad \frac{\partial y_m}{\partial q}, \quad \frac{\partial^2 y_m}{\partial p^2}, \quad \frac{\partial^2 y_m}{\partial p\,\partial q}, \quad \frac{\partial^2 y_m}{\partial q^2},$$

in a row, and there are five rows for $m=1, 2, 3, 4, 5$; thus

$$J^2=\begin{vmatrix} A, & H, & [11,1], & [12,1], & [22,1] \\ H, & B, & [11,2], & [12,2], & [22,2] \\ [11,1], & [11,2], & s_1, & p_3, & p_2 \\ [12,1], & [12,2], & p_3, & s_2, & p_1 \\ [22,1], & [22,2], & p_2, & p_1, & s_3 \end{vmatrix},$$

where

$$s_1=\sum\left(\frac{\partial^2 y}{\partial p^2}\right)^2, \quad s_2=\sum\left(\frac{\partial^2 y}{\partial p\,\partial q}\right)^2, \quad s_3=\sum\left(\frac{\partial^2 y}{\partial q^2}\right)^2,$$

$$p_1=\sum\frac{\partial^2 y}{\partial p\,\partial q}\frac{\partial^2 y}{\partial q^2}, \quad p_2=\sum\frac{\partial^2 y}{\partial q^2}\frac{\partial^2 y}{\partial p^2}, \quad p_3=\sum\frac{\partial^2 y}{\partial p^2}\frac{\partial^2 y}{\partial p\,\partial q}.$$

When the magnitudes η_{ij} of §§ 14, 21 are introduced, it is not difficult to shew that

$$J^2=\begin{vmatrix} A, & H, & [11,1], & [12,1], & [22,1] \\ H, & B, & [11,2], & [12,2], & [22,2] \\ 0, & 0, & \sum\eta_{11}^2, & \sum\eta_{11}\eta_{12}, & \sum\eta_{11}\eta_{22} \\ 0, & 0, & \sum\eta_{11}\eta_{12}, & \sum\eta_{12}^2, & \sum\eta_{12}\eta_{22} \\ 0, & 0, & \sum\eta_{11}\eta_{22}, & \sum\eta_{12}\eta_{22}, & \sum\eta_{22}^2 \end{vmatrix}$$

$$=V^2\begin{vmatrix} \sum\eta_{11}^2, & \sum\eta_{11}\eta_{12}, & \sum\eta_{11}\eta_{22} \\ \sum\eta_{11}\eta_{12}, & \sum\eta_{12}^2, & \sum\eta_{12}\eta_{22} \\ \sum\eta_{11}\eta_{22}, & \sum\eta_{12}\eta_{22}, & \sum\eta_{22}^2 \end{vmatrix},$$

a quantity which will be found (§ 105) not to vanish for a surface existing freely in a plenary homaloidal space of more than four dimensions. The assumption accordingly is justified: and the single point so obtained is called the normal centre of the surface, the locus of which is obviously another surface.

If $N=4$, so that the plenary homaloidal space is quadruple, the preceding five equations cannot coexist unless the condition

$$\begin{vmatrix} \frac{\partial y_1}{\partial p}, & \frac{\partial y_2}{\partial p}, & \frac{\partial y_3}{\partial p}, & \frac{\partial y_4}{\partial p}, & 0 \\ \frac{\partial y_1}{\partial q}, & \frac{\partial y_2}{\partial q}, & \frac{\partial y_3}{\partial q}, & \frac{\partial y_4}{\partial q}, & 0 \\ \frac{\partial^2 y_1}{\partial p^2}, & \frac{\partial^2 y_2}{\partial p^2}, & \frac{\partial^2 y_3}{\partial p^2}, & \frac{\partial^2 y_4}{\partial p^2}, & A \\ \frac{\partial^2 y_1}{\partial p\,\partial q}, & \frac{\partial^2 y_2}{\partial p\,\partial q}, & \frac{\partial^2 y_3}{\partial p\,\partial q}, & \frac{\partial^2 y_4}{\partial p\,\partial q}, & H \\ \frac{\partial^2 y_1}{\partial q^2}, & \frac{\partial^2 y_2}{\partial q^2}, & \frac{\partial^2 y_3}{\partial q^2}, & \frac{\partial^2 y_4}{\partial q^2}, & B \end{vmatrix}=0$$

is satisfied. This relation implies * the vanishing of an invariant appertaining to the surface and consequently requires a limitation to the generality of the surface. The orthogonal homaloid of the surface is a plane; and the vanishing of the invariant is the condition that this homaloid plane shall possess an envelope.

If $N=3$, so that the plenary space is triple, the orthogonal homaloid of the surface is its unique normal. We know that, except along the directions of curves of curvature, the normals at points consecutive to a given point do not intersect; there is no general envelope of the normal.

But there is another aspect of regarding the possible envelope of the orthogonal homaloid of a surface; and it is taken into consideration, not merely in general when $N \geqslant 5$, but also when $N=4$ and when $N=3$. We may require the envelope of that homaloid for successive positions along a curve on the surface, whether the curve be organic to the surface alone, or be such as to provide some property, or be required solely to secure the existence of such envelope. Accordingly, we take successive orthogonal homaloids of the surface at points along the direction p', q', in the tangent plane; and then, to obtain the envelope of the normal homaloid

$$\sum (\bar{y}-y)\frac{\partial y}{\partial p}=0, \quad \sum (\bar{y}-y)\frac{\partial y}{\partial q}=0,$$

we associate, with these two equations, the two further relations

$$\left.\begin{aligned}\sum (\bar{y}-y)\left(\frac{\partial^2 y}{\partial p^2}p'+\frac{\partial^2 y}{\partial p\,\partial q}q'\right)=\sum \frac{dy}{ds}\frac{\partial y}{\partial p}=Ap'+Hq'\\ \sum (\bar{y}-y)\left(\frac{\partial^2 y}{\partial p\,\partial q}p'+\frac{\partial^2 y}{\partial q^2}q'\right)=\sum \frac{dy}{ds}\frac{\partial y}{\partial q}=Hp'+Bq'\end{aligned}\right\}.$$

When $N \geqslant 5$, the envelope is a configuration, which lies within the earlier configuration when $N>5$; and when $N=5$, it is a curve passing through the former point.

When $N=4$, there are four equations involving the four quantities $\bar{y}_1-y_1$, $\bar{y}_2-y_2$, $\bar{y}_3-y_3$, $\bar{y}_4-y_4$, linearly. In general, the determinant of the coefficients on the left-hand side is different from zero; the four equations then determine unique finite values for the four magnitudes which, accordingly, define the orthogonal centre † of the surface for the direction p', q', the locus of this orthogonal centre for different directions p', q', in the surface being a conic in the normal homaloid (now the orthogonal plane) of the surface. But it may happen that, for a direction p', q', the determinant is zero: when this occurs,

* *G.F.D.*, vol. i, §§ 232, 252.

† *G.F.D.*, vol. i, §§ 247, 251.

its square also is zero : that is *,

$$\begin{vmatrix} A, & H, & 0, & 0 \\ H, & B, & 0, & 0 \\ 0, & 0, & \frac{L}{\rho}-(\mathrm{g}-\mathrm{b})q'^2, & \frac{M}{\rho}+(\mathrm{g}-\mathrm{b})p'q' \\ 0, & 0, & \frac{M}{\rho}+(\mathrm{g}-\mathrm{b})p'q', & \frac{N}{\rho}-(\mathrm{g}-\mathrm{b})p'^2 \end{vmatrix}=0$$

or

$$V^2\left\{\frac{LN-M^2}{\rho^2}-\frac{1}{\rho^2}(\mathrm{g}-\mathrm{b})\right\}=0,$$

that is,

$$\frac{V^2}{\rho^2\tau^2}=0.$$

Thus the tilt of the superficial geodesic in the direction p', q', must be zero : the direction on the surface is (§ 57) that of a curve of spherical curvature.

When $N=3$, there are four equations involving only three magnitudes $\bar{y}_1-y_1, \bar{y}_2-y_2, \bar{y}_3-y_3$; in general, they cannot coexist. The normal homaloid now is a line, being the unique normal to the surface in triple space : it is, of course, a known property that, except along a curve of curvature, consecutive normals to the surface do not intersect. Consequently some condition must be satisfied to allow the coexistence of the four equations, and it is easily obtained. We write

$$X, Y, Z, =\frac{1}{V}\left\|\begin{matrix} \frac{\partial y_1}{\partial p}, & \frac{\partial y_2}{\partial p}, & \frac{\partial y_3}{\partial p} \\ \frac{\partial y_1}{\partial q}, & \frac{\partial y_2}{\partial q}, & \frac{\partial y_3}{\partial q} \end{matrix}\right\|,$$

$$\sum X\frac{\partial^2 y_1}{\partial p^2}=\bar{A}, \quad \sum X\frac{\partial^2 y_1}{\partial p\,\partial q}=\bar{H}, \quad \sum X\frac{\partial^2 y_1}{\partial q^2}=\bar{B};$$

and the necessary condition is found to be

$$\begin{vmatrix} \bar{A}p'+\bar{H}q', & Ap'+Hq' \\ \bar{H}p'+\bar{B}q', & Hp'+Bq' \end{vmatrix}=0.$$

Thus in order that the normals may meet, one or other of the two directions thus determined must be chosen. The equation is, in fact, the equation of the curves of circular curvature ; and it also expresses the property that the torsion of the superficial geodesic in the direction p', q', must be zero.

* The notation is that used in the foregoing reference : for the geometrical result as regards the tilt, see *G.F.D.*, vol. i, § 236.

Equations of geodesics on a surface.

93. The intrinsic equations of a geodesic on a surface, which exists freely in multiple space, can be obtained for a surface in the same manner as for a geodesic in any free amplitude. We now take p and q, equal to x_1 and x_2, as the parameters of the surface ; and the general result, with the modifications of notation indicated in § 12, gives the intrinsic equations of the superficial geodesic in the form

$$\left.\begin{aligned} p'' + \Gamma_{11}p'^2 + 2\Gamma_{12}p'q' + \Gamma_{22}q'^2 = 0 \\ q'' + \Delta_{11}p'^2 + 2\Delta_{12}p'q' + \Delta_{22}q'^2 = 0 \end{aligned}\right\}.$$

It has been noted that these two equations must amount to only a single independent equation, when the permanent arc-relation

$$Ap'^2 + 2Hp'q' + Gq'^2 = 1$$

is retained ; in fact, when this relation is differentiated along any curve in the surface and not merely along a geodesic, we have

$$(Ap' + Hq')(p'' + \Gamma_{11}p'^2 + 2\Gamma_{12}p'q' + \Gamma_{22}q'^2) \\ + (Hp' + Bq')(q'' + \Delta_{11}p'^2 + 2\Delta_{12}p'q' + \Delta_{22}q'^2) = 0,$$

the values of p'' and q'' being taken in connection with the curve. From this relation it follows that each of the two geodesic equations is a consequence of the other.

Ex. Shew that, if $\theta(p, q) = 0$ is everywhere geodesic upon the surface, the equation

$$\theta_2^2(\theta_{11} - \theta_1\Gamma_{11} - \theta_2\Delta_{11}) - 2\theta_2\theta_1(\theta_{12} - \theta_1\Gamma_{12} - \theta_2\Delta_{12}) + \theta_1^2(\theta_{22} - \theta_1\Gamma_{22} - \theta_2\Delta_{22}) = 0$$

must be satisfied. (This partial differential equation in θ can also be regarded as the equation of superficial geodesics.)

Thus geodesics on a free surface are subject to the foregoing two differential equations, which have a special integral in the permanent arc-relation.

In the equations, the quantities Γ_{ij} and Δ_{ij} are functions of p and q alone ; they do not involve the quantities p' and q'. They can be regarded as arising through derivatives of the primary magnitudes A, H, B, being given by relations such as

$$\Gamma_{11} = \frac{1}{V^2}\left\{\tfrac{1}{2}B\frac{\partial A}{\partial p} - H\left(\frac{\partial H}{\partial p} - \tfrac{1}{2}\frac{\partial A}{\partial q}\right)\right\}.$$

The quantity V^2 is essentially positive for a surface, so that there is no deviation from finiteness to be produced by a zero value of V. Consequently the only singularities of the quantities Γ_{ij} and Δ_{ij} must arise through singularities of A, H, B. If therefore (as will be assumed to be the fact) we consider only ordinary ranges of the surface, meaning thereby ranges where no singularities of A, H, B,

occur, then we can regard the coefficients Γ_{ij} and Δ_{ij} in the intrinsic equations of the superficial geodesics as holomorphic functions of their arguments.

To such a pair of equations Cauchy's existence-theorem * applies, with the result that there exists a unique integral set (that is, a value of p and a value of q) for the equations: it exists for any assigned initial values p_0 and q_0 of p and q (that is, it exists at any point of the surface), and for any assigned initial values p_0' and q_0' at p_0, q_0, such that

$$A_0 p_0'^2 + 2H_0 p_0' q_0' + B_0 q_0'^2 = 1,$$

that is, for any assigned direction lying in the tangent plane of the surface and passing through the arbitrary initial place p_0, q_0, on the surface. In other words, an assigned direction in the tangent plane at any arbitrarily chosen place on the surface can be chosen: it will determine one, and only one, geodesic of the surface so that, at the place, the geodesic shall have the assigned direction. Thus at any point on a surface, a superficial geodesic is uniquely determinate by the assignment of its superficial direction at that point; and the geodesic will be said to originate in that direction. (The result is in accordance with the general theorem of § 19.)

It is known that, for real surfaces, the Legendre test and the Weierstrass test are satisfied, both these being qualitative †; and we shall not here be concerned with the quantitative † Jacobi test, which relates to the limit of range to be imposed in order to secure the minimum of superficial distance between two points. It is the qualitative character of the geodesic curve which here is of immediate importance for our investigations.

Ex. A surface of constant sphericity (§ 65) has its arc-element represented by the equations

$$D^2\, ds^2 = dp^2 + dq^2, \quad 4\kappa(D-1) = p^2 + q^2,$$

$1/\kappa$ being the measure of sphericity.

Shew that, when the sphericity is positive, the general integral parametric equations of its geodesics are

$$\frac{p}{\sin\alpha\cos\beta\cos t - \sin\beta\sin t} = \frac{q}{\sin\alpha\sin\beta\cos t + \cos\beta\sin t} = \frac{2\kappa^{\frac{1}{2}}}{1+\cos\alpha\cos t},$$

where α and β are arbitrary constants, $dt = \kappa^{-\frac{1}{2}}\, ds$, and an arbitrary constant is absorbed into the variable t.

Obtain, in real terms, the corresponding integral equations when the sphericity $1/\kappa$ is negative.

Now along any curve in the surface, we have

$$y_m'' = \frac{\partial y_m}{\partial p} p'' + \frac{\partial y_m}{\partial q} q'' + \frac{\partial^2 y_m}{\partial p^2} p'^2 + 2\frac{\partial^2 y_m}{\partial p\,\partial q} p'q' + \frac{\partial^2 y_m}{\partial q^2} q'^2,$$

* See my *Theory of Differential Equations*, vol. ii, chap. ii.

† See my *Calculus of Variations*, chap. viii.

for each of the N values of m: where p', q', are direction-variables of the tangent; and p'', q'', are determined, certainly in part by the surface, and usually in part by the curve. In the case when the curve is organic on the surface and actually geodesic there, the values of p'', q'', are determined entirely by the surface, being given by the foregoing equations; and therefore, if we write

$$\left.\begin{aligned}\eta_{11}{}^{(m)}&=\frac{\partial^2 y_m}{\partial p^2}-\frac{\partial y_m}{\partial p}\Gamma_{11}-\frac{\partial y_m}{\partial q}\Delta_{11}\\ \eta_{12}{}^{(m)}&=\frac{\partial^2 y_m}{\partial p\,\partial q}-\frac{\partial y_m}{\partial p}\Gamma_{12}-\frac{\partial y_m}{\partial q}\Delta_{12}\\ \eta_{22}{}^{(m)}&=\frac{\partial^2 y_m}{\partial q^2}-\frac{\partial y_m}{\partial p}\Gamma_{22}-\frac{\partial y_m}{\partial q}\Delta_{22}\end{aligned}\right\},$$

for all the values of m, we have

$$y_m''=\eta_{11}{}^{(m)}p'^2+2\eta_{12}{}^{(m)}p'q'+\eta_{22}{}^{(m)}q'^2.$$

But if $1/\rho$ be the circular curvature of the geodesic at the point, and if Y_1, Y_2, ... be the spatial directions of its prime normal, then

$$\rho y_m''=Y_m, \quad (m=1, 2, \ldots, N),$$

so that

$$\frac{Y_m}{\rho}=\eta_{11}{}^{(m)}p'^2+2\eta_{12}{}^{(m)}p'q'+\eta_{22}{}^{(m)}q'^2.$$

Taking a typical equation, in accordance with the practice already (§ 10) adopted, we have the equations connected with the circular curvature of a geodesic on a free surface represented by the typical equation

$$\frac{Y}{\rho}=\eta_{11}p'^2+2\eta_{12}p'q'+\eta_{22}q'^2,$$

the quantities Y being subject to the permanent equation $\sum Y^2=1$.

Tangent plane, and prime normal of a geodesic.

94. Before proceeding further, it is desirable to establish the geometrical relation between the tangent plane to the surface and the prime normal of a superficial geodesic. This relation is inferred at once from the general property (§§ 20, 21) of all free amplitudes in association with their tangent homaloids. Here, the relation expresses the property that the direction of the perpendicular drawn from a point Q, on a geodesic in the immediate vicinity of the originating point P of the geodesic, upon the tangent plane of the surface at P, coincides (in the limit when Q coincides with P along the geodesic) with the direction of the prime normal of the geodesic at P.

After the earlier general analysis, the analysis for a surface can be stated briefly. We denote the length of the geodesic arc QP by t, so that t tends to zero in the indicated limit; we denote the coordinates of Q by η_1, η_2, ...; and we

denote the coordinates of the foot of the perpendicular from Q on the tangent plane at P by $\bar{y}_1, \bar{y}_2, \ldots$, so that there exist parameters α and β such that

$$\bar{y}_m - y_m = \alpha \frac{\partial y_m}{\partial p} + \beta \frac{\partial y_m}{\partial q}.$$

The requirement of perpendicularity is secured, by making the magnitude

$$\sum (\eta_m - \bar{y}_m)^2$$

a minimum among all the values that can arise for all values of the parameters α and β; and the critical equations, necessary and sufficient for this purpose, are

$$\sum \frac{\partial y_m}{\partial p} (\eta_m - \bar{y}_m) = 0, \quad \sum \frac{\partial y_m}{\partial q} (\eta_m - \bar{y}_m) = 0,$$

or, alternatively,

$$\sum \frac{\partial y_m}{\partial p} (\eta_m - y_m) = \sum \frac{\partial y_m}{\partial p} \left(\alpha \frac{\partial y_m}{\partial p} + \beta \frac{\partial y_m}{\partial q} \right) = A\alpha + H\beta,$$

$$\sum \frac{\partial y_m}{\partial q} (\eta_m - y_m) = \sum \frac{\partial y_m}{\partial q} \left(\alpha \frac{\partial y_m}{\partial p} + \beta \frac{\partial y_m}{\partial q} \right) = H\alpha + B\beta.$$

The equations, in their first form, give the first form of the required relation. For if $\bar{Y}_1, \bar{Y}_2, \ldots$ denote the direction-cosines of the perpendicular and if Π denote its length, then

$$\eta_m - \bar{y}_m = \bar{Y}_m \Pi,$$

so that they become

$$\sum \bar{Y}_m \frac{\partial y_m}{\partial p} = 0, \quad \sum \bar{Y}_m \frac{\partial y_m}{\partial q} = 0:$$

that is, the perpendicular in question is at right angles to every direction in the tangent plane * and therefore is orthogonal to it.

When the second form of the equations is used, we can obtain the values of α and β expressed in ascending powers of t. We have

$$\eta_m - y_m = t y_m' + \tfrac{1}{2} t^2 y_m'' + \tfrac{1}{6} t^3 y_m''' + \ldots;$$

and therefore

$$A\alpha + H\beta = \sum \frac{\partial y_m}{\partial p} (t y_m' + \tfrac{1}{2} t^2 y_m'' + \ldots).$$

On the right-hand side, the coefficient of t

$$= \sum \frac{\partial y_m}{\partial p} y_m'$$

$$= \sum \frac{\partial y_m}{\partial p} \left(\frac{\partial y_m}{\partial p} p' + \frac{\partial y_m}{\partial q} q' \right) = Ap' + Hq'.$$

* When the plenary space is of more than three dimensions, this result does not secure any unique direction for the perpendicular: the property holds for every direction which lies in the orthogonal homaloid of the surface.

In the same expression, the coefficient of $\frac{1}{2}t^2$

$$\begin{aligned}
&= \sum \frac{\partial y_m}{\partial p} y_m'' \\
&= \sum \frac{\partial y_m}{\partial p}\left(\frac{\partial y_m}{\partial p} p'' + \frac{\partial y_m}{\partial q} q'' + \frac{\partial^2 y_m}{\partial p^2} p'^2 + 2\frac{\partial^2 y_m}{\partial p\,\partial q} p'q' + \frac{\partial^2 y_m}{\partial q^2} q'^2\right) \\
&= Ap'' + Hq'' + (A\Gamma_{11} + H\Delta_{11})p'^2 + 2(A\Gamma_{12} + H\Delta_{12})p'q' + (A\Gamma_{22} + H\Gamma_{22})q'^2 \\
&= A(p'' + \Gamma_{11}p'^2 + 2\Gamma_{12}p'q' + \Gamma_{22}q'^2) + H(q'' + \Delta_{11}p'^2 + 2\Delta_{12}p'q' + \Delta_{22}q'^2) \\
&= 0,
\end{aligned}$$

because the curve is a geodesic *. Accordingly, up to the second power of t inclusive, the equation gives

$$A\alpha + H\beta = t(Ap' + Hq');$$

and the second equation similarly is found to give

$$H\alpha + B\beta = t(Hp' + Bq'),$$

up to the same power of t inclusive. Hence, accurately up to t^2 inclusive, we have

$$\alpha = tp', \quad \beta = tq'.$$

Now for the perpendicular in question, we have

$$\begin{aligned}
\overline{Y}_m \Pi &= \eta_m - \bar{y}_m \\
&= \eta_m - \left(y_m + \alpha\frac{\partial y_m}{\partial p} + \beta\frac{\partial y_m}{\partial q}\right) \\
&= \eta_m - y_m - \left(\alpha\frac{\partial y_m}{\partial p} + \beta\frac{\partial y_m}{\partial q}\right);
\end{aligned}$$

up to the second power of t inclusive, we have

$$\eta_m - y_m = ty_m' + \tfrac{1}{2}t^2 y_m'';$$

and we have proved that, to the same order,

$$\alpha\frac{\partial y_m}{\partial p} + \beta\frac{\partial y_m}{\partial q} = ty_m'.$$

Consequently, up to the second power of t inclusive,

$$\overline{Y}_m \Pi = \tfrac{1}{2}t^2 y_m'' = Y_m \frac{t^2}{2\rho}:$$

* It should be noted that the investigation differs from the investigation in § 20. There, the purpose is to find the direction of the perpendicular from any contiguous point of the amplitude upon its tangent homaloid: and the limit of that direction is proved to coincide with the direction of the prime normal of the geodesic, drawn through the initial point and the contiguous point. Here, the investigation is simplified by taking the contiguous point lying on a superficial geodesic. The ultimate property is, of course, the same.

or in the limit, as t tends to zero, $\bar{Y}_m = Y_m$, for all values of m. Thus the limiting position of the specified perpendicular coincides with the prime normal of the geodesic. Moreover, up to the second order inclusive, we have

$$\Pi = \frac{t^2}{2\rho},$$

as is to be expected.

95. It thus appears that the prime normal of any superficial geodesic (the surface existing freely in a plenary homaloidal space) is orthogonal to the tangent plane of the surface at the point. The tangent line of the geodesic lies in that tangent plane; and thus the osculating plane of the superficial geodesic is represented by the equations typified by the form

$$\| \ \bar{y} - y, \ \ y', \ \ Y \ \| = 0,$$

where Y has the significance obtained in § 93. Now, as we have seen in § 91, this tangent plane contains a direction, with the typical direction-cosine

$$\frac{1}{V}\left\{(Ap' + Hq')\frac{\partial y}{\partial q} - (Hp' + Bq')\frac{\partial y}{\partial p}\right\},$$

which is at right angles to the tangent; being in the tangent plane to the surface, it is at right angles to the prime normal of the geodesic. The last property is characteristic of all the remaining principal lines in the orthogonal frame of the geodesic. We shall prove that the direction in question is that of the binormal of the geodesic, in accordance with the property established in § 33, now with the limitation that $n=2$.

Consider the flat, represented by the typical equation

$$\left\| \ \bar{y} - y, \ \ \frac{\partial y}{\partial p}, \ \ \frac{\partial y}{\partial q}, \ \ Y \ \right\| = 0.$$

It manifestly contains the tangent line to the geodesic in the direction p', q'; it manifestly contains the prime normal of that geodesic: thus it contains the osculating plane of the geodesic. It also contains the tangent at a contiguous point of the geodesic, having $y_1 + ty_1'$, $y_2 + ty_2'$, ... for its point-coordinates; because the equations of such a tangent are typically represented by

$$\bar{y} - (y + ty') = \lambda(y' + ty''),$$

that is, by

$$\begin{aligned} \bar{y} - y &= y'(t+\lambda) + y''t\lambda \\ &= (t+\lambda)p'\frac{\partial y}{\partial p} + (t+\lambda)q'\frac{\partial y}{\partial q} + \frac{t\lambda}{\rho}Y, \end{aligned}$$

manifestly satisfied by the equations of the flat for all values of the line-parameter λ. Thus the flat contains, in addition to the osculating plane of the geodesic at

the initial point, the tangent at a consecutive point of the curve: it therefore is the osculating flat of the geodesic. The binormal of any geodesic lies in its osculating flat; and therefore the binormal of the superficial geodesic lies in the foregoing flat. Such binormal is at right angles to the tangent to the geodesic and also is at right angles to the prime normal; its direction is therefore that of the line in the tangent plane of the surface, at right angles to the geodesic tangent in that plane. With the customary notation l_3, to denote the typical direction-cosine of the binormal of a geodesic, we therefore have

$$l_3 = \frac{1}{V}\left\{(Ap' + Hq')\frac{\partial y}{\partial q} - (Hp' + Bq')\frac{\partial y}{\partial p}\right\}$$

as the typical equation for the direction-cosines of the binormal of a geodesic on a free surface.

For any curve in multiple space, we have

$$\frac{l_3}{\sigma} = \rho y''' + \rho' y'' - \frac{y'}{\rho}:$$

the determination of the value of σ, and the verification of the formulæ, will be obtained after further discussion of the circular curvature.

Ex. 1. Utilising this property that the binormal of a superficial geodesic lies in the tangent plane of the surface, we can obtain approximations, to the length of the perpendicular from a neighbouring point upon the tangent plane and to the direction of that perpendicular, closer than those obtained in § 94. We take the tangent plane of the surface as a plane, organic to the geodesic, its leading lines being the tangent line and the binormal of the geodesic; and then its equations have the form

$$\| \; \bar{y} - y, \quad y', \quad l_3 \; \| = 0.$$

Any point in this plane, such as the foot of the perpendicular from a point at an arc-distance t along the geodesic from the central point, is given by coordinates

$$\bar{y} = y + ay' + \beta l_3;$$

and the property of perpendicularity from the selected point $\eta_1, \eta_2, \ldots$ on the geodesic upon the plane requires that

$$\sum (\eta - \bar{y})^2$$

shall be a minimum for all values of the parameters a and β. The conditions are

$$\sum y'(\eta - \bar{y}) = 0, \quad \sum l_3(\eta - \bar{y}) = 0;$$

on substitution, these become

$$a = \sum y'(\eta - y), \quad \beta = \sum l_3(\eta - y).$$

Now for the point η at an arc-distance t along the geodesic, we have

$$\eta - y = ty' + \tfrac{1}{2}t^2 y'' + \tfrac{1}{6}t^3 y''' + \tfrac{1}{24}t^4 y'''',$$

accurately up to t^4 inclusive. In general *, along any curve, we have

$$y'' = \frac{1}{\rho} Y,$$

$$y''' = \frac{1}{\rho\sigma} l_3 - \frac{\rho'}{\rho^2} Y - \frac{1}{\rho^2} y',$$

$$y'''' = \frac{1}{\rho\sigma\tau} l_4 - \frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right) l_3 + \left\{\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) - \frac{1}{\rho^3} - \frac{1}{\rho\sigma^2}\right\} Y + 3\frac{\rho'}{\rho^3} y',$$

so far as the retained terms are concerned. Hence

$$a = t - \frac{t^3}{6\rho^2} + \frac{\rho'}{8\rho^3} t^4,$$

$$\beta = \frac{t^3}{6\rho\sigma} - \frac{t^4}{24\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right),$$

accurately up to t^4 inclusive.

For the length Π of the perpendicular, and for its direction-cosines typified by $\overline{Y}$, we have

$$\begin{aligned} \overline{Y}\Pi &= \eta - \bar{y} \\ &= \eta - (y + ay' + \beta l_3) \\ &= ty' + \tfrac{1}{2}t^2 y'' + \tfrac{1}{6}t^3 y''' + \tfrac{1}{24}t^4 y'''' - (ay' + \beta l_3), \end{aligned}$$

up to t^4 inclusive; and therefore, after re-arrangement, and still up to the same order inclusive, we have

$$\begin{aligned} \overline{Y}\Pi &= Y\left[\frac{t^2}{2\rho} - \frac{\rho'}{6\rho^2} t^3 + \tfrac{1}{24}t^4 \left\{\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) - \frac{1}{\rho}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)\right\}\right] + \frac{l_4}{24\rho\sigma\tau} t^4 \\ &= YT + \frac{l_4}{24\rho\sigma\tau} t^4, \end{aligned}$$

with an obvious significance for T.

Thus the direction of the perpendicular Π is not exactly the same as that of the geodesic prime normal. If θ denote the small inclination of the two directions, we have (always up to the retained order of small quantities),

$$\Pi \cos\theta = T, \quad \Pi \sin\theta = \frac{t^4}{24\rho\sigma\tau},$$

and therefore

$$\theta = \frac{t^2}{12\sigma\tau},$$

approximately. Hence, up to the fourth order of small quantities inclusive,

$$\Pi = \frac{1}{2\rho} t^2 - \frac{\rho'}{6\rho^2} t^3 + \tfrac{1}{24}\left\{\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) - \frac{1}{\rho}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)\right\} t^4.$$

We thus obtain a more approximate value of the length of the perpendicular from a consecutive point of the geodesic upon the tangent plane of the surface, as well as a

* *G.F.D.*, vol. i, pp. 278, 279.

measure of the deviation between the direction of this perpendicular and the prime normal of the geodesic at the initial point.

When the surface exists in triple homaloidal space, all directions normal to the tangent plane are the same ; and the angle θ then is zero, a result secured analytically by the fact that there can be no tilt for any curve in such a plenary space.

The property, however, is as much a geometrical property of the curve in general as it is of the surface : but the analytical expressions to be obtained later for ρ, σ, τ, render the property characteristic of the surface.

Ex. 2. In the same way, the following property may be established. When a perpendicular, from the same contiguous point of a geodesic, is drawn upon the osculating plane of the geodesic, the length of that perpendicular is

$$\frac{t^3}{6\rho\sigma}\left\{1-\tfrac{1}{4}t\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)\right\}$$

accurately up to the fourth order in t inclusive ; and the inclination of the perpendicular to the binormal of the geodesic at the initial point is approximately

$$\tfrac{1}{4}\frac{t}{\tau}.$$

As in the preceding example, the property is as much a geometrical property of any curve as it is of the surface.

Geodesic polar coordinates.

96. The intrinsic equations of a geodesic on a surface are

$$p''+\Gamma_{11}p'^2+2\Gamma_{12}\beta'q'+\Gamma_{22}q'^2=0, \quad q''+\Delta_{11}p'^2+2\Delta_{12}p'q'+\Delta_{22}q'^2=0,$$

the parametric curves on the surface being unrestricted as regards properties on the surface.

Occasionally it is convenient to have a geodesic through the point O as one of the parametric curves. Accordingly, we seek the modifications which are required to make one set of parametric curves of a geodesic character. Let the curve, $p=$variable, $q=$constant, be a geodesic, so that q' vanishes along the curve ; the intrinsic equations become

$$p''+\Gamma_{11}p'^2=0, \quad 0=\Delta_{11}p'^2,$$

so that we must have $\Delta_{11}=0$. Hence (§ 90)

$$\tfrac{1}{2}A_1=A\Gamma_{11}, \quad H_1-\tfrac{1}{2}A_2=H\Gamma_{11},$$

and therefore

$$H_1-\tfrac{1}{2}A_2=\tfrac{1}{2}\frac{H}{A}A_1,$$

so that

$$\frac{\partial}{\partial p}\left(\frac{H}{A^{\frac{1}{2}}}\right)=\frac{\partial}{\partial q}(A^{\frac{1}{2}}).$$

Consequently, there must exist a function of p and q—let it be denoted by l—such that

$$A^{\frac{1}{2}}=\frac{\partial l}{\partial p}, \quad \frac{H}{A^{\frac{1}{2}}}=\frac{\partial l}{\partial q}.$$

Thus, for the arc-element on the surface, we have

$$\begin{aligned} ds^2 &= A\,dp^2+2H\,dp\,dq+B\,dq^2 \\ &= \left(\frac{\partial l}{\partial p}\right)^2 dp^2+2\frac{\partial l}{\partial p}\frac{\partial l}{\partial q}dp\,dq+B\,dq^2 \\ &= dl^2+P\,dq^2, \end{aligned}$$

where

$$P=B-\left(\frac{\partial l}{\partial q}\right)^2.$$

Obviously along the geodesic, for which q is constant, the quantity l is the length of the geodesic arc itself.

In the first place, we note that, along the parametric curve q=constant, the direction-variables p', q', are

$$p'=\frac{1}{A^{\frac{1}{2}}}, \quad q'=0.$$

Along the curve l=constant, the direction-variables $\frac{dp}{dt}$ and $\frac{dq}{dt}$ satisfy the relation

$$\frac{\partial l}{\partial p}\frac{dp}{dt}+\frac{\partial l}{\partial q}\frac{dq}{dt}=0,$$

and therefore

$$A\frac{dp}{dt}+H\frac{dq}{dt}=0.$$

Hence

$$Ap'\frac{dp}{dt}+H\left(p'\frac{dq}{dt}+q'\frac{dp}{dt}\right)+Bq'\frac{dq}{dt}=0;$$

verifying that the two curves l=constant, q=constant, are orthogonal to one another, as is necessary because the expression for ds^2 in terms of dl and dq does not involve the product $dl\,dq$. Moreover, on the surface, the geodesic distance of one curve q=constant from another curve q=constant, is solely the difference in the values of l at the points on the two curves where they are cut by the geodesic: that is, the curves

$$l=\text{parametric constant}$$

are geodesically parallel to one another.

The function P, originally a function of p and q, becomes a function of l and q,

when these are taken as the parametric variables for the surface. Owing to the similarity of its form to the representation of the arc of a plane curve when referred to polar coordinates, such a representation of an arc of a surface in any plenary homaloidal space is often called a polar geodesic representation ; and the parameters l and q are called geodesic polar coordinates.

But, for a general surface, it is only the curve $l=$variable, $q=$constant, which is a geodesic: the other parametric curve $l=$constant, $q=$variable, is not a geodesic. If the geodesic quality of the latter were possible, the intrinsic equations would be satisfied by $l=$constant, $q=$variable : that is,

$$0=\Gamma_{22}q'^2, \quad q''=\Delta_{22}q'^2,$$

so that we should have $\Gamma_{22}=0$ in the representation

$$ds^2=dl^2+P\,dq^2.$$

Hence, as generally (§ 90)

$$\frac{\partial H}{\partial q}-\frac{1}{2}\frac{\partial B}{\partial p}=A\Gamma_{22}+H\Delta_{22},$$

and as, here, $H=0$, $B=P$, it follows that

$$\frac{\partial P}{\partial p}=0,$$

that is, P is a function of q alone ; and by changing the variable, we have

$$ds^2=dl^2+dm^2.$$

Thus comparing with the general form (§ 90), we have

$$A=1, \quad H=0, \quad B=1,$$
$$\Gamma_{ij}=0, \quad \Delta_{ij}=0,$$

for $ij=11$, 12, 22 ; and therefore the magnitude $(12, 12)=0$, that is,

$$K=0.$$

As will be seen later (§ 112), this result implies that the sphericity of the surface is zero ; the surface ceases to rank as a general surface.

Ex. 1. Shew that, when the arc on a surface is expressed in the form

$$ds^2=dp^2+T^2\,dq^2,$$

the quantity K, later (§ 112) shewn to be the sphericity of the surface, is given by

$$K=-\frac{1}{T}\frac{\partial^2 T}{\partial p^2}.$$

Ex. 2. Shew that the partial differential equation of geodesically parallel curves on the surface is

$$A\left(\frac{\partial l}{\partial q}\right)^2 - 2H\frac{\partial l}{\partial p}\frac{\partial l}{\partial q} + B\left(\frac{\partial l}{\partial p}\right)^2 = V^2.$$

Values of p''', q'''.

97. Expressions were obtained (§ 90) for the parametric derivatives of the quantities Γ_{ij} and Δ_{ij}; and these expressions were found to provide relations for differences of some of these derivatives. All the expressions involved the primary magnitudes A, H, B, and their derivatives, without the introduction of new magnitudes; and the indicated relations gave rise to a quantity K, known as to its analytical form, but not interpreted geometrically in that connection. Partly as a preliminary to such interpretation, and partly on grounds of convenience, we proceed to the values of p''', q''', estimated along the superficial geodesic originating in the direction p', q'.

When the relation

$$p'' = -(\Gamma_{11}p'^2 + 2\Gamma_{12}p'q' + \Gamma_{22}q'^2)$$

is differentiated along the geodesic, we obtain an expression on the right-hand side which, after substitution for p'', q'', is homogeneous in p', q', of the third degree. If this be taken in the form

$$p''' = -(\Gamma_{111}p'^3 + 3\Gamma_{112}p'^2q' + 3\Gamma_{122}p'q'^2 + \Gamma_{222}q'^3),$$

we have, in the first place,

$$\Gamma_{111} = \frac{\partial \Gamma_{11}}{\partial p} - 2\Gamma_{11}^2 - 2\Gamma_{12}\Delta_{11}.$$

Next, we have

$$3\Gamma_{112} = \frac{\partial \Gamma_{11}}{\partial q} + 2\frac{\partial \Gamma_{12}}{\partial p} - 6\Gamma_{11}\Gamma_{12} - 4\Gamma_{12}\Delta_{12} - 2\Gamma_{22}\Delta_{11};$$

$$3\Gamma_{122} = 2\frac{\partial \Gamma_{12}}{\partial q} + \frac{\partial \Gamma_{22}}{\partial p} - 2\Gamma_{11}\Gamma_{22} - 4\Gamma_{12}^2 - 2\Gamma_{12}\Delta_{22} - 4\Gamma_{22}\Delta_{12};$$

$$\Gamma_{222} = \frac{\partial \Gamma_{22}}{\partial q} - 2\Gamma_{12}\Gamma_{22} - 2\Gamma_{22}\Delta_{22}.$$

But (p. 238) we had the relation

$$\left(\frac{\partial \Gamma_{11}}{\partial q} + \Gamma_{22}\Delta_{11}\right) - \left(\frac{\partial \Gamma_{12}}{\partial p} + \Gamma_{12}\Delta_{12}\right) = -HK;$$

and therefore we can express both $\dfrac{\partial \Gamma_{11}}{\partial q}$ and $\dfrac{\partial \Gamma_{12}}{\partial p}$ in terms of Γ_{112} and K, while on the other hand Γ_{112} is expressible in terms of the primary magnitudes A, H, B, and their derivatives.

We had the similar relation

$$\left(\frac{\partial \Gamma_{22}}{\partial p}+\Gamma_{11}\Gamma_{22}+\Gamma_{12}\Delta_{22}\right)-\left(\frac{\partial \Gamma_{12}}{\partial q}+\Gamma_{12}{}^2+\Gamma_{22}\Delta_{12}\right)=BK,$$

so that $\frac{\partial \Gamma_{12}}{\partial q}$ and $\frac{\partial \Gamma_{22}}{\partial p}$ can be expressed in terms of Γ_{122} and K, while Γ_{122} itself is expressible also in terms of A, H, B, and their derivatives. The full tale of these relations is

$$\left.\begin{aligned}
\frac{\partial \Gamma_{11}}{\partial p} &= \Gamma_{111}+2\Gamma_{11}{}^2+2\Gamma_{12}\Delta_{11}\\
\frac{\partial \Gamma_{11}}{\partial q} &= -\tfrac{2}{3}HK+\Gamma_{112}+2\Gamma_{11}\Gamma_{12}+2\Gamma_{12}\Delta_{12}\\
\frac{\partial \Gamma_{12}}{\partial p} &= \tfrac{1}{3}HK+\Gamma_{112}+2\Gamma_{11}\Gamma_{12}+\Gamma_{12}\Delta_{12}+\Gamma_{22}\Delta_{11}\\
\frac{\partial \Gamma_{12}}{\partial q} &= -\tfrac{1}{3}BK+\Gamma_{122}+\Gamma_{11}\Gamma_{22}+\Gamma_{12}{}^2+\Gamma_{12}\Delta_{22}+\Gamma_{22}\Delta_{12}\\
\frac{\partial \Gamma_{22}}{\partial p} &= \tfrac{2}{3}BK+\Gamma_{122}+2\Gamma_{12}{}^2+2\Gamma_{22}\Delta_{12}\\
\frac{\partial \Gamma_{22}}{\partial q} &= \Gamma_{222}+2\Gamma_{12}\Gamma_{22}+2\Gamma_{22}\Delta_{22}
\end{aligned}\right\}.$$

Similarly, we have

$$q'''=-(\Delta_{111}p'^3+3\Delta_{112}p'^2q'+3\Delta_{122}p'q'^2+\Delta_{222}q'^3),$$

where

$$\begin{aligned}
\Delta_{111} &= \frac{\partial \Delta_{11}}{\partial p}-2\Gamma_{11}\Delta_{11}-2\Delta_{11}\Delta_{12},\\
3\Delta_{112} &= \frac{\partial \Delta_{11}}{\partial q}+2\frac{\partial \Delta_{12}}{\partial p}-2\Gamma_{11}\Delta_{12}-4\Gamma_{12}\Delta_{11}-2\Delta_{11}\Delta_{22}-4\Delta_{12}{}^2,\\
3\Delta_{122} &= 2\frac{\partial \Delta_{12}}{\partial q}+\frac{\partial \Delta_{22}}{\partial p}-4\Gamma_{12}\Delta_{12}-2\Gamma_{22}\Delta_{11}-6\Delta_{12}\Delta_{22},\\
\Delta_{222} &= \frac{\partial \Delta_{22}}{\partial q}-2\Gamma_{22}\Delta_{12}-2\Delta_{22}{}^2.
\end{aligned}$$

leading to expressions for the quantities Δ_{ijk}, after substitution for the derivatives of the quantities Δ_{ij}, in terms of A, H, B, and their derivatives. By the use of the other two relations

$$\left(\frac{\partial \Delta_{11}}{\partial q}+\Gamma_{11}\Delta_{12}+\Delta_{11}\Delta_{22}\right)-\left(\frac{\partial \Delta_{12}}{\partial p}+\Gamma_{12}\Delta_{11}+\Delta_{12}{}^2\right)=AK,$$

$$\left(\frac{\partial \Delta_{22}}{\partial p}+\Gamma_{22}\Delta_{11}\right)-\left(\frac{\partial \Delta_{12}}{\partial q}+\Gamma_{12}\Delta_{12}\right)=-HK,$$

in § 90, it is possible to express the first derivatives of the quantities Δ_{ij} in terms of the quantities Δ_{ijk} and of K: and the full tale of these expressions is

$$\left.\begin{aligned}
\frac{\partial\Delta_{11}}{\partial p} &= \Delta_{111}+2\Gamma_{11}\Delta_{11}+2\Delta_{11}\Delta_{12}\\
\frac{\partial\Delta_{11}}{\partial q} &= \tfrac{2}{3}AK+\Delta_{112}+2\Gamma_{12}\Delta_{11}+2\Delta_{12}^2\\
\frac{\partial\Delta_{12}}{\partial p} &= -\tfrac{1}{3}AK+\Delta_{112}+\Gamma_{11}\Delta_{12}+\Gamma_{12}\Delta_{11}+\Delta_{11}\Delta_{22}+\Delta_{12}^2\\
\frac{\partial\Delta_{12}}{\partial q} &= \tfrac{1}{3}HK+\Delta_{122}+\Gamma_{12}\Delta_{12}+\Gamma_{22}\Delta_{11}+2\Delta_{11}\Delta_{22}\\
\frac{\partial\Delta_{22}}{\partial p} &= -\tfrac{2}{3}HK+\Delta_{122}+2\Gamma_{12}\Delta_{12}+2\Delta_{12}\Delta_{22}\\
\frac{\partial\Delta_{22}}{\partial q} &= \Delta_{222}+2\Gamma_{22}\Delta_{12}+2\Delta_{22}^2
\end{aligned}\right\}.$$

The values of p''' and q''' are given by

$$\left.\begin{aligned}
p'''+\Gamma_{111}p'^3+3\Gamma_{112}p'^2q'+3\Gamma_{122}p'q'^2+\Gamma_{222}q'^3=0\\
q'''+\Delta_{111}p'^3+3\Delta_{112}p'^2q'+3\Delta_{122}p'q'^2+\Delta_{222}q'^3=0
\end{aligned}\right\}.$$

When these expressions, connected with a geodesic on a surface, are compared with the corresponding expressions (§ 23), connected with a geodesic in a general amplitude, so that $n=2$ for the comparison, they agree, under the conventions

$$\Gamma_{ijk}=\{ijk,\,1\},\quad \Delta_{ijk}=\{ijk,\,2\};$$

and the expressions for the derivatives of Γ_{ij} and Δ_{ij} similarly agree with the former expressions (§ 23) under the further convention

$$(12,\,12)=K,$$

together with the properties (§ 16) of the Riemann four-index symbol $(ij,\,kl)$.

Magnitudes for a geodesic range.

98. In a later investigation, connected with geodesic ranges along a surface, we shall require to consider arcs in various directions and, connected with them, certain magnitudes involving the first derivatives of the quantities Γ_{ij} and Δ_{ij}. For greatest generality as regards these magnitudes, we shall denote three such elements of arc by ds_i, ds_j, ds_k; and the corresponding direction-variables along those elements of arc by p_i', q_i'; p_j', q_j'; p_k', q_k'; respectively. For some formulæ, two of the three directions may be the same; the modifications in the formulæ are then obvious.

For the briefer expression of the results, we use (as equivalent to the symbols $g_{ij}^{(p)}$ of § 60 when $n=2$) certain symbols with the assigned denotations

$$\left.\begin{aligned}\alpha_l=\Gamma_{11}p_l'+\Gamma_{12}q_l'\\ \beta_l=\Gamma_{12}p_l'+\Gamma_{22}q_l'\end{aligned}\right\},\quad \left.\begin{aligned}\epsilon_l=\Delta_{11}p_l'+\Delta_{12}q_l'\\ \eta_l=\Delta_{12}p_l'+\Delta_{22}q_l'\end{aligned}\right\},$$

for $l=i, j, k$. Then, by direct substitution, we find

$$\left.\begin{aligned}
p_i'p_j'\frac{\partial \Gamma_{11}}{\partial s_k}&+(p_i'q_j'+q_i'p_j')\frac{\partial \Gamma_{12}}{\partial s_k}+q_i'q_j'\frac{\partial \Gamma_{22}}{\partial s_k}\\
&=-\tfrac{1}{3}K\{(Hp_i'+Bq_i')(p_j'q_k'-p_k'q_j')+(Hp_j'+Bq_j')(p_i'q_k'-p_k'q_i')\}\\
&\qquad+(\Gamma_{111}, \Gamma_{112}, \Gamma_{122}, \Gamma_{222}\between p_i', q_i'\between p_j', q_j'\between p_k', q_k')\\
&\qquad+p_k'(2\alpha_i\alpha_j+\beta_i\epsilon_j+\beta_j\epsilon_i)+q_k'(\alpha_i\beta_j+\alpha_j\beta_i+\beta_i\eta_j+\beta_j\eta_i)\\
p_i'p_j'\frac{\partial \Delta_{11}}{\partial s_k}&+(p_i'q_j'+q_i'p_j')\frac{\partial \Delta_{12}}{d s_k}+q_i'q_j'\frac{\partial \Delta_{22}}{\partial s_k}\\
&=\tfrac{1}{3}K\{(Ap_i'+Hq_i')(p_j'q_k'-p_k'q_j')+(Ap_j'+Hq_j')(p_i'q_k'-p_k'q_i')\}\\
&\qquad+(\Delta_{111}, \Delta_{112}, \Delta_{122}, \Delta_{222}\between p_i', q_i'\between p_j', q_j'\between p_k', q_k')\\
&\qquad+p_k'(\alpha_i\epsilon_j+\alpha_j\epsilon_i+\epsilon_i\eta_j+\epsilon_j\eta_i)+q_k'(\beta_i\epsilon_j+\beta_j\epsilon_i+2\eta_i\eta_j)
\end{aligned}\right\},$$

whatever be the values of i, j, k, distinct from one another, or the same as one another, wholly or in part.

In the same investigations connected with geodesic ranges on a surface, we shall require the values of

$$\frac{dA}{ds_i},\quad \frac{dH}{ds_i},\quad \frac{dB}{ds_i},\quad \frac{d^2A}{ds_i^2},\quad \frac{d^2H}{ds_i^2},\quad \frac{d^2B}{ds_i^2},$$

these second derivatives being taken along a geodesic in any direction p_i', q_i'. We have

$$\frac{\partial A}{\partial p}=2(A\Gamma_{11}+H\Delta_{11}),\quad \frac{\partial A}{\partial q}=2(A\Gamma_{12}+H\Delta_{12});$$

and therefore

$$\begin{aligned}
\frac{dA}{ds_i}&=2A(\Gamma_{11}p_i'+\Gamma_{12}q_i')+2H(\Delta_{11}p_i'+\Delta_{12}q_i')\\
&=2(A\alpha_i+H\epsilon_i),
\end{aligned}$$

with the preceding symbols as defined on p. 260. Similarly

$$\frac{dH}{ds_i}=A\beta_i+H\eta_i+H\alpha_i+B\epsilon_i,$$

$$\frac{dB}{ds_i}=2(H\beta_i+B\eta_i).$$

For the second derivatives along a geodesic, we have

$$\begin{aligned}
\frac{d^2A}{ds_i^2}=\frac{\partial^2 A}{\partial p^2}p_i'^2&+2\frac{\partial^2 A}{\partial p\,\partial q}p_i'q_i'+\frac{\partial^2 A}{\partial q^2}q_i'^2\\
&-\frac{\partial A}{\partial p}(\Gamma_{11}p'^2+2\Gamma_{12}p'q'+\Gamma_{22}q'^2)-\frac{\partial A}{\partial q}(\Delta_{11}p'^2+2\Delta_{11}p'q'+\Delta_{22}q'^2).
\end{aligned}$$

When the second parametric derivatives of A are formed, they involve first

parametric derivatives of Γ_{11}, Γ_{12}, Δ_{11}, Δ_{12}. The values of these first derivatives, in terms of the magnitudes Γ_{ijk} and Δ_{ijk} as obtained in § 97, are to be substituted; also the due values of the first parametric derivatives of A, H, B, in terms of the quantities Γ_{ij} and Δ_{ij}. Then when the terms are gathered together, and we use the symbols of p. 260, the value of $\dfrac{d^2A}{ds_i^2}$ can be expressed in the form

$$\begin{aligned}\frac{d^2A}{ds_i^2}= &-\tfrac{2}{3}V^2Kq_i'^2\\ &+2A(\Gamma_{111}p_i'^2+2\Gamma_{112}p_i'q_i'+\Gamma_{122}q_i'^2)+2H(\Delta_{111}p_i'^2+2\Delta_{112}p_i'q_i'+\Delta_{122}q_i'^2)\\ &+A(6\alpha_i^2+4\beta_i\epsilon_i)+H(8\alpha_i\epsilon_i+4\epsilon_i\eta_i)+2B\epsilon_i^2.\end{aligned}$$

Proceeding in the same way with the second derivatives of H and B, we find

$$\begin{aligned}\frac{d^2H}{ds_i^2}= &\tfrac{2}{3}V^2Kp_i'q_i'\\ &+A(\Gamma_{112}p_i'^2+2\Gamma_{122}p_i'q_i'+\Gamma_{222}q_i'^2)+H(\Gamma_{111}p_i'^2+2\Gamma_{112}p_i'q_i'+\Gamma_{122}q_i'^2)\\ &+H(\Delta_{112}p_i'^2+2\Delta_{122}p_i'q_i'+\Delta_{222}q_i'^2)+B(\Delta_{111}p_i'^2+2\Delta_{112}p_i'q_i'+\Delta_{122}q_i'^2)\\ &+A(4\alpha_i\beta_i+2\beta_i\eta_i)+H(2\alpha_i^2+2\alpha_i\eta_i+6\beta_i\epsilon_i+2\eta_i^2)+B(2\alpha_i\epsilon_i+4\epsilon_i\eta_i),\end{aligned}$$

$$\begin{aligned}\frac{d^2B}{ds_i^2}= &-\tfrac{2}{3}V^2Kp_i'^2\\ &+2H(\Gamma_{112}p_i'^2+2\Gamma_{122}p_i'q_i'+\Gamma_{222}q_i'^2)+2B(\Delta_{112}p_i'^2+2\Delta_{122}p_i'q_i'+\Delta_{222}q_i'^2)\\ &+2A\beta_i^2+H(4\alpha_i\beta_i+8\beta_i\eta_i)+B(4\beta_i\epsilon_i+6\eta_i^2).\end{aligned}$$

One special kind of combination of these magnitudes will be required, in various cases that are included in

$$\Theta=p_j'p_k'\frac{d^2A}{ds_i^2}+(p_j'q_k'+p_k'q_j')\frac{d^2H}{ds_i^2}+q_j'q_k'\frac{d^2B}{ds_i^2};$$

the value of this expression can be arranged in the form

$$\begin{aligned}\Theta= &-\tfrac{2}{3}V^2K(p_i'q_j'-p_j'q_i')(p_i'q_k'-p_k'q_i')\\ &+(Ap_j'+Hq_j')(\Gamma_{111}\between p_k',\, q_k'\between p_i',\, q_i')^2+(Ap_k'+Hq_k')(\Gamma_{111}\between p_j',\, q_j'\between p_i',\, q_i')^2\\ &+(Hp_j'+Bq_j')(\Delta_{111}\between p_k',\, q_k'\between p_i',\, q_i')^2+(Hp_k'+Bq_k')(\Delta_{111}\between p_j',\, q_j'\between p_i',\, q_i')^2\\ &+2(Ap_j'+Hq_j')(\alpha_i\gamma_{ik}+\beta_i\delta_{ik})+2(Ap_k'+Hq_k')(\alpha_i\gamma_{ij}+\beta_i\delta_{ij})\\ &+2(Hp_j'+Bq_j')(\epsilon_i\gamma_{ik}+\eta_i\delta_{ik})+2(Hp_k'+Bq_k')(\epsilon_i\gamma_{ij}+\eta_i\delta_{ij})\\ &+2\{A\gamma_{ij}\gamma_{ik}+H(\gamma_{ij}\delta_{ik}+\gamma_{ik}\delta_{ij})+B\delta_{ij}\delta_{ik}\},\end{aligned}$$

where

$$\left.\begin{aligned}\gamma_{ab}&=\alpha_ap_b'+\beta_aq_b'\\ &=\alpha_bp_a'+\beta_bq_a'=\Gamma_{11}p_a'p_b'+\Gamma_{12}(p_a'q_b'+p_b'q_a')+\Gamma_{22}q_a'q_b'\\ \delta_{ab}&=\epsilon_ap_b'+\eta_aq_b'\\ &=\epsilon_bp_a'+\eta_bq_a'=\Delta_{11}p_a'p_b'+\Delta_{12}(p_a'q_b'+p_b'q_a')+\Delta_{22}q_a'q_b'\end{aligned}\right\}.$$

We may add three other expressions which will occasionally be useful. We have

$$\left.\begin{aligned} p_i'\frac{dA}{ds_k}+q_i'\frac{dH}{ds_k}&=A\gamma_{ik}+H\delta_{ik}\\ &\qquad+(Ap_i'+Hq_i')\alpha_k+(Hp_i'+Bq_i')\epsilon_k\\ p_i'\frac{dH}{ds_k}+q_i'\frac{dB}{ds_k}&=H\gamma_{ik}+B\delta_{ik}\\ &\qquad+(Ap_i'+Hq_i')\beta_k+(Hp_i'+Bq_i')\eta_k \end{aligned}\right\},$$

and

$$\left.\begin{aligned} p_i'p_j'\frac{dA}{ds_k}+(p_i'q_j'+p_j'q_i')\frac{dH}{ds_k}+q_i'q_j'\frac{dB}{ds_k}&\\ =(Ap_j'+Hq_j')\gamma_{ik}+(Ap_i'+Hq_i')\gamma_{jk}&\\ +(Hp_j'+Bq_j')\delta_{ik}+(Hp_i'+Bq_i')\delta_{jk}& \end{aligned}\right\}.$$

Magnitudes for ranges along two geodesics.

99. The foregoing formulæ give the values of the successive arc-derivatives of the primary magnitudes of the surface along any one and the same geodesic, the values being derived through the intrinsic equations of the geodesic. But occasions arise (*e.g.* in connection with the geodesic parallelograms on the surface, in Chapter X), when it is necessary to evaluate derivatives of the primary magnitudes taken with respect to different geodesics; thus we shall require the quantities

$$\frac{d^2A}{ds_1ds_2},\quad \frac{d^2H}{ds_1ds_2},\quad \frac{d^2B}{ds_1\,ds_2}.$$

For instance, we have

$$\frac{dA}{ds_2}=2(A\Gamma_{11}+H\Delta_{11})p_2'+2(A\Gamma_{12}+H\Delta_{12})q_2';$$

in order to form the derivative with regard to the arc ds_1 of a different geodesic, it is necessary to have the laws of variation of p_2' and q_2' along this different geodesic.

As indicated, the immediate need arises in association with sets of parallel geodesics. But, even on surfaces, the geodesic parallelograms can be defined in a variety of ways; and, as will be seen later in the case of amplitudes more extensive than surfaces, the parallels can be drawn according to different definitions even when connected with angles alone. It happens, however, that, as regards the immediate requirements, all the definitions and all the constructions give the same first two terms in the expression of each of the direction-variables. Thus if p_1' and q_1' are the direction-variables of OA at O;

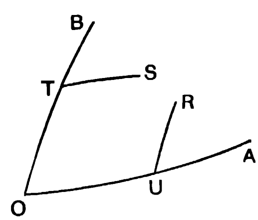

FIG. 4.

p_2' and q_2' those of OB at O; if $OU=x$, $OT=y$, both x and y being small; the common terms (under all definitions and constructions) of the direction-variables of UR, as parallel to OT, are

$$P_2'=p_2'-x\sum\Gamma_{11}p_1'p_2',\quad Q_2'=q_2'-x\sum\Delta_{11}p_1'p_2',$$

and similarly the common terms in the direction-variables of TS, as parallel to OU, are

$$P_1'=p_1'-y\sum\Gamma_{11}p_1'p_2',\quad Q_1'=q_1'-y\sum\Delta_{11}p_1'p_2';$$

all subsequent terms in higher powers of x and of y depending upon the law of parallelism adopted.

From the former, we have

$$\operatorname*{Lim}_{x\to 0}\left\{\frac{1}{x}(P_2'-p_2')\right\}=-\sum\Gamma_{11}p_1'p_2',$$

that is,

$$\frac{dp_2'}{ds_1}=-\sum\Gamma_{11}p_1'p_2';$$

and from the latter, we have

$$\frac{dp_1'}{ds_2}=-\sum\Gamma_{11}p_1'p_2'.$$

Similarly,

$$\frac{dq_2'}{ds_1}=-\sum\Delta_{11}q_1'q_2'=-\frac{dq_1'}{ds_2}.$$

With this convention as to differentiation with respect to the arc-lengths along two different geodesics, we have

$$\frac{d^2p}{ds_1\,ds_2}=\frac{d^2p}{ds_2\,ds_1}=-\sum\Gamma_{11}p_1'p_2',$$
$$\frac{d^2q}{ds_1\,ds_2}=\frac{d^2q}{ds_2\,ds_1}=-\sum\Delta_{11}p_1'p_2',$$

and, for any function of position on the surface,

$$\frac{d^2}{ds_1\,ds_2}\{f(p,q)\}=\frac{d^2}{ds_2\,ds_1}\{f(p,q)\}.$$

But except along one and the same geodesic arc, all differentiations of higher order are affected by the type of parallelism postulated.

In particular, we have

$$\begin{aligned}\frac{d^2A}{ds_i\,ds_j}=-\tfrac{2}{3}V^2Kq_i'q_j'&+2A\{\Gamma_{111}p_i'p_j'+\Gamma_{112}(p_i'q_j'+p_j'q_i')+\Gamma_{122}q_i'q_j'\}\\&+2H\{\Delta_{111}p_i'p_j'+\Delta_{112}(p_i'q_j'+p_j'q_i')+\Delta_{122}q_i'q_j'\}\\&+A\{6a_ia_j+2(\beta_i\epsilon_j+\beta_j\epsilon_i)\}\\&+H\{4(a_i\epsilon_j+a_j\epsilon_i)+2(\epsilon_i\eta_j+\epsilon_j\eta_i)\}+2B\epsilon_i\epsilon_j,\end{aligned}$$

with corresponding expressions for the same derivatives of H and of B.

A comprehensive expression, in the general form

$$p_k'p_l'\frac{d^2A}{ds_i\,ds_j}+(p_k'q_l'+p_l'q_k')\frac{d^2H}{ds_i\,ds_j}+q_k'q_l'\frac{d^2B}{ds_i\,ds_j},$$

is of frequent use in connection with geodesic triangles and geodesic parallels, for different values of i, j, k, l, and for special combinations of equal values for i, j, k, l: its value

$$\begin{aligned}
= -\tfrac{1}{3}V^2K\{&(p_i'q_k'-p_k'q_i')(p_j'q_l'-p_l'q_j')+(p_i'q_l'-p_l'q_i')(p_j'q_k'-p_k'q_j')\}\\
&+(Ap_k'+Hq_k')(\Gamma_{111}\between p_i',q_i'\between p_j',q_j'\between p_l',q_l')\\
&\qquad+(Ap_l'+Hq_l')(\Gamma_{111}\between p_i',q_i'\between p_j',q_j'\between p_k',q_k')\\
&+(Hp_k'+Bq_k')(\Delta_{111}\between p_i',q_i'\between p_j',q_j'\between p_l',q_l')\\
&\qquad+(Hp_l'+Bq_l')(\Delta_{111}\between p_i',q_i'\between p_j',q_j'\between p_k',q_k')\\
&+(Ap_k'+Hq_k')(\alpha_i\gamma_{jk}+\beta_i\delta_{jk}+\alpha_j\gamma_{ik}+\beta_j\delta_{ik})\\
&+(Ap_l'+Hq_l')(\alpha_i\gamma_{jl}+\beta_i\delta_{jl}+\alpha_j\gamma_{il}+\beta_i\delta_{il})\\
&+(Hp_k'+Bq_k')(\epsilon_i\gamma_{jk}+\eta_i\delta_{jk}+\epsilon_j\gamma_{ik}+\eta_j\delta_{ik})\\
&+(Hp_l'+Bq_l')(\epsilon_i\gamma_{jl}+\eta_i\delta_{jl}+\epsilon_j\gamma_{il}+\eta_j\delta_{il})\\
&+A(\gamma_{ik}\gamma_{jl}+\gamma_{il}\gamma_{jk})\\
&\qquad+H(\gamma_{ik}\delta_{jl}+\gamma_{ik}\delta_{jk}+\gamma_{jl}\delta_{ik}+\gamma_{jk}\delta_{il})+B(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}).
\end{aligned}$$

In particular, it is to be noted that when $i=j$, and either k or l is equal to the common value of i and j, the term involving K disappears; and in the special combination $\sum p_1'p_2'\dfrac{d^2A}{ds_1\,ds_2}$, which occurs (§ 128) in connection with one of the geodesic parallelograms, the term involving K

$$=-\tfrac{1}{3}V^2K(p_1'q_2'-q_1'p_2')^2=-\tfrac{1}{3}K\sin^2\epsilon,$$

where ϵ is the angle at O between the two geodesics.

The quantities y''', $\dfrac{d}{ds}\left(\dfrac{1}{\rho}\right)$.

100. It is possible to deduce the value of a typical quantity y''', the continued derivation being effected along a geodesic. In accordance with the notation adopted (§ 24) for a general amplitude, we write

$$y'''=\eta_{111}p'^3+3\eta_{112}p'^2q'+3\eta_{122}p'q'^2+\eta_{222}q'^3\,;$$

and it is obtained, simply, as follows.

The third arc-derivative along a curve is

$$\begin{aligned}
y'''=\frac{\partial y}{\partial p}p'''+\frac{\partial y}{\partial q}q'''+3\left\{\frac{\partial^2y}{\partial p^2}p'p''+\frac{\partial^2y}{\partial p\,\partial q}(p'q''+q'p'')+\frac{\partial^2y}{\partial q^2}q'q''\right\}\\
+\frac{\partial^3y}{\partial p^3}p'^3+3\frac{\partial^3y}{\partial p^2\partial q}p'^2q'+3\frac{\partial^3y}{\partial p\,\partial q^2}p'q'^2+\frac{\partial^3y}{\partial q^3}q'^3.
\end{aligned}$$

When the curve is a geodesic,

$$\begin{aligned}
-p'' &= \Gamma_{11}p'^2 + 2\Gamma_{12}p'q' + \Gamma_{22}q'^2,\\
-q'' &= \Delta_{11}p'^2 + 2\Delta_{12}p'q' + \Delta_{22}q'^2,\\
-p''' &= \Gamma_{111}p'^3 + 3\Gamma_{112}p'^2q' + 3\Gamma_{122}p'q'^2 + \Gamma_{222}q'^3,\\
-q''' &= \Delta_{111}p'^3 + 3\Delta_{112}p'^2q' + 3\Delta_{122}p'q'^2 + \Delta_{222}q'^3;
\end{aligned}$$

after these values are substituted in the second expression, and coefficients are collected, it becomes a homogeneous cubic in p' and q', agreeing with the first expression, with the coefficients η_{ijk} defined according to the scheme :

$$\left.\begin{aligned}
\eta_{111} &= \frac{\partial^3 y}{\partial p^3} - \frac{\partial y}{\partial p}\{\Gamma_{111} + 3(\Gamma_{11}\Gamma_{11} + \Delta_{11}\Gamma_{12})\} - \frac{\partial y}{\partial q}\{\Delta_{111} + 3(\Delta_{11}\Gamma_{11} + \Delta_{12}\Delta_{11})\}\\
&\qquad - 3(\eta_{11}\Gamma_{11} + \eta_{12}\Delta_{11})\\
\eta_{112} &= \frac{\partial^3 y}{\partial p^2\,\partial q} - \frac{\partial y}{\partial p}\{\Gamma_{112} + 2(\Gamma_{11}\Gamma_{12} + \Gamma_{12}\Delta_{12}) + (\Gamma_{12}\Gamma_{11} + \Gamma_{22}\Delta_{11})\}\\
&\qquad - \frac{\partial y}{\partial q}\{\Delta_{112} + 2(\Delta_{11}\Gamma_{12} + \Delta_{12}\Delta_{12}) + (\Delta_{12}\Gamma_{11} + \Delta_{22}\Delta_{11})\}\\
&\qquad - 2(\eta_{11}\Gamma_{12} + \eta_{12}\Delta_{12}) - (\eta_{12}\Gamma_{11} + \eta_{22}\Delta_{11})\\
\eta_{122} &= \frac{\partial^3 y}{\partial p\,\partial q^2} - \frac{\partial y}{\partial p}\{\Gamma_{122} + (\Gamma_{11}\Gamma_{22} + \Gamma_{12}\Delta_{22}) + 2(\Gamma_{12}\Gamma_{12} + \Gamma_{22}\Delta_{12})\}\\
&\qquad - \frac{\partial y}{\partial q}\{\Delta_{122} + (\Delta_{11}\Gamma_{22} + \Delta_{12}\Delta_{22}) + 2(\Delta_{12}\Gamma_{12} + \Delta_{22}\Delta_{12})\}\\
&\qquad - (\eta_{11}\Gamma_{22} + \eta_{12}\Delta_{22}) - 2(\eta_{12}\Gamma_{12} + \eta_{22}\Delta_{12})\\
\eta_{222} &= \frac{\partial^3 y}{\partial q^3} - \frac{\partial y}{\partial p}\{\Gamma_{222} + 3(\Gamma_{12}\Gamma_{22} + \Gamma_{22}\Delta_{22})\} - \frac{\partial y}{\partial q}\{\Delta_{222} + 3(\Delta_{12}\Gamma_{22} + \Delta_{22}\Delta_{22})\}\\
&\qquad - 3(\eta_{12}\Gamma_{22} + \eta_{22}\Delta_{22}).
\end{aligned}\right\}.$$

These results are in agreement with the general results for any amplitude in § 24 when, in the latter, $n=2$.

It will be found convenient to use symbols θ_1 and θ_2, according to the denotations

$$\left.\begin{aligned}
\theta_1 &= \eta_{111}p'^2 + 2\eta_{112}p'q' + \eta_{122}q'^2\\
\theta_2 &= \eta_{112}p'^2 + 2\eta_{122}p'q' + \eta_{222}q'^2
\end{aligned}\right\},$$

so that $y''' = \theta_1 p' + \theta_2 q'$, while θ_1 and θ_2 bear the same analytical relation to y''' as ξ_1 and ξ_2 bear to y''.

101. The arc-derivative of the circular curvature of a superficial geodesic can be constructed from the foregoing typical relation

$$y''' = \eta_{111}p'^3 + 3\eta_{112}p'^2q' + 3\eta_{122}p'q'^2 + \eta_{222}q'^3.$$

We have

$$Y = \rho y'',$$

and therefore

$$Y'=\rho y'''+\rho' y''=\rho y'''+\frac{\rho'}{\rho}Y;$$

hence, as $\sum Y^2=1$, $\sum YY'=0$, it follows that, on multiplying this equation by Y and adding for all space-dimensions, we have

$$\rho\sum Yy'''+\frac{\rho'}{\rho}=0,$$

so that

$$\frac{d}{ds}\left(\frac{1}{\rho}\right)=\sum Yy'''.$$

In accordance with the notation for the general amplitude (§ 87), we write

$$e_{ijk}=\sum Y\eta_{ijk},$$

for the various combinations ijk; when the values of the coefficients η_{ijk} are substituted, the values of the coefficients e_{ijk} are

$$\left.\begin{aligned}
e_{111}&=\sum Y\frac{\partial^3 y}{\partial p^3}-3(\bar{A}\Gamma_{11}+\bar{H}\Delta_{11})\\
e_{112}&=\sum Y\frac{\partial^3 y}{\partial p^2\partial q}-2(\bar{A}\Gamma_{12}+\bar{H}\Delta_{12})-(\bar{H}\Gamma_{11}+\bar{B}\Delta_{11})\\
e_{122}&=\sum Y\frac{\partial^3 y}{\partial p\,\partial q^2}-(\bar{A}\Gamma_{22}+\bar{H}\Delta_{22})-2(\bar{H}\Gamma_{12}+\bar{B}\Delta_{12})\\
e_{222}&=\sum Y\frac{\partial^3 y}{\partial q^3}-3(\bar{H}\Gamma_{22}+\bar{B}\Delta_{22})
\end{aligned}\right\}.$$

Consequently, the value of the arc-derivative of the curvature is given by the expression

$$w=\frac{d}{ds}\left(\frac{1}{\rho}\right)=(e_{111},\ e_{112},\ e_{122},\ e_{222}\between p',\ q')^3,$$

analogous to the corresponding expression (§ 36) for the like magnitude in a general amplitude.

It will be convenient to write

$$\left.\begin{aligned}
w_1&=(e_{111},\ e_{112},\ e_{122}\between p',\ q')^2\\
w_2&=(e_{112},\ e_{122},\ e_{222}\between p',\ q')^2
\end{aligned}\right\}.$$

Again, we have

$$\eta_{11}=\frac{\partial^2 y}{\partial p^2}-\Gamma_{11}\frac{\partial y}{\partial p}-\Delta_{11}\frac{\partial y}{\partial q},$$

and therefore

$$\begin{aligned}
\frac{\partial\eta_{11}}{\partial p}=\frac{\partial^3 y}{\partial p^3}&-\Gamma_{11}\left(\eta_{11}+\Gamma_{11}\frac{\partial y}{\partial p}+\Delta_{11}\frac{\partial y}{\partial q}\right)-\Delta_{11}\left(\eta_{12}+\Gamma_{12}\frac{\partial q}{\partial p}+\Delta_{12}\frac{\partial y}{\partial q}\right)\\
&-\frac{\partial y}{\partial p}(\Gamma_{111}+2\Gamma_{11}{}^2+2\Gamma_{12}\Delta_{11})-\frac{\partial y}{\partial q}(\Delta_{111}+2\Gamma_{11}\Delta_{11}+2\Delta_{11}\Delta_{12})\\
=\eta_{111}&+2\eta_{11}\Gamma_{11}+2\eta_{12}\Delta_{11},
\end{aligned}$$

by the results in § 100. Similarly for all the first parametric derivatives of η_{11}, η_{12}, η_{22}; the full set of results, in addition to the above, is

$$\frac{\partial\eta_{11}}{\partial q}=\eta_{112}+2\eta_{11}\Gamma_{12}+2\eta_{12}\Delta_{12}+\tfrac{2}{3}K\left(H\frac{\partial y}{\partial p}-A\frac{\partial y}{\partial q}\right),$$

$$\frac{\partial\eta_{12}}{\partial p}=\eta_{112}+\eta_{11}\Gamma_{12}+\eta_{12}\Delta_{12}+\eta_{12}\Gamma_{11}+\eta_{22}\Delta_{11}-\tfrac{1}{3}K\left(H\frac{\partial y}{\partial p}-A\frac{\partial y}{\partial q}\right),$$

$$\frac{\partial\eta_{12}}{\partial q}=\eta_{122}+\eta_{11}\Gamma_{22}+\eta_{12}\Delta_{22}+\eta_{12}\Gamma_{12}+\eta_{22}\Delta_{12}-\tfrac{1}{3}K\left(H\frac{\partial y}{\partial q}-B\frac{\partial y}{\partial p}\right),$$

$$\frac{\partial\eta_{22}}{\partial p}=\eta_{122}+2\eta_{12}\Gamma_{12}+2\eta_{22}\Delta_{12}+\tfrac{2}{3}K\left(H\frac{\partial y}{\partial q}-B\frac{\partial y}{\partial p}\right),$$

$$\frac{\partial\eta_{22}}{\partial q}=\eta_{222}+2\eta_{12}\Gamma_{22}+2\eta_{22}\Delta_{22}.$$

Further, when we differentiate the relations

$$\xi_1=\eta_{11}p'+\eta_{12}q',\quad \xi_2=\eta_{12}p'+\eta_{22}q',$$

along a geodesic, and introduce the typical direction-cosine l_3 of the binormal, we have

$$\left.\begin{aligned}\frac{d\xi_1}{ds}&=\frac{\partial\eta_{11}}{\partial p}p'^2+\left(\frac{\partial\eta_{11}}{\partial q}+\frac{\partial\eta_{12}}{\partial p}\right)p'q'+\frac{\partial\eta_{12}}{\partial q}q'^2+\eta_{11}p''+\eta_{12}q''\\&=\theta_1+\alpha\xi_1+\epsilon\xi_2-\tfrac{1}{3}q'l_3VK\\\frac{d\xi_2}{ds}&=\theta_2+\beta\xi_1+\eta\xi_2+\tfrac{1}{3}p'l_3VK\end{aligned}\right\}.$$

Later, we shall require expressions for the quantities $\sum\eta_{ab}\eta_{ijk}$, where $a, b, i, j, k, =1, 2$, in all combinations: they can be obtained in terms of the first parametric derivatives of the quantities $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{k}$, defined in § 104. We have

$$\mathbf{a}=\sum\eta_{11}^2,$$

and therefore

$$\tfrac{1}{2}\frac{\partial\mathbf{a}}{\partial p}=\sum\eta_{11}\frac{\partial\eta_{11}}{\partial p}=\sum\eta_{11}\eta_{111}+2\mathbf{a}\Gamma_{11}+2\mathbf{h}\Delta_{11}.$$

Similarly for the other expressions. In all, we have

$$\left.\begin{aligned}\sum\eta_{11}\eta_{111}&=\tfrac{1}{2}\frac{\partial\mathbf{a}}{\partial p}-2\mathbf{a}\Gamma_{11}-2\mathbf{h}\Delta_{11}\\\sum\eta_{12}\eta_{111}&=\frac{\partial\mathbf{h}}{\partial p}-\tfrac{1}{2}\frac{\partial\mathbf{a}}{\partial q}-3\mathbf{h}\Gamma_{11}+\mathbf{a}\Gamma_{12}-3\mathbf{k}\Delta_{11}+\mathbf{h}\Delta_{12}\\\sum\eta_{22}\eta_{111}&=\frac{\partial\mathbf{g}}{\partial p}-\frac{\partial\mathbf{h}}{\partial q}+\tfrac{1}{2}\frac{\partial\mathbf{b}}{\partial p}-(2\mathbf{g}+\mathbf{b})\Gamma_{11}+\mathbf{a}\Gamma_{22}-3\mathbf{f}\Delta_{11}-(\mathbf{g}-\mathbf{b})\Delta_{12}+\mathbf{h}\Delta_{22}\end{aligned}\right\},$$

$$\left.\begin{aligned}
\sum\eta_{11}\eta_{112}&=\tfrac{1}{2}\frac{\partial \mathbf{a}}{\partial q}-2\mathbf{a}\Gamma_{12}-2\mathbf{h}\Delta_{12}\\
\sum\eta_{12}\eta_{112}&=\tfrac{1}{2}\frac{\partial \mathbf{b}}{\partial p}-\mathbf{b}\Gamma_{11}-\mathbf{h}\Gamma_{12}-\mathbf{f}\Delta_{11}-\mathbf{b}\Delta_{12}\\
\sum\eta_{22}\eta_{112}&=\frac{\partial \mathbf{f}}{\partial p}-\tfrac{1}{2}\frac{\partial \mathbf{b}}{\partial q}-\mathbf{f}\Gamma_{11}-(\mathbf{g}+\mathbf{b})\Gamma_{12}+\mathbf{h}\Gamma_{22}-\mathbf{c}\Delta_{11}-2\mathbf{f}\Delta_{12}+\mathbf{b}\Delta_{22}
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\sum\eta_{11}\eta_{122}&=\frac{\partial \mathbf{h}}{\partial q}-\tfrac{1}{2}\frac{\partial \mathbf{b}}{\partial p}+\mathbf{b}\Gamma_{11}-2\mathbf{h}\Gamma_{12}-\mathbf{a}\Gamma_{22}+\mathbf{f}\Delta_{11}-(\mathbf{g}+\mathbf{b})\Delta_{12}-\mathbf{h}\Delta_{22}\\
\sum\eta_{12}\eta_{122}&=\tfrac{1}{2}\frac{\partial \mathbf{b}}{\partial q}-\mathbf{b}\Gamma_{12}-\mathbf{h}\Gamma_{22}-\mathbf{f}\Delta_{12}-\mathbf{b}\Delta_{22}\\
\sum\eta_{22}\eta_{122}&=\tfrac{1}{2}\frac{\partial \mathbf{c}}{\partial p}-2\mathbf{f}\Gamma_{12}-2\mathbf{c}\Delta_{12}
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\sum\eta_{11}\eta_{222}&=\frac{\partial \mathbf{g}}{\partial q}-\frac{\partial \mathbf{f}}{\partial p}+\tfrac{1}{2}\frac{\partial \mathbf{b}}{\partial q}+\mathbf{f}\Gamma_{11}-(\mathbf{g}-\mathbf{b})\Gamma_{12}-3\mathbf{h}\Gamma_{22}+\mathbf{c}\Delta_{11}-(2\mathbf{g}+\mathbf{b})\Delta_{22}\\
\sum\eta_{12}\eta_{222}&=\frac{\partial \mathbf{f}}{\partial q}-\tfrac{1}{2}\frac{\partial \mathbf{c}}{\partial p}+\mathbf{f}\Gamma_{12}-3\mathbf{k}\Gamma_{22}+\mathbf{c}\Delta_{12}-3\mathbf{f}\Delta_{22}\\
\sum\eta_{22}\eta_{222}&=\tfrac{1}{2}\frac{\partial \mathbf{c}}{\partial q}-2\mathbf{f}\Gamma_{22}-2\mathbf{c}\Delta_{22}
\end{aligned}\right\}.$$

We thus have twelve magnitudes of the type $\sum\eta_{ab}\eta_{ijk}$, with the three possible combinations 11, 12, 22, for ab, and the four possible combinations 111, 112, 122, 222, for ijk. Just as it proved convenient to introduce the six magnitudes Γ_{ab}, Δ_{ab}, instead of the six first parametric derivatives of A, H, B, it proves convenient to introduce twelve symbols to denote magnitudes in terms of which the twelve first derivatives of **a**, **b**, **c**, **f**, **g**, **h**, can be expressed: we write

$$\sum\eta_{11}\eta_{ijk}=A_{ijk},\quad \sum\eta_{12}\eta_{ijk}=H_{ijk},\quad \sum\eta_{22}\eta_{ijk}=B_{ijk},$$

for the four possible combinations ijk. The values of these new magnitudes are given, in terms of **a**, **b**, **c**, **f**, **g**, **h**, **k**, in the preceding table: and conversely, these first derivatives are expressible in terms of A_{ijk}, H_{ijk}, B_{ijk}, and additive terms linear in **a**, **b**, **c**, **f**, **g**, **h**, **k**.

Also, we have a magnitude K, later obtained as the Riemann sphericity, defined in the relation

$$V^2K=\mathbf{g}-\mathbf{b}\,;$$

we easily verify the values of the parametric derivatives of K in the form

$$\left.\begin{aligned}
V^2\frac{\partial K}{\partial p}&=A_{122}-2H_{112}+B_{111}\\
V^2\frac{\partial K}{\partial q}&=A_{222}-2H_{122}+B_{112}
\end{aligned}\right\}.$$

These quantities A_{ijk}, H_{ijk}, B_{ijk}, will be used at a later stage. Meanwhile, we have

$$\begin{aligned}\frac{1}{\rho}e_{ijk} &= \sum \frac{Y}{\rho}\eta_{ijk}\\ &= \sum(\eta_{11}p'^2+2\eta_{12}p'q'+\eta_{22}q'^2)\eta_{ijk}\\ &= A_{ijk}p'^2+2H_{ijk}p'q'+B_{ijk}q'^2,\end{aligned}$$

for the four combinations of $i, j, k, =1, 2$.

Certain summations involving products of first parametric derivatives and third parametric derivatives of the point-variable will arise later, and are expressible in terms of the magnitudes Γ_{ijk}, Δ_{ijk}. Thus we have

$$\sum \frac{\partial y}{\partial p}\frac{\partial^2 y}{\partial p^2} = A\Gamma_{11}+H\Delta_{11},$$

and therefore

$$\begin{aligned}\sum\left(\frac{\partial y}{\partial p}\frac{\partial^3 y}{\partial p^3}\right)+\sum\left(\frac{\partial^2 y}{\partial p^2}\right)^2 = A\frac{\partial \Gamma_{11}}{\partial p}+H\frac{\partial \Delta_{11}}{\partial p}&+\Gamma_{11}(2A\Gamma_{11}+2H\Delta_{11})\\ &+\Delta_{11}(A\Gamma_{12}+H\Delta_{12}+H\Gamma_{11}+B\Delta_{11}).\end{aligned}$$

Now

$$\sum\left(\frac{\partial^2 y}{\partial p^2}\right)^2 = \sum\left(\eta_{11}+\frac{\partial y}{\partial p}\Gamma_{11}+\frac{\partial y}{\partial q}\Delta_{11}\right)^2 = \mathbf{a}+A\Gamma_{11}^2+2H\Gamma_{11}\Delta_{11}+B\Delta_{11}^2;$$

when this value and the values of $\frac{\partial \Gamma_{11}}{\partial p}$ and $\frac{\partial \Delta_{11}}{\partial p}$ are substituted, we find

$$\sum\left(\frac{\partial y}{\partial p}\frac{\partial^3 y}{\partial p^3}\right) = -\mathbf{a}+A(\Gamma_{111}+3\Gamma_{11}^2+3\Gamma_{12}\Delta_{11})+H(\Delta_{111}+3\Delta_{11}\Gamma_{12}+3\Delta_{11}\Delta_{12}).$$

Similarly for the like expressions with the other third parametric derivatives. The full tale of results is

$$\left.\begin{aligned}\sum\left(\frac{\partial y}{\partial p}\frac{\partial^3 y}{\partial p^3}\right)+\mathbf{a} &= A(\Gamma_{111}+3\Gamma_{11}^2+3\Gamma_{12}\Delta_{11})+H(\Delta_{111}+3\Delta_{11}\Gamma_{11}+3\Delta_{11}\Delta_{12})\\ \sum\left(\frac{\partial y}{\partial q}\frac{\partial^3 y}{\partial p^3}\right)+\mathbf{h} &= H(\Gamma_{111}+3\Gamma_{11}^2+3\Gamma_{12}\Delta_{11})+B(\Delta_{111}+3\Delta_{11}\Gamma_{11}+3\Delta_{11}\Delta_{12})\end{aligned}\right\},$$

$$\left.\begin{aligned}\sum\left(\frac{\partial y}{\partial p}\frac{\partial^3 y}{\partial p^2\partial q}\right)+\mathbf{h} &= A(\Gamma_{112}+3\Gamma_{11}\Gamma_{12}+2\Gamma_{12}\Delta_{12}+\Gamma_{22}\Delta_{11})\\ &\quad+H(\Delta_{112}+\Gamma_{11}\Delta_{12}+2\Gamma_{12}\Delta_{11}+\Delta_{11}\Delta_{22}+2\Delta_{12}^2)\\ \sum\left(\frac{\partial y}{\partial q}\frac{\partial^3 y}{\partial p^2\partial q}\right)+\mathbf{k} &= H(\Gamma_{112}+3\Gamma_{11}\Gamma_{12}+2\Gamma_{12}\Delta_{12}+\Gamma_{22}\Delta_{11})\\ &\quad+B(\Delta_{112}+\Gamma_{11}\Delta_{12}+2\Gamma_{12}\Delta_{11}+\Delta_{11}\Delta_{22}+2\Delta_{12}^2)\end{aligned}\right\},$$

$$\left.\begin{aligned}\sum\left(\frac{\partial y}{\partial p}\frac{\partial^3 y}{\partial p\partial q^2}\right)+\mathbf{k} &= A(\Gamma_{122}+\Gamma_{11}\Gamma_{22}+2\Gamma_{12}^2+\Gamma_{12}\Delta_{22}+\Gamma_{22}\Delta_{12})\\ &\quad+H(\Delta_{122}+\Delta_{11}\Gamma_{22}+2\Delta_{12}\Gamma_{12}+3\Delta_{12}\Delta_{22})\\ \sum\left(\frac{\partial y}{\partial q}\frac{\partial^3 y}{\partial p\partial q^2}\right)+\mathbf{f} &= H(\Gamma_{122}+\Gamma_{11}\Gamma_{22}+2\Gamma_{12}^2+\Gamma_{12}\Delta_{22}+\Gamma_{22}\Delta_{12})\\ &\quad+B(\Delta_{122}+\Delta_{11}\Gamma_{22}+2\Delta_{12}\Gamma_{12}+3\Delta_{12}\Delta_{22})\end{aligned}\right\},$$

$$\left.\begin{aligned}\sum\left(\frac{\partial y}{\partial p}\frac{\partial^3 y}{\partial q^3}\right)+\mathbf{f}&=A\left(\Gamma_{222}+3\Gamma_{12}\Gamma_{22}+3\Gamma_{22}\Delta_{22}\right)+H\left(\Delta_{222}+3\Gamma_{22}\Delta_{12}+3\Delta_{22}{}^2\right)\\ \sum\left(\frac{\partial y}{\partial q}\frac{\partial^3 y}{\partial q^3}\right)+\mathbf{c}&=H\left(\Gamma_{222}+3\Gamma_{12}\Gamma_{22}+3\Gamma_{22}\Delta_{22}\right)+B\left(\Delta_{222}+3\Gamma_{22}\Delta_{12}+3\Delta_{22}{}^2\right)\end{aligned}\right\}.$$

From the relations connecting the third parametric derivatives of the space-variables with the magnitudes η_{ijk}, we at once deduce the relations

$$\left.\begin{aligned}\sum\frac{\partial y}{\partial p}\eta_{111}&=-\mathbf{a}, & \sum\frac{\partial y}{\partial q}\eta_{111}&=-\mathbf{h}\\ \sum\frac{\partial y}{\partial p}\eta_{112}&=-\mathbf{h}, & \sum\frac{\partial y}{\partial q}\eta_{112}&=-\mathbf{k}\\ \sum\frac{\partial y}{\partial p}\eta_{122}&=-\mathbf{k}, & \sum\frac{\partial y}{\partial q}\eta_{122}&=-\mathbf{f}\\ \sum\frac{\partial y}{\partial p}\eta_{222}&=-\mathbf{f}, & \sum\frac{\partial y}{\partial q}\eta_{222}&=-\mathbf{c}\end{aligned}\right\}.$$

Values of p'''', q''''.

102. The expressions for the quantities Γ_{ijk} and Δ_{ijk}, in terms of the derivatives of Γ_{ij} and Δ_{ij}, shew that magnitudes of the type

$$\frac{\partial\Gamma_{111}}{\partial q}-\frac{\partial\Gamma_{112}}{\partial p}$$

are expressible in terms of magnitudes of no higher order of derivation than Γ_{ijk} and Δ_{ijk} themselves; thus the foregoing magnitude is easily found to be equal to

$$-\tfrac{2}{3}H\frac{\partial K}{\partial p}+\tfrac{8}{3}(H\Gamma_{11}-A\Gamma_{12})K+2(\Gamma_{12}\Gamma_{111}+\Delta_{12}\Gamma_{112})-2(\Gamma_{11}\Gamma_{112}+\Delta_{11}\Gamma_{122}).$$

But such relations merely give expressions for the differences of some of the derivatives of Γ_{ijk} and Δ_{ijk}; in order to have the expressions for the derivatives individually, we use the values of p'''' and q'''', which can be deduced from p''' and q''' in the same manner as these were deduced from p'' and q''.

It will suffice to state the results: we find

$$\left.\begin{aligned}-p''''&=(\Gamma_{1111},\Gamma_{1112},\Gamma_{1122},\Gamma_{1222},\Gamma_{2222}\between p',q')^4\\ -q''''&=(\Delta_{1111},\Delta_{1112},\Delta_{1122},\Delta_{1222},\Delta_{2222}\between p',q')^4\end{aligned}\right\},$$

where

$$\left.\begin{aligned}\frac{\partial\Gamma_{111}}{\partial p}&=\Gamma_{1111}+3\Gamma_{11}\Gamma_{111}+3\Delta_{11}\Gamma_{112}\\ \frac{\partial\Gamma_{111}}{\partial q}&=\Gamma_{1112}+3\Gamma_{12}\Gamma_{111}+3\Delta_{12}\Gamma_{112}-\tfrac{1}{2}H\frac{\partial K}{\partial p}+2(\Gamma_{11}H-\Gamma_{12}A)K\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{\partial \Gamma_{112}}{\partial p}&=\Gamma_{1112}+\Gamma_{12}\Gamma_{111}+\Delta_{12}\Gamma_{112}+2\Gamma_{11}\Gamma_{112}+2\Delta_{11}\Gamma_{122}\\&\qquad+\tfrac{1}{6}H\frac{\partial K}{\partial p}-\tfrac{2}{3}(\Gamma_{11}H-\Gamma_{12}A)K\\\frac{\partial \Gamma_{112}}{\partial q}&=\Gamma_{1122}+\Gamma_{22}\Gamma_{111}+\Delta_{22}\Gamma_{112}+2\Gamma_{12}\Gamma_{112}+2\Delta_{12}\Gamma_{122}\\&\qquad-\tfrac{1}{6}H\frac{\partial K}{\partial q}-\tfrac{1}{6}B\frac{\partial K}{\partial p}+\tfrac{2}{3}(\Gamma_{11}B-\Gamma_{22}A)K\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{\partial \Gamma_{122}}{\partial p}&=\Gamma_{1122}+2\Gamma_{12}\Gamma_{112}+2\Delta_{12}\Gamma_{122}+\Gamma_{11}\Gamma_{122}+\Delta_{11}\Gamma_{222}\\&\qquad+\tfrac{1}{6}H\frac{\partial K}{\partial q}+\tfrac{1}{6}B\frac{\partial K}{\partial p}-\tfrac{2}{3}(\Gamma_{11}B-\Gamma_{22}A)K\\\frac{\partial \Gamma_{122}}{\partial q}&=\Gamma_{1222}+2\Gamma_{22}\Gamma_{112}+2\Delta_{22}\Gamma_{122}+\Gamma_{12}\Gamma_{122}+\Delta_{12}\Gamma_{222}\\&\qquad-\tfrac{1}{6}B\frac{\partial K}{\partial q}+\tfrac{2}{3}(\Gamma_{12}B-\Gamma_{22}H)K\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{\partial \Gamma_{222}}{\partial p}&=\Gamma_{1222}+3\Gamma_{12}\Gamma_{122}+3\Delta_{12}\Gamma_{222}+\tfrac{1}{2}B\frac{\partial K}{\partial q}-2(\Gamma_{12}B-\Gamma_{22}H)K\\\frac{\partial \Gamma_{222}}{\partial q}&=\Gamma_{2222}+3\Gamma_{22}\Gamma_{122}+3\Delta_{22}\Gamma_{222}\end{aligned}\right\},$$

for the derivatives of Γ_{ijk}; and

$$\left.\begin{aligned}\frac{\partial \Delta_{111}}{\partial p}&=\Delta_{1111}+3\Gamma_{11}\Delta_{111}+3\Delta_{11}\Delta_{112}\\\frac{\partial \Delta_{111}}{\partial q}&=\Delta_{1112}+3\Gamma_{12}\Delta_{111}+3\Delta_{12}\Delta_{112}+\tfrac{1}{2}A\frac{\partial K}{\partial p}+2(\Delta_{11}H-\Delta_{12}A)K\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{\partial \Delta_{112}}{\partial p}&=\Delta_{1112}+2\Gamma_{11}\Delta_{112}+2\Delta_{11}\Delta_{122}+\Gamma_{12}\Delta_{111}+\Delta_{12}\Delta_{112}\\&\qquad-\tfrac{1}{6}A\frac{\partial K}{\partial p}-\tfrac{2}{3}(\Delta_{11}H-\Delta_{12}A)K\\\frac{\partial \Delta_{112}}{\partial q}&=\Delta_{1122}+2\Gamma_{12}\Delta_{112}+2\Delta_{12}\Delta_{122}+\Gamma_{22}\Delta_{111}+\Delta_{22}\Delta_{112}\\&\qquad+\tfrac{1}{6}H\frac{\partial K}{\partial p}+\tfrac{1}{6}A\frac{\partial K}{\partial q}+\tfrac{2}{3}(\Delta_{11}B-\Delta_{22}A)K\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{\partial \Delta_{122}}{\partial p}&=\Delta_{1122}+\Gamma_{11}\Delta_{122}+\Delta_{11}\Delta_{222}+2\Gamma_{12}\Delta_{112}+2\Delta_{12}\Delta_{122}\\&\qquad-\tfrac{1}{6}H\frac{\partial K}{\partial p}-\tfrac{1}{6}A\frac{\partial K}{\partial q}-\tfrac{2}{3}(\Delta_{11}B-\Delta_{22}A)K\\\frac{\partial \Delta_{122}}{\partial q}&=\Delta_{1222}+\Gamma_{12}\Delta_{122}+\Delta_{12}\Delta_{222}+2\Gamma_{22}\Delta_{112}+2\Delta_{22}\Delta_{122}\\&\qquad+\tfrac{1}{6}H\frac{\partial K}{\partial q}+\tfrac{2}{3}(\Delta_{12}B-\Delta_{22}H)K\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{\partial\Delta_{222}}{\partial p}&=\Delta_{1222}+3\Gamma_{12}\Delta_{122}+3\Delta_{12}\Delta_{222}-\tfrac{1}{3}H\frac{\partial K}{\partial q}-2(\Delta_{12}B-\Delta_{22}H)K\\ \frac{\partial\Delta_{222}}{\partial q}&=\Delta_{2222}+3\Gamma_{22}\Delta_{122}+3\Delta_{22}\Delta_{222}\end{aligned}\right\}.$$

These formulæ can serve two purposes. They serve to express the quantities Γ_{ijkl} and Δ_{ijkl}: all that is necessary is substitution of the known values (§ 97) of Γ_{ijk} and Δ_{ijk}. They also imply the relations between differences of derivatives of Γ_{ijk} and of Δ_{ijk}: the relations themselves follow from forming the appropriate differences.

Minimal surfaces in general plenary homaloidal space.

103. The general equations of minimal surfaces in a plenary homaloidal space of any dimensionality* N can be obtained by an extension of the methods effective for the determination of such surfaces in a plenary triple space or in a plenary quadruple space. As always, the space-coordinates of a point on the surface are expressed in terms of two parameters u and v; these parameters will be specialised for minimal surfaces. Then, with

$$A=\sum y_u^2,\quad H=\sum y_u y_v,\quad B=\sum y_v^2,$$

the element of arc on the surface is

$$ds^2=A\,du^2+2H\,du\,dv+B\,dv^2,$$

and the element of area on the surface is

$$(AB-H^2)^{\frac{1}{2}}\,du\,dv.$$

Accordingly, if V denotes $(AB-H^2)^{\frac{1}{2}}$, it is necessary to make the double integral

$$\iint V du\,dv$$

a minimum in order to obtain a minimal surface, defined as the surface of least area bounded by two curves (usually closed curves). In the integral, the dependent variables are the space-coordinates of a point on the surface; and therefore the critical equations † are

$$-\frac{\partial V}{\partial y}+\frac{d}{du}\left(\frac{\partial V}{\partial y_u}\right)+\frac{d}{dv}\left(\frac{\partial V}{\partial y_v}\right)=0$$

for each of the space-variables typically represented by y. The quantity V

* For the following investigation, see the paper by Beckenbach, *Amer. Journ. Math.*, vol. lv (1933), pp. 458-468.

† See my *Calculus of Variations*, chaps. ix, x.

involves the derivatives of the space-variables but not the variables themselves; and thus the typical critical equation is

$$\frac{d}{du}\left(\frac{\partial V}{\partial y_u}\right)+\frac{d}{dv}\left(\frac{\partial V}{\partial y_v}\right)=0.$$

When the value of V, in the form $(AB-H^2)^{\frac{1}{2}}$, is substituted, account being taken of the constitution of A, H, B, the equation becomes

$$\frac{d}{du}\left\{\frac{1}{V}\left(By_u-Hy_v\right)\right\}+\frac{d}{dv}\left\{\frac{1}{V}(Ay_v-Hy_u)\right\}=0.$$

As the parametric variables may be specialised, we use the form first used by Weierstrass which selects them so as to make the (conjugate imaginary) nul-lines of the surface to be its parametric curves. Then

$$A=0, \quad B=0, \quad V=iH\,;$$

and therefore the typical critical equation becomes

$$y_{uv}=0.$$

The primitive of this equation is

$$y=f(u)+g(v)=f+g,$$

where f and g are arbitrary functions, each of its own single argument; and therefore the space-variables of a point on the surface are given by

$$y_m=f_m(u)+g_m(v)=f_m+g_m,$$

for $m=1, \ldots, N$.

In these expressions, all the functions f and g are arbitrary. So far as the critical equations are concerned, these functions are independent. But they are functions of the parameters of the nul-lines, $A=0$, $B=0$: that is, they are subject to the relations

$$\sum y_u^2=0, \quad \sum y_v^2=0.$$

When we write

$$f_m'=\frac{df_m}{du}, \quad g_m'=\frac{dg_m}{dv}, \quad (m=1, \ldots, N),$$

so that the quantities f_m' are functions of u alone and the quantities g_m' are functions of v alone, we have relations

$$f_1'^2+f_2'^2+\ldots+f_N'^2=0,$$
$$g_1'^2+g_2'^2+\ldots+g_N'^2=0,$$

which are the limiting restrictions upon the set of functions f_m and the set of functions g_m, otherwise independent.

To obtain expressions which shall be free from conditions, these equations must be resolved. We write

$$\sum_{\lambda}^{N} f_m'^2 = S_\lambda,$$

so that

$$f_1'^2 + f_2'^2 + S_3 = 0\ ;$$

and therefore, as in Weierstrass's resolution for the case $N=3$, we can take functions of u, denoted by F_1 and F_2, such that

$$f_1' = \tfrac{1}{2}(1-F_1)F_2, \quad f_2' = \tfrac{1}{2}i(1+F_1)F_2, \quad S_3 = F_1 F_2^2.$$

Similarly, because

$$f_3'^2 + f_4'^2 + S_5 - F_1F_2^2 = 0,$$

we can take

$$f_3' = \tfrac{1}{2}(1-F_3)F_4, \quad f_4' = \tfrac{1}{2}i(1+F_3)F_4, \quad S_5 = F_1F_2^2 + F_3F_4^2.$$

And so on in succession. The concluding part of the whole resolution varies according as N is even or is odd.

In the former event, let $N=2k$; the last resolution is

$$f_{2k-1}'^2 + f_{2k}'^2 = \sum_{1}^{k-1} (F_{2r-1}F_{2r}^2)$$

so that, if we introduce new functions F_{2k-1} and F_{2k}, and write

$$\sum_{1}^{k} (F_{2r-1}F_{2r}^2) = 0,$$

we can take

$$f_{2k-1}' = \tfrac{1}{2}(1-F_{2k-1})F_{2k}, \quad f_{2k}' = \tfrac{1}{2}i(1+F_{2k-1})F_{2k}.$$

In the latter event, let $N=2k+1$; the last resolution is

$$f_{2k-1}'^2 + f_{2k}'^2 + f_{2k+1}'^2 - \sum_{1}^{k-1} (F_{2r-1}F_{2r}^2) = 0,$$

and we can take

$$f_{2k-1}' = \tfrac{1}{2}(1-F_{2k-1})F_{2k}, \quad f_{2k}' = \tfrac{1}{2}i(1+F_{2k-1})F_{2k},$$

$$f_{2k+1}'^2 = F_{2k-1}F_{2k}^2 + \sum_{1}^{k-1}(F_{2r-1}F_{2r}^2) = \sum_{1}^{k}(F_{2r-1}F_{2r}^2).$$

All the functions F are functions of u alone. Thus we have

$$f_{2r-1}' = \tfrac{1}{2}(1-F_{2r-1})F_{2r}, \quad f_{2r}' = \tfrac{1}{2}i(1+F_{2r-1})F_{2r}:$$

when the range of subscript indices is $1, \ldots, 2k$, there is a condition

$$\sum_{r=1}^{k} F_{2r-1}F_{2r}^2 = 0\ ;$$

and when the range is 1, ..., $2k+1$, there is no condition, but the final term is

$$f'_{2k+1}=\left\{\sum_1^k(F_{2r-1}F_{2r}^2)\right\}^{\frac{1}{2}}.$$

The resolution of the equation involving the functions of g' is of the same type. But the variable of these functions is v, which is the conjugate of u when we are dealing with real surfaces; and therefore, in the resolution, the sign of i must be changed throughout*. We can take functions of G, involving the parameter v alone, and we can resolve the equation by the values

$$g'_{2r-1}=\tfrac{1}{2}(1-G_{2r-1})G_{2r},\quad g'_{2r}=-\tfrac{1}{2}i(1+G_{2r-1})G_{2r}:$$

when the range of indices is 1, ..., $2k$, there is a condition

$$\sum_{r=1}^k G_{2r-1}G_{2r}^2=0;$$

and when the range is 1, ..., $2k+1$, again there is no condition, but the final term is

$$g'_{2k+1}=\left\{\sum_1^k(G_{2r-1}G_{2r}^2)\right\}^{\frac{1}{2}}.$$

For the space-variables of a point on the surface, we have

$$y_m=f_m+g_m:$$

and therefore

$$y_{2r-1}=\tfrac{1}{2}\int(1-F_{2r-1})F_{2r}\,du+\tfrac{1}{2}\int(1-G_{2r-1})G_{2r}\,dv,$$

$$y_{2r}=\tfrac{1}{2}i\int(1+F_{2r-1})F_{2r}\,du-\tfrac{1}{2}i\int(1+G_{2r-1})G_{2r}\,dv.$$

When the range of subscript indices is 1, ..., $2k$, there are conditions

$$\sum_{r=1}^k F_{2r-1}F_{2r}^2=0,\quad \sum_{r=1}^k G_{2r-1}G_{2r}^2=0;$$

when the range of these indices is 1, ..., $2k+1$, there are no conditions, but the last point-variable is given by

$$y_{2k+1}=\int\left\{\sum_1^k(F_{2r-1}F_{2r}^2)\right\}^{\frac{1}{2}}du+\int\left\{\sum_1^k(G_{2r-1}G_{2r}^2)\right\}^{\frac{1}{2}}dv.$$

In the most general case, all the functions F and G are arbitrary functions of their respective variables u and v. These variables have not been settled precisely; in the most general case, it is convenient to take $F_{2r-1}=u^2$ and $G_{2r-1}=v^2$, for

* Whether the variables u and v be actually conjugate or not, the resolution of the g'-equation is made distinct from that of the f'-equation by the forms adopted for the functions g_2', g_4', ..., g_{2k}'.

some value of r; and consequently, for a real surface, the full expression of the point-coordinates involves $N-2$ arbitrary functions, any function F_m determining the function G_m in the conjugate expression.

This last formulation is not effective for any function F_{2r-1} which could be a constant (or, of course, equally for any function G_{2r-1} which could be a constant). In the event of the former hypothesis, we have

$$y_{2r-1} = \tfrac{1}{2}\,(1+c_{2r-1})\,F_{2r} - \tfrac{1}{2}\int(1-G_{2r-1})\,G_{2r}\,dv,$$

$$y_{2r} = \tfrac{1}{2}i\,(1-c_{2r-1})\,F_{2r} + \tfrac{1}{2}i\int(1+G_{2r-1})\,G_{2r}\,dv;$$

and therefore the nul-lines $v=$constant on the surface, when projected on the plane containing the axes of y_{2r-1} and y_{2r}, become straight lines meeting the circle at infinity in that plane.

Ex. 1. In a triple homaloidal space, the resolved equations are

$$f_1' = \tfrac{1}{2}(1-F_1)F_2,\quad f_2' = \tfrac{1}{2}i(1+F_1)F_2,\quad f_3' = F_1^{\frac{1}{2}}F_2,$$
$$g_1' = \tfrac{1}{2}(1-G_1)G_2,\quad g_2' = -\tfrac{1}{2}i(1+G_1)G_2,\quad g_3' = G_1^{\frac{1}{2}}G_2.$$

Excluding the exceptional case when F_1 or G_1 is a constant, we take

$$F_1 = u^2,\quad G_1 = v^2,\quad F_2 = F(u),\quad G_2 = G(v),$$

where $G(v)$ is the function conjugate to $F(u)$; and then

$$y_1 = \tfrac{1}{2}\int(1-u^2)F(u)\,du + \tfrac{1}{2}\int(1-v^2)G(v)\,dv,$$

$$y_2 = \tfrac{1}{2}i\int(1+u^2)F(u)\,du - \tfrac{1}{2}i\int(1+v^2)G(v)\,dv,$$

$$y_3 = \int uF(u)\,du + \int vG(v)\,dv,$$

one of the Weierstrass forms * of the equations of a minimal surface in triple homaloidal space.

Ex. 2. In a quadruple homaloidal space, the resolved equations are

$$f_1' = \tfrac{1}{2}\,(1-F_1)F_2,\quad g_1' = \tfrac{1}{2}\,(1-G_1)G_2,$$
$$f_2' = \tfrac{1}{2}i(1+F_1)F_2,\quad g_2' = -\tfrac{1}{2}i(1+G_1)G_2,$$
$$f_3' = \tfrac{1}{2}\,(1-F_3)F_4,\quad g_3' = \tfrac{1}{2}\,(1-G_3)G_4,$$
$$f_4' = \tfrac{1}{2}i(1+F_3)F_4,\quad g_4' = -\tfrac{1}{2}i(1+G_3)G_4,$$

with the limitations

$$F_1F_2^2 + F_3F_4^2 = 0,\quad G_1G_2^2 + G_3G_4^2 = 0,$$

upon the otherwise arbitrary functions.

* Apparently first published in the *Berl. Monatsber.* (1866), pp. 612-625, 855-856.

As the variable has not been specialised, we take

$$\frac{F_1F_2}{F_3F_4}=u, \quad \frac{F_4}{F_2}=-u,$$

thus satisfying the F-limitation. Also we assume

$$f_3'+if_4'=-F_3F_4=2f'', \quad f_1'-if_2'=F_2=-2g'',$$

where f and g are substituted functions: and now

$$f_1'+if_2'=-F_1F_2=2uf'', \quad f_3'-if_4'=F_4=2ug''.$$

Consequently

$$\begin{aligned} f_1'&=uf''-g'', \quad & if_2'&=uf''+g'', \\ f_3'&=f''+ug'', \quad & if_4'&=f''-ug'', \end{aligned}$$

where f and g now denote arbitrary functions of u.

Similarly,

$$\begin{aligned} g_1'&=v\phi''-\psi'', \quad & -ig_2'&=v\phi''+\psi'', \\ g_3'&=\phi''+v\psi'', \quad & -ig_4'&=\phi''-v\psi'', \end{aligned}$$

where ϕ and ψ are arbitrary functions of v. When the surface is real, $f(u)$ and $\phi(v)$ are conjugate to one another, likewise $g(u)$ and $\psi(v)$.

Finally, we have

$$\left.\begin{aligned} y_1 &= \text{Real part of } 2(uf'-f-g') \\ y_2 &= \quad ,, \qquad ,, \quad \frac{2}{i}(uf'-f+g') \\ y_3 &= \quad ,, \qquad ,, \quad 2(f'+ug'-g) \\ y_4 &= \quad ,, \qquad ,, \quad \frac{2}{i}(f'-ug'+g) \end{aligned}\right\},$$

the equations of a minimal surface in quadruple homaloidal space *.

The surface is algebraical when f and g are algebraic. For example, if

$$f=au^3, \quad g=du^2,$$

we have

$$y_1=R\,[4(au^3-du)], \qquad y_3=R\,[2(3a+d)u^2],$$

$$y_2=R\left[\frac{4}{i}(au^3+du)\right], \quad y_4=R\left[\frac{2}{i}(3a-d)u^2\right].$$

In the particular case $d=3a$, so that $y_4=0$ and the surface lies in the flat of the variables y_1, y_2, y_3, effectively the surface is Enneper's simple algebraical minimal surface in triple space †.

* Apparently first due to Eisenhart, *Amer. Journ. Math.*, vol. xxxiv (1912), pp. 214-236; see also my *G.F.D.*, vol. ii, § 406.

† *Zeitschr. f. Math. u. Phys.*, t. ix (1864), p. 108.

Ex. 3. Shew that, for a minimal surface in a quintuple homaloidal space, the equations can be expressed in the form

$$\left.\begin{aligned} f_1' &= \tfrac{1}{2}k(1-h^2)(1+u^2)\Theta \\ f_2' &= \tfrac{1}{2}ik(1+h^2)(1+u^2)\Theta \\ f_3' &= \tfrac{1}{2}ih(1-k^2)(1-u^2)\Theta \\ f_4' &= \tfrac{1}{2}h(1+k^2)(1-u^2)\Theta \\ f_5' &= 2uhk\Theta \end{aligned}\right\},$$

where h, k, Θ are arbitrary functions of the variable u, there being corresponding expressions for g_1', g_2', g_3', g_4', g_5'.

Ex. 4. Shew that, for a minimal surface in a sextuple homaloidal space, the equations can be expressed in the form

$$\left.\begin{aligned} f_1' &= i(1-h^2)(1+l^2)kmu \\ f_2' &= (1+h^2)(1+l^2)kmu \\ f_3' &= (1-k^2)(1-l^2)hmu \\ f_4' &= i(1+k^2)(1-l^2)hmu \\ f_5' &= 2ihklm(1+u^2) \\ f_6' &= 2hklm(1-u^2) \end{aligned}\right\},$$

where h, k, l, m, are arbitrary functions of the variable u, there being corresponding expressions for g_1', g_2', g_3', g_4', g_5', g_6'.

In this example, as in *Ex.* 3, while expressions for the space-variables y cannot in general be obtained free from quadratures, such expressions can be obtained by taking the functions h, k, l, equal to algebraical polynomials while leaving m an arbitrary function.

CHAPTER IX

Ranges on Surfaces: Riemann Measure of Curvature

Secondary magnitudes.

104. As proved in § 93, the circular curvature and the direction-cosines of the prime normal of a geodesic in the direction p', q', are given by the typical equation

$$\frac{Y}{\rho} = \eta_{11}p'^2 + 2\eta_{12}p'q' + \eta_{22}q'^2,$$

where the quantities η are free from the direction-variables, while the typical direction-cosine Y varies from one geodesic to another. Moreover, we have had a magnitude K, initially expressible (§ 16) as a Riemann four-index symbol (12, 12), being the only such symbol appertaining to a free surface, and also (§ 14) expressible in the form

$$V^2K = \sum \eta_{11}\eta_{22} - \sum \eta^2_{12},$$

the summation being over the range of the plenary space.

In connection with the circular curvature of a geodesic, and on the analogy of the Gauss theory of surfaces in triple space, we define secondary magnitudes, denoted by $\bar{A}$, $\bar{H}$, $\bar{B}$, of the surface according to the laws

$$\bar{A} = \sum Y\eta_{11}, \quad \bar{H} = \sum Y\eta_{12}, \quad \bar{B} = \sum Y\eta_{22};$$

and because of the relations

$$\sum Y \frac{\partial y}{\partial p} = 0, \quad \sum Y \frac{\partial y}{\partial q} = 0,$$

we also have

$$\bar{A} = \sum Y \frac{\partial^2 y}{\partial p^2}, \quad \bar{H} = \sum Y \frac{\partial^2 y}{\partial p\, \partial q}, \quad \bar{B} = \sum Y \frac{\partial^2 y}{\partial q^2}.$$

Unlike the secondary magnitudes in the Gauss theory, where they do not involve the direction-variables of the geodesic, these magnitudes $\bar{A}$, $\bar{H}$, $\bar{B}$, do involve p', q'.

In the first place, we have

$$\frac{1}{\rho} = \sum Y \frac{Y}{\rho} = \bar{A}p'^2 + 2\bar{H}p'q' + \bar{B}q'^2,$$

formally analogous to the Gauss expression for surfaces in triple homaloidal space, but essentially different because $\bar{A}$, $\bar{H}$, $\bar{B}$, themselves involve p', q'. To obtain the explicit expression of the circular curvature, we have

$$\frac{1}{\rho^2} = \sum \left(\frac{Y}{\rho}\right)^2 = \sum (\eta_{11}p'^2 + 2\eta_{12}p'q' + \eta_{22}q'^2)^2;$$

and therefore, if

$$\begin{array}{lll} \mathbf{a}=\sum \eta_{11}{}^2, & \mathbf{b}=\sum \eta_{12}{}^2, & \mathbf{c}=\sum \eta_{22}{}^2, \\ \mathbf{f}=\sum \eta_{12}\eta_{22}, & \mathbf{g}=\sum \eta_{22}\eta_{11}, & \mathbf{h}=\sum \eta_{11}\eta_{12}, \end{array}$$

$$\mathbf{g}+2\mathbf{b}=3\mathbf{k},$$

so that also

$$V^2K=\mathbf{g}-\mathbf{b},$$

we have

$$\frac{1}{\rho^2}=\mathbf{a}p'^4+4\mathbf{h}p'^3q'+6\mathbf{k}p'^2q'^2+4\mathbf{f}p'q'^3+\mathbf{c}q'^4.$$

It is worth while re-stating the formulæ

$$\sum \frac{\partial y}{\partial p}\eta_{ij}=0, \quad \sum \frac{\partial y}{\partial q}\eta_{ij}=0,$$

for $i, j, =1, 2$, which are equivalent to the definitions of the Christoffel symbols $\{ij, 1\}$ and $\{ij, 2\}$ of § 12 ; by means of them, we have results such as

$$\begin{aligned} \mathbf{a} &= \sum \left(\frac{\partial^2 y}{\partial p^2}-\Gamma_{11}\frac{\partial y}{\partial p}-\Delta_{11}\frac{\partial y}{\partial q}\right)^2 \\ &= \sum \left(\frac{\partial^2 y}{\partial p^2}\right)^2-(A, H, B \between \Gamma_{11}, \Delta_{11})^2, \end{aligned}$$

as well as the expression for V^2K, which is equal to

$$\sum \left(\frac{\partial^2 y}{\partial p^2}\frac{\partial^2 y}{\partial q^2}\right)-\sum \left(\frac{\partial^2 y}{\partial p\,\partial q}\right)^2-(A, H, B \between \Gamma_{11}, \Delta_{11} \between \Gamma_{22}, \Delta_{22})+(A, H, B \between \Gamma_{12}, \Delta_{12})^2.$$

Next, returning to the definitions of $\bar{A}$, $\bar{H}$, $\bar{B}$, we have

$$\left.\begin{aligned} \frac{\bar{A}}{\rho} &= \frac{\sum Y\eta_{11}}{\rho}=\mathbf{a}p'^2+2\mathbf{h}p'q'+\mathbf{g}q'^2 \\ \frac{\bar{H}}{\rho} &= \frac{\sum Y\eta_{12}}{\rho}=\mathbf{h}p'^2+2\mathbf{b}p'q'+\mathbf{f}q'^2 \\ \frac{\bar{B}}{\rho} &= \frac{\sum Y\eta_{22}}{\rho}=\mathbf{g}p'^2+2\mathbf{f}p'q'+\mathbf{c}q'^2 \end{aligned}\right\}.$$

The substitution of these values of $\bar{A}$, $\bar{H}$, $\bar{B}$, in the value

$$\frac{1}{\rho}=\bar{A}p'^2+2\bar{H}p'q'+\bar{B}q'^2$$

merely leads to the expression for $1/\rho^2$ already given.

Further, we have

$$\begin{aligned} \frac{\partial}{\partial p'}\left(\frac{1}{\rho^2}\right) &= 4(\mathbf{a}p'^3+3\mathbf{h}p'^2q'+3\mathbf{k}p'q'^2+\mathbf{f}q'^3) \\ &= 4\{p'(\mathbf{a}p'^2+2\mathbf{h}p'q'+\mathbf{g}q'^2)+q'(\mathbf{h}p'^2+2\mathbf{b}p'q'+\mathbf{f}q'^2)\} \\ &= \frac{4}{\rho}(\bar{A}p'+\bar{H}q')\,; \end{aligned}$$

and similarly

$$\frac{\partial}{\partial q'}\left(\frac{1}{\rho^2}\right)=\frac{4}{\rho}(\bar{H}p'+\bar{B}q').$$

Accordingly,

$$\frac{\partial}{\partial p'}\left(\frac{1}{\rho}\right)=2(\bar{A}p'+\bar{H}q'),\quad \frac{\partial}{\partial q'}\left(\frac{1}{\rho}\right)=2(\bar{H}p'+\bar{B}q'),$$

being the form of the general results for an amplitude in § 30 now applicable to a surface.

Also, by direct substitution, we find

$$\frac{1}{\rho^2}(\bar{A}\bar{B}-\bar{H}^2)=\frac{1}{\rho^2}(\mathbf{g}-\mathbf{b})+\mathbf{H},$$

where

$$\mathbf{H}=(\mathbf{ab}-\mathbf{h}^2)p'^4+2(\mathbf{af}-\mathbf{kh})p'^3q'+(\mathbf{ac}+2\mathbf{fh}-3\mathbf{k}^2)p'^2q'^2$$
$$+2(\mathbf{ch}-\mathbf{fk})p'q'^3+(\mathbf{ck}-\mathbf{f}^2)q'^4,$$

so that $\mathbf{H}$ is the Hessian of the binary quartic in p', q', which is the value of $1/\rho^2$. As will be seen later (§ 132), this result leads to the relation

$$\bar{A}\bar{B}-\bar{H}^2=\mathbf{g}-\mathbf{b}+\frac{V^2}{\tau^2},$$

or

$$\bar{A}\bar{B}-\bar{H}^2-V^2K=\frac{V^2}{\tau^2},$$

where $1/\tau$ is the tilt of the superficial geodesic.

105. The equations, which determine the direction-cosines of the prime normal of a superficial geodesic, lead to the relations

$$\| \; Y, \quad \eta_{11}, \quad \eta_{12}, \quad \eta_{22} \; \|=0\,;$$

and therefore the prime normals of all the superficial geodesics through the originating point $y_1, y_2, \ldots$ lie in the flat

$$\| \; \bar{y}-y, \quad \eta_{11}, \quad \eta_{12}, \quad \eta_{22} \; \|=0.$$

(Incidentally we infer the property that the locus of the centre of circular curvature of all these geodesics is a curve in this flat: the locus will be considered later.) Also, the equations of the tangent plane to the surface are

$$\left\| \; \bar{y}-y, \quad \frac{\partial y}{\partial p}, \quad \frac{\partial y}{\partial q} \; \right\|=0.$$

Now the relations

$$\sum\frac{\partial y}{\partial p}\eta_{ij}=0,\quad \sum\frac{\partial y}{\partial q}\eta_{ij}=0,$$

are satisfied for $i, j, =1, 2$: the plane and the flat are therefore orthogonal to one another: or the flat is orthogonal to the surface. Manifestly the flat lies in the orthogonal homaloid (§ 92) of the surface.

We know that when the plenary homaloidal space of the surface is triple, the orthogonal homaloid is a line: the line is the unique normal to the tangent plane. Also, when the plenary homaloidal space of the surface is quadruple, the orthogonal homaloid is a plane * which is the orthogonal plane of the surface. In the latter case, the determinant Υ, where

$$\Upsilon = \begin{vmatrix} \mathrm{a}, & \mathrm{h}, & \mathrm{g} \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f} \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c} \end{vmatrix},$$

vanishes; in the former case, each of its first minors vanishes.

Passing now to the more general case when the plenary space is of more than four dimensions, we have the relations

$$\| \; Y, \quad \eta_{11}, \quad \eta_{12}, \quad \eta_{22} \; \| = 0$$

satisfied; and therefore, by a known theorem in determinants,

$$\sum_i \sum_j \sum_k \sum_l \begin{vmatrix} Y_i, & \eta_{11}{}^{(i)}, & \eta_{12}{}^{(i)}, & \eta_{22}{}^{(i)} \\ Y_j, & \eta_{11}{}^{(j)}, & \eta_{12}{}^{(j)}, & \eta_{22}{}^{(j)} \\ Y_k, & \eta_{11}{}^{(k)}, & \eta_{12}{}^{(k)}, & \eta_{22}{}^{(k)} \\ Y_l, & \eta_{11}{}^{(l)}, & \eta_{12}{}^{(l)}, & \eta_{22}{}^{(l)} \end{vmatrix}^2 = 0.$$

Hence we have the relation

$$\begin{vmatrix} 1, & \bar{A}, & \bar{H}, & \bar{B} \\ \bar{A}, & \mathrm{a}, & \mathrm{h}, & \mathrm{g} \\ \bar{H}, & \mathrm{h}, & \mathrm{b}, & \mathrm{f} \\ \bar{B}, & \mathrm{g}, & \mathrm{f}, & \mathrm{c} \end{vmatrix} = 0,$$

a relation which can be derived at once from the expressions for $1/\rho$, $\bar{A}/\rho$, $\bar{H}/\rho$, $\bar{B}/\rho$, by eliminating p' and q'. Now, as the orthogonal homaloid is a flat, so that relations

$$\begin{vmatrix} \eta_{11}{}^{(i)}, & \eta_{12}{}^{(i)}, & \eta_{22}{}^{(i)} \\ \eta_{11}{}^{(j)}, & \eta_{12}{}^{(j)}, & \eta_{22}{}^{(j)} \\ \eta_{11}{}^{(k)}, & \eta_{12}{}^{(k)}, & \eta_{22}{}^{(k)} \end{vmatrix} = 0$$

are no longer satisfied for all values of i, j, k, and therefore the determinant Υ no longer vanishes, the foregoing relation becomes

$$\sum \bar{A}^2 (\mathrm{bc} - \mathrm{f}^2) = \Upsilon.$$

* *G.F.D.*, vol. i, § 209.

It is a relation connecting the magnitudes $\bar{A}$, $\bar{H}$, $\bar{B}$, which involve the direction-variables. Also, we have

$$\frac{1}{\rho}\begin{vmatrix} \bar{A}, & \bar{H}, & \bar{B} \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f} \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c} \end{vmatrix} = Yp'^2, \quad \frac{1}{\rho}\begin{vmatrix} \bar{A}, & \bar{H}, & \bar{B} \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c} \\ \mathrm{a}, & \mathrm{h}, & \mathrm{b} \end{vmatrix} = 2Yp'q', \quad \frac{1}{\rho}\begin{vmatrix} \bar{A}, & \bar{H}, & \bar{B} \\ \mathrm{a}, & \mathrm{h}, & \mathrm{b} \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f} \end{vmatrix} = Yq'^2.$$

We shall have to return to the consideration of the foregoing flat, which is orthogonal to the surface.

Torsion of a superficial geodesic.

106. Before considering the circular curvature of a geodesic in detail, we shall obtain an expression for the torsion of the geodesic. It has been proved (§ 95) that the binormal lies in the tangent plane of the surface, and that its typical direction-cosine l_3 can be taken in the form

$$l_3 = \frac{1}{V}\left\{(Ap' + Hq')\frac{\partial y}{\partial q} - (Hp' + Bq')\frac{\partial y}{\partial p}\right\}.$$

The relevant Frenet equations, connecting the torsion with the circular curvature, are represented by

$$\frac{l_3}{\sigma} = \frac{dl_2}{ds} + \frac{l_1}{\rho}$$

in general, and therefore by

$$\frac{l_3}{\sigma} = \frac{dY}{ds} + \frac{y'}{\rho} = Y' + \frac{y'}{\rho}$$

for the superficial geodesic ; consequently, there exist the equations typified by

$$\frac{1}{V\sigma}\left\{(Ap' + Hq')\frac{\partial y}{\partial q} - (Hp' + Bq')\frac{\partial y}{\partial p}\right\} = Y' + \frac{y'}{\rho},$$

there being one such equation for every space-coordinate.

Now we have

$$\sum Yy' = 0, \quad \sum Y\frac{\partial y}{\partial p} = 0, \quad \sum Y\frac{\partial y}{\partial q} = 0.$$

From the first of these, we have

$$\sum Y'y' = -\sum Yy'' = -\frac{1}{\rho};$$

from the second, we have

$$\begin{aligned}\sum Y'\frac{\partial y}{\partial p} &= -\sum Y\frac{d}{ds}\left(\frac{\partial y}{\partial p}\right) \\ &= -\sum Y\left(\frac{\partial^2 y}{\partial p^2}p' + \frac{\partial^2 y}{\partial p\,\partial q}q'\right) = -(\bar{A}p' + \bar{H}q');\end{aligned}$$

and similarly, from the third,

$$\sum Y' \frac{\partial y}{\partial q} = -(\bar{H}p' + \bar{B}q').$$

Also, we have

$$\sum Y'^2 = \sum \left(\frac{l_3}{\sigma} + \frac{y'}{\rho}\right)^2 = \frac{1}{\rho^2} + \frac{1}{\sigma^2}.$$

Hence, multiplying the foregoing typical direction-cosine equation by Y' and adding for the whole set of equations, we have

$$\frac{1}{V\sigma}\{(\bar{A}p' + \bar{H}q')(Hp' + Bq') - (\bar{H}p' + \bar{B}q')(Ap' + Bq')\} = \frac{1}{\rho^2} + \frac{1}{\sigma^2} - \frac{1}{\rho^2} = \frac{1}{\sigma^2},$$

and therefore

$$\frac{V}{\sigma} = \begin{vmatrix} \bar{A}p' + \bar{H}q', & \bar{H}p' + \bar{B}q' \\ Ap' + Hq', & Hp' + Bq' \end{vmatrix}$$

$$= \begin{vmatrix} \bar{A}, & A, & q'^2 \\ \bar{H}, & H, & -q'p' \\ \bar{B}, & B, & p'^2 \end{vmatrix},$$

which accordingly give expressions for the torsion of the superficial geodesic drawn in the direction p', q', through O.

Owing to the implicit occurrence of p' and q' in the values of $\bar{A}$, $\bar{H}$, $\bar{B}$, neither expression gives the fully explicit value of the torsion in terms of p' and q'; but we at once have

$$\frac{V}{\rho\sigma} = \begin{vmatrix} \frac{\bar{A}}{\rho}p' + \frac{\bar{H}}{\rho}q', & \frac{\bar{H}}{\rho} + \frac{\bar{B}}{\rho}q' \\ Ap' + Hq', & Hp' + Bq' \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + 3\mathbf{k}p'q'^2 + \mathbf{f}q'^3, & Ap' + Hq' \\ \mathbf{h}p'^3 + 3\mathbf{k}p'^2q' + 3\mathbf{f}p'q'^2 + \mathbf{c}q'^3, & Hp' + Bq' \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}p'^2, & A, & q'^2 \\ \mathbf{h}p'^2 + 2\mathbf{b}p'q' + \mathbf{f}q'^2, & H, & -q'p' \\ \mathbf{g}p'^2 + 2\mathbf{f}p'q' + \mathbf{c}q'^2, & B, & p'^2 \end{vmatrix}.$$

Also, because

$$\frac{1}{\rho^2} = \mathbf{a}p'^4 + 4\mathbf{h}p'^3q' + 6\mathbf{k}p'^2q'^2 + 4\mathbf{f}p'q'^3 + \mathbf{c}q'^4;$$

we infer an explicit expression for the torsion alone.

Further, by comparing the general curve-expression with the specific surface-expression for the typical direction-cosine l_3, we have

$$Y' = -\left(\frac{p'}{\rho} + \frac{Hp' + Bq'}{V\sigma}\right)\frac{\partial y}{\partial p} - \left(\frac{q'}{\rho} - \frac{Ap' + Hq'}{V\sigma}\right)\frac{\partial y}{\partial q};$$

and by using the foregoing value of σ, we can deduce the equivalent form

$$V^2 Y' = -\begin{vmatrix} \bar{A}p' + \bar{H}q', & H \\ \bar{H}p' + \bar{B}q', & B \end{vmatrix} \frac{\partial y}{\partial p} + \begin{vmatrix} \bar{A}p' + \bar{H}q', & A \\ \bar{H}p' + \bar{B}q', & H \end{vmatrix} \frac{\partial y}{\partial q}.$$

Ex. Establish the relations:

(i) $\sum\left(l_3 \frac{\partial y}{\partial p}\right) = -Vq'$;

(ii) $\sum\left(l_3 \frac{\partial y}{\partial q}\right) = \quad Vp'$;

(iii) $\sum\left(y''' \frac{\partial y}{\partial p}\right) = -\frac{1}{\rho}(\bar{A}p' + \bar{H}q')$;

(iv) $\sum\left(y''' \frac{\partial y}{\partial q}\right) = -\frac{1}{\rho}(\bar{H}p' + \bar{B}q')$;

(v) $\sum(y_{ij} l_3) \; = V(p'\Delta_{ij} - q'\Gamma_{ij})$;

(vi) $\sum(y_{ij} Y') = -(v_1 \Gamma_{ij} + v_2 \Delta_{ij})$;

(vii) $\frac{\sum \eta_{11} y'''}{\bar{A}} = \frac{\sum \eta_{12} y'''}{\bar{H}} = \frac{\sum \eta_{22} y'''}{\bar{B}} = \frac{d}{ds}\left(\frac{1}{\rho}\right)$.

107. For the sake of brevity, it is convenient to use the symbols u and v, in accordance with their general significance (§ 31), and defined for a surface by the equations

$$\left.\begin{aligned} u_1 &= Ap' + Hq' \\ u_2 &= Hp' + Bq' \end{aligned}\right\}, \quad \left.\begin{aligned} v_1 &= \bar{A}p' + \bar{H}q' \\ v_2 &= \bar{H}p' + \bar{B}q' \end{aligned}\right\}.$$

The preceding two expressions, as values for Y', thus become

$$Y' = -\left(\frac{p'}{\rho} + \frac{u_2}{V\sigma}\right)\frac{\partial y}{\partial p} - \left(\frac{q'}{\rho} - \frac{u_1}{V\sigma}\right)\frac{\partial y}{\partial q},$$

$$V^2 Y' = -(Bv_1 - Hv_2)\frac{\partial y}{\partial p} + (Hv_1 - Av_2)\frac{\partial y}{\partial q},$$

both of them appertaining solely to superficial geodesics and not to any skew curve. We also had (§ 38) the quite general relation, in a Frenet system,

$$Y'' = \frac{l_4}{\sigma\tau} + l_3 \frac{d}{ds}\left(\frac{1}{\sigma}\right) - Y\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) - y'\frac{d}{ds}\left(\frac{1}{\rho}\right),$$

appertaining to a geodesic in any configuration; and an equivalent relation is obtained by differentiating the relation

$$l_3 = \frac{\sigma}{\rho} y' + \sigma Y',$$

which leads to the form

$$\frac{l_4}{\tau} - \frac{Y}{\sigma} = y' \frac{d}{ds}\left(\frac{\sigma}{\rho}\right) + Y\frac{\sigma}{\rho^2} + \sigma' Y' + \sigma Y''.$$

Now, at various stages, we have obtained general equations

$$\sum Y\frac{\partial y}{\partial p}=0,\quad \sum Y\frac{\partial y}{\partial q}=0,\quad \sum l_4\frac{\partial y}{\partial p}=0,\quad \sum l_4\frac{\partial y}{\partial q}=0,$$

$$\sum y'\frac{\partial y}{\partial p}=u_1,\quad \sum Y'\frac{\partial y}{\partial p}=-v_1,\quad \sum Y''\frac{\partial y}{\partial p}=-w_1,$$

$$\sum y'\frac{\partial y}{\partial q}=u_2,\quad \sum Y'\frac{\partial y}{\partial q}=-v_2,\quad \sum Y''\frac{\partial y}{\partial q}=-w_2.$$

Hence, on multiplying the second equation for l_4 by $\frac{\partial y}{\partial p}$ and $\frac{\partial y}{\partial q}$ in turn, and adding separately for all the spatial dimensions, we find

$$u_1\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-v_1\sigma'-w_1\sigma=0,$$

$$u_2\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-v_2\sigma'-w_2\sigma=0.$$

Also the superficial torsion is given by the equation

$$\frac{V}{\sigma}=v_1u_2-v_2u_1\,;$$

hence

$$\frac{-\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)}{w_1v_2-w_2v_1}=\frac{\sigma'}{w_2u_1-w_1u_2}=\frac{\sigma}{v_1u_2-v_2u_1}=\frac{\sigma^2}{V}.$$

Consequently, we have

$$V\frac{d}{ds}\left(\frac{1}{\sigma}\right)=w_1u_2-w_2u_1.$$

The other relation need not be retained ; for, because

$$\frac{1}{\rho}=v_1p'+v_2q',\quad \frac{d}{ds}\left(\frac{1}{\rho}\right)=w_1p'+w_2q',\quad 1=u_1p'+u_2q',$$

we have

$$\begin{aligned}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)&=\frac{1}{\rho}\sigma'+\sigma\frac{d}{ds}\left(\frac{1}{\rho}\right)\\&=\frac{\sigma^2}{V}\{(w_2u_1-w_1u_2)(v_1p'+v_2q')+(v_1u_2-v_2u_1)(w_1p'+w_2q')\}\\&=\frac{\sigma^2}{V}(w_2v_1-w_1v_2)(u_1p'+u_2q')\\&=\frac{\sigma^2}{V}(w_2v_1-w_1v_2),\end{aligned}$$

and therefore the value of the first fraction is derivable from the values of ρ, ρ', σ, σ', as given.

Again, substituting the results

$$\frac{d}{ds}\left(\frac{1}{\rho}\right)=w_1 p'+w_2 q', \quad \frac{d}{ds}\left(\frac{1}{\sigma}\right)=-\frac{1}{V}(u_1 w_2-u_2 w_1),$$

in the relation

$$Y''+Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)=\frac{l_4}{\sigma\tau}+l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right)-y'\frac{d}{ds}\left(\frac{1}{\rho}\right),$$

as well as the values

$$y'=\frac{\partial y}{\partial p}p'+\frac{\partial y}{\partial q}q', \quad Vl_3=u_1\frac{\partial y}{\partial q}-u_2\frac{\partial y}{\partial p},$$

we have

$$Y''+Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)=\frac{l_4}{\sigma\tau}+\frac{1}{V^2}\left\{\frac{\partial y}{\partial p}(Hw_2-Bw_1)-\frac{\partial y}{\partial q}(Aw_2-Hw_1)\right\},$$

both forms of which will be found useful for Y''.

Squaring the former relation, and adding for all the space-dimensions, we have

$$\sum Y''^2=\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)^2+\frac{1}{\sigma^2\tau^2}+\frac{\rho'^2}{\rho^4}+\frac{\sigma'^2}{\sigma^4};$$

and, similarly from the second,

$$\sum Y''^2=\frac{1}{\rho^2}+\frac{1}{\sigma^2}+\frac{1}{\sigma^2\tau^2}+\frac{1}{V^2}(Aw_2{}^2-2Hw_2w_1+Bw_1{}^2).$$

It follows that

$$Aw_2{}^2-2Hw_2w_1+Bw_1{}^2=V^2\left(\frac{\rho'^2}{\rho^4}+\frac{\sigma'^2}{\sigma^4}\right),$$

a result easily verified from the expressions for $\frac{d}{ds}\left(\frac{1}{\rho}\right)$ and $\frac{d}{ds}\left(\frac{1}{\sigma}\right)$; it provides a geometrical interpretation of a covariant belonging to the complete system of concomitants of a surface.

Historical note on the Riemann measure.

108. The circular curvature of a geodesic on a surface, existing freely in a multiple plenary space, is given by an equation

$$\frac{1}{\rho^2}=(\mathbf{a}, \mathbf{h}, \mathbf{k}, \mathbf{f}, \mathbf{c}\between p', q')^4,$$

where $3\mathbf{k}=\mathbf{g}+2\mathbf{b}$; and the maximum and the minimum values of this circular curvature, together with all combinations of such values, are expressible in terms of these magnitudes **a**, **h**, **k**, **f**, **c**, together with the primary magnitudes A, H, B.

On the other hand, a magnitude K has been obtained, initially arising out of derivatives of the primary magnitudes, and later expressible in various ways

(§§ 14, 16, 104), the simple formal expressions, when the amplitude is a surface, being

$$V^2K=(12,\,12)=\sum\eta_{11}\eta_{22}-\sum\eta_{12}{}^2=\mathbf{g}-\mathbf{b}.$$

Hence K is independent of combinations of the principal circular curvatures for a surface in multiple plenary space of more than four dimensions, the determinant

$$\begin{vmatrix} \mathbf{a}, & \mathbf{h}, & \mathbf{g} \\ \mathbf{h}, & \mathbf{b}, & \mathbf{f} \\ \mathbf{g}, & \mathbf{f}, & \mathbf{c} \end{vmatrix}$$

being assumed not to vanish.

Even in quadruple plenary space, for which this determinant does vanish, and for which K therefore has the value

$$\frac{1}{V^2(\mathbf{ac}-\mathbf{g}^2)}\begin{vmatrix} \mathbf{a}, & \mathbf{h}, & \mathbf{g} \\ \mathbf{h}, & \mathbf{g}, & \mathbf{f} \\ \mathbf{g}, & \mathbf{f}, & \mathbf{c} \end{vmatrix},$$

it appears * that the quantity K is independent of the four principal measures of circular curvature and is unconnected with the (four) curves of circular curvature.

In triple plenary space, the quantity K is the Gauss measure of curvature of the surface, being the product of the two principal circular curvatures, so that $1/K$ is the product of the two principal radii of circular curvature.

Now Riemann, among manuscripts unpublished at the time of his death †, propounded an expression ‡ which he described as a "measure of curvature of a surface" in an amplitude of n dimensions; and, in the notation of § 14 with the four-index symbols, this expression is

$$\frac{\sum\sum(ij,\,kl)\,\xi_{ij}\xi_{kl}}{(\sum\sum A_{ij}\,dx_i\,dx_j)(\sum\sum A_{ij}\delta x_i\delta x_j)-(\sum\sum A_{ij}\,dx_i\,\delta x_j)^2},$$

where

$$\xi_{ij}=dx_i\,\delta x_j-dx_j\,\delta x_i,\quad \xi_{kl}=dx_k\,\delta x_l-dx_l\,\delta x_k.$$

But no indication was left by Riemann, as to the character of the curvature thus measured, or of the significance of the measure, or of the actual construction of the expression. A note **, due to Weber, and reproduced in substance by other writers, establishes the property that, for such a surface in an amplitude, the

* *G.F.D.*, vol. i, chap. xiii.

† They are included in the 1876 edition of Riemann's *Gesammelte mathematische Werke*, edited by Dedekind and Weber; and explanatory amplifications, due to Weber, are added in the 1892 edition. Usually, the later edition is cited here, in the references about to be made.

‡ *l.c.* p. 403. The surface, described by Riemann, is effectively what is termed a geodesic surface in the amplitude.

** *l.c.* pp. 405-414.

propounded analytical measure is formally the same as the analytical measure for a surface existing freely in plenary space.

For the immediate purpose, it will suffice to consider the Riemann measure for a surface existing freely in multiple space. Though it agrees in formal expression with the Gauss measure for a surface in triple space and has the significance of the Gauss measure for such a surface, the agreement continues only in formal expression for a surface in four or more dimensions; the significance often assigned to the Gauss measure (as the product of the two principal radii of curvature of the surface at the place) can no longer be assigned, if only for the reason that there are four curves of curvature at every point on a surface in the more extended space. In these circumstances, writers * are accustomed to declare that the magnitude is *called* the measure of curvature; but though the label is affixed, the geometrical significance remains undiscussed. Later, in connection with the Levi-Civita theory of parallels, some geometrical properties of a surface which is geodesic to an amplitude are obtained † by Levi-Civita himself, by Severi, Bompiani, and others, leading to an interpretation of the quantity K; but the significance, for the most part, remains generically analytical and is not made specifically geometrical. Some specific interpretations have been obtained, additional to the Gauss measure for a surface in triple space. Thus for a region in a plenary quadruple space, as for a general primary n-fold amplitude, the general Riemann measure is proved ‡ to be one of the two principal measures of superficial curvature of the region and the amplitude respectively. Again, for a parametric surface contained in a primary region, the Riemann measure can be interpreted ** as the sum of (i), the foregoing principal measure of the superficial curvature of the region in the orientation of the parametric surface and (ii), the Gauss measure of regional flexure of the parametric surface; and this last property can be extended to a parametric configuration in a primary amplitude.

But it proves possible to obtain an interpretation of the Riemann measure K for a surface existing freely in any plenary space; for the purpose, it is sufficient to deal with properties of triangles on a surface bounded by superficial geodesics as sides, without invoking any theory of geodesic parallels. The same interpretation can then be assigned to a geodesic surface in an amplitude, that is, it can be assigned to the general measure as propounded by Riemann.

To this investigation we now proceed. Much of the analysis is practically the

* Bianchi, *Lezioni di geometria differenziale*, i, p. 101, ii, pp. 426, 427; Levi-Civita, *The absolute differential calculus* (Eng. trans., 1927), p. 196; Eddington, *The mathematical theory of relativity*, p. 151.

† References will be found later, in Chapter X, when the theory of geodesic parallels is discussed.

‡ *G.F.D.*, vol. ii, §§ 319, 431.

** *l.c.* vol. ii, § 368.

same as the corresponding analysis in Chapter V. But there, as will appear also later when the matter is discussed for regions and for domains, the three geodesic surfaces of the amplitude at O, P, Q, in Fig. 1, p. 145, are distinct from one another, and the geodesic PQ does not lie in the surface at O. Accordingly, instead of merely citing the analytical results of §§ 62, 64, the investigation (as concerned with one and the same surface throughout) is repeated from the beginning.

Small geodesic triangle on a surface: the third side.

109. At any point O on a surface existing freely in a multiple plenary space, let OR and OT be two superficial geodesics in directions p_1', q_1', and p_2', q_2', respectively. Along OR, let a small geodesic arc u be measured, up to the point R; and along OT, let a small geodesic arc v be measured up to the point T. Let the superficial geodesic RT be drawn. As u and v are small, there is no question of any conjugate relation (in the Jacobi sense) between the points R and T; and the arc RT itself is small. Thus there is a small geodesic triangle ORT on the surface; we have to find its third side TR, and its angles at R and T, while the angle at O, denoted by ϵ, is given by the relations

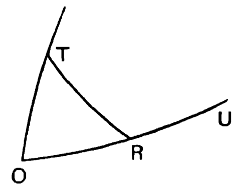

FIG. 5.

$$\cos\epsilon = Ap_1'p_2' + H(p_1'q_2' + q_1'p_2') + Bq_1'q_2',$$
$$\sin\epsilon = V(p_1'q_2' - q_1'p_2'),$$

under the usual convention as to the positive direction of angular measurement.

At R, let the direction-variables of the geodesic RT in the direction RT be denoted by p_5', q_5'; let the angles at R and T be denoted by these letters R and T; and let the length of the geodesic arc RT be denoted by z.

The superficial parameters at O are p, q. The values of these parameters at T are the same, whether we use the geodesic arcs OR, RT, in succession, or we use the geodesic arc OT direct, to pass from O to T. To obtain certain subsidiary quantities, we retain in the first instance only first powers of the small magnitudes u, v, z. By the path OR, RT, the value of p at T is $p_R + zp_5' + \frac{1}{2}z^2p_5''$, and p_R itself is equal to $p + up_1' + \frac{1}{2}u^2p_1''$: by the direct path OT, the value of p obtained at T is $p + vp_2' + \frac{1}{2}v^2p_2''$. We therefore have an equation

$$zp_5' + \tfrac{1}{2}z^2p_5'' = vp_2' - up_1' + \tfrac{1}{2}v^2p_2'' - \tfrac{1}{2}u^2p_1'';$$

and by proceeding similarly with the parameter q, there is a like equation

$$zq_5' + \tfrac{1}{2}z^2q_5'' = vq_2' - uq_1' + \tfrac{1}{2}v^2q_2'' - \tfrac{1}{2}u^2q_1'',$$

both equations being accurate up to (but only up to) the second order of small quantities inclusive.

For the resolution of these equations, we take

$$p_5' = p_0' + P, \quad q_5' = q_0' + Q, \quad z = t + Z.$$

where t is to be of the same order of magnitude as u and v, while Z is certainly of the second order: where p_0' and q_0' satisfy the condition

$$Ap_0'^2+2Hp_0'q_0'+Bq_0'^2=1:$$

and where P and Q are of the first order of small quantities. The terms of the first order in the two parametric equations now balance, provided

$$tp_0'=vp_2'-up_1', \quad tq_0'=vq_2'-uq_1';$$

and therefore

$$\begin{aligned} t^2&=t^2(Ap_0'^2+2Hp_0'q_0'+Bq_0'^2)\\ &=v^2+u^2-2uv\cos\epsilon. \end{aligned}$$

Also, we take

$$\sin R_0=V(p_1'q_0'-p_0'q_1'), \quad \sin T_0=V(p_2'q_0'-p_0'q_2'),$$

regard being paid to the positive direction in the measurement of angles: it is easy to see that R_0 and T_0 are approximations to R and T. Moreover, there are relations

$$\frac{\sin T_0}{u}=\frac{\sin R_0}{v}=\frac{\sin\epsilon}{t},$$

$$t\cos T_0=v-u\cos\epsilon, \quad t\cos R_0=u-v\cos\epsilon, \quad u\cos R_0+v\cos T_0=t,$$

being in effect properties connecting the sides and angles of a plane rectilinear triangle.

We now consider the second-order terms in the same two parametric relations; in the process, terms of the third and higher orders are to be neglected. We have

$$z^2p_5''=-z^2\{\Gamma_{11}^{(R)}p_5'^2+2\Gamma_{12}^{(R)}p_5'q_5'+\Gamma_{22}^{(R)}q_5'^2\},$$

where, for the accurate values of p_5'' and q_5'', we should require the values of Γ_{ij} and Δ_{ij} at R; but as there already exists a factor z^2 on the right-hand side, we can take (up to the second order inclusive)

$$z^2p_5''=-t^2(\Gamma_{11}p_0'^2+2\Gamma_{12}p_0'q_0'+\Gamma_{22}q_0'^2).$$

Thus, when substitution for tp_0' and tq_0' is effected,

$$z^2p_5''=v^2p_2''+u^2p_1''+2uv\{\Gamma_{11}p_1'p_2'+2\Gamma_{12}(p_1'q_2'+q_1'p_2')+\Gamma_{22}q_1'q_2'\};$$

and therefore, for the combination of terms in the p-relation,

$$\begin{aligned} z^2p_5''-v^2p_2''+u^2p_1''&\\ =-2u^2\{\Gamma_{11}p_1'^2+2\Gamma_{12}p_1'q_1'+\Gamma_{22}q_1'^2\}&\\ +2uv\{\Gamma_{11}p_1'p_2'+2\Gamma_{12}(p_1'q_2'+q_1'p_2')+\Gamma_{22}q_1'q_2'\}&\\ =2u[(vp_2'-up_1')(\Gamma_{11}p_1'+\Gamma_{12}q_1')+(vq_2'-uq_1')(\Gamma_{12}p_1'+\Gamma_{22}q_1')]&\\ =2ut(\alpha_1p_0'+\beta_1q_0'),& \end{aligned}$$

on using the symbols (§ 98)

$$\alpha_l=\Gamma_{11}p_l'+\Gamma_{12}q_l', \quad \beta_l=\Gamma_{12}p_l'+\Gamma_{22}q_l'.$$

Similarly, with the like symbols (*l.c.*)

$$\epsilon_l = \Delta_{11} p_l' + \Delta_{12} q_l', \quad \eta_l = \Delta_{12} p_l' + \Delta_{22} q_l',$$

we find, also up to the second order of small quantities,

$$z^2 q_5'' - v^2 q_2'' + u^2 q_1'' = 2ut(\epsilon_1 p_0' + \eta_1 q_0').$$

Also we have

$$z p_5' = t p_0' + Z p_0' + t P,$$

the term ZP being negligible at this stage. Hence the condition, arising from the p-parameter, is

$$Z p_0' + t\{P + u(\alpha_1 p_0' + \beta_1 q_0')\} = 0;$$

and similarly the condition, arising from the q-parameter, is

$$Z q_0' + t\{Q + u(\epsilon_1 p_0' + \eta_1 q_0')\} = 0.$$

But there is also, at the place R, the permanent arc-relation for all directions, so that the relation

$$A^{(R)} p_5'^2 + 2H^{(R)} p_5' q_5' + B^{(R)} q_5'^2 = 1,$$

where $A^{(R)}$, $H^{(R)}$, $B^{(R)}$, denote the values of A, H, B, at R, must be satisfied. Now at R, which is at an arc-distance u from O along the geodesic OR,

$$A^{(R)} = A + u\frac{dA}{ds_1},$$

up to the first order of small quantities inclusive, and similarly for $H^{(R)}$ and $B^{(R)}$: that is,

$$\begin{aligned} A^{(R)} &= A + 2u(A\alpha_1 + H\epsilon_1), \\ H^{(R)} &= H + u(A\beta_1 + H\eta_1 + H\alpha_1 + B\epsilon_1), \\ B^{(R)} &= B + 2u(H\beta_1 + B\eta_1). \end{aligned}$$

Also substituting $p_0' + P$, $q_0' + Q$, for p_5', q_5', respectively: noting that the finite terms disappear because of the relation $A p_0'^2 + 2H p_0' q_0' + B q_0'^2 = 1$: and keeping only terms apparently up to the first order, we obtain the arc-relation at R in the form

$$(Ap_0' + Hq_0')\{P + u(\alpha_1 p_0' + \beta_1 q_0')\} \\ + (Hp_0' + Bq_0')\{Q + u(\epsilon_1 p_0' + \eta_1 q_0')\} = 0.$$

Now multiply the deduced forms of the two parameter equations by $Ap_0' + Hq_0'$, $Hp_0' + Bq_0'$, respectively and add: then

$$Z = 0,$$

that is, up to the second order of small quantities inclusive. Accordingly, Z is of the third order of small quantities at least. With this inference as to the order of magnitude of Z, the two parametric conditions become

$$P + u(\alpha_1 p_0' + \beta_1 q_0') = 0, \quad Q + u(\epsilon_1 p_0' + \eta_1 q_0') = 0,$$

up to the first order of small quantities inclusive * ; and therefore we can take, more generally,

$$\left.\begin{aligned} p_5' &= p_0' + P = p_0' - u(\alpha_1 p_0' + \beta_1 q_0') + \mathbf{P}_2 \\ q_5' &= q_0' + Q = q_0' - u(\epsilon_1 p_0' + \eta_1 q_0') + \mathbf{Q}_2 \end{aligned}\right\},$$

where $\mathbf{P}_2$, $\mathbf{Q}_2$, denote magnitudes of order higher than the first in small quantities; as we require only their contributions of the second order for the present purpose, $\mathbf{P}_2$ and $\mathbf{Q}_2$ will be described as of the second order.

Moreover, in the expression

$$z = t + Z,$$

the quantity Z is of order higher than the second in small quantities; as we require only its contribution of the third order for the present purpose, Z will be described as of the third order.

110. We now proceed to the next approximation, so that relevant magnitudes of the third order inclusive must be retained.

First of all, the parametric equations must be taken accurately up to this order. The method of construction is the same as before, the sole difference being the detail that terms of the third order must be retained explicitly. The two parametric equations now become

$$\left.\begin{aligned} zp_5' + \tfrac{1}{2}z^2 p_5'' + \tfrac{1}{6}z^3 p_5''' &= vp_2' - up_1' + \tfrac{1}{2}(v^2 p_2'' - u^2 p_1'') + \tfrac{1}{6}(v^3 p_2''' - u^3 p_1''') \\ zq_5' + \tfrac{1}{2}z^2 q_5'' + \tfrac{1}{6}z^3 q_5''' &= vq_2' - uq_1' + \tfrac{1}{2}(v^2 q_2'' - u^2 q_1'') + \tfrac{1}{6}(v^3 q_2''' - u^3 q_1''') \end{aligned}\right\};$$

and, for this further approximation, we have to determine the values of $\mathbf{P}_2$ and $\mathbf{Q}_2$, which are of the second order, and the value of Z which is of the third order.

Now, up to the third order, we have $z^2 = t^2$, because tZ is of the fourth order and Z^2 is of higher order still; so we can take

$$z^2 p_5'' = t^2 p_5'',$$

where it is necessary to retain, in the expression of p_5'', small quantities of the first order. The quantity p_5'' is to be taken at R; and therefore

$$\begin{aligned} p_5'' &= -\Gamma_{11}^{(R)} p_5'^2 - 2\Gamma_{12}^{(R)} p_5' q_5' - \Gamma_{22}^{(R)} q_5'^2 \\ &= -\left(\Gamma_{11} + u\frac{d\Gamma_{11}}{ds_1}\right) p_5'^2 - 2\left(\Gamma_{12} + u\frac{d\Gamma_{12}}{ds_1}\right) p_5' q_5' - \left(\Gamma_{22} + u\frac{d\Gamma_{22}}{ds_1}\right) q_5'^2 \\ &= -(\Gamma_{11} p_0'^2 + 2\Gamma_{12} p_0' q_0' + \Gamma_{22} q_0'^2) \\ &\qquad - u\left(p_0'^2 \frac{d\Gamma_{11}}{ds_1} + 2p_0' q_0' \frac{d\Gamma_{12}}{ds_1} + q_0'^2 \frac{d\Gamma_{22}}{ds_1}\right) \\ &\qquad + 2u(\Gamma_{11} p_0' + \Gamma_{12} q_0')(\alpha_1 p_0' + \beta_1 q_0') + 2u(\Gamma_{12} p_0' + \Gamma_{22} q_0')(\epsilon_1 p_0' + \eta_1 q_0'). \end{aligned}$$

* These two results shew that the surface-direction p_5', q_5', at R is geodesically parallel to the surface-direction p_0', q_0', at O; see § 119, *post*.

In the last line of this expression, we use the relations

$$\alpha_1 p_0' + \beta_1 q_0' = \alpha_0 p_1' + \beta_0 q_1', \quad \epsilon_1 p_0' + \eta_1 q_0' = \epsilon_0 p_1' + \eta_0 q_1',$$

so as to make that line

$$= 2u\{(\alpha_0^2 + \beta_0\epsilon_0) p_1' + (\alpha_0\beta_0 + \beta_0\eta_0) q_1'\}.$$

Further, by the first result in § 98, when we take $i=j=0$, $k=1$, we have

$$p_0'^2 \frac{d\Gamma_{11}}{ds_1} + 2p_0'q_0' \frac{d\Gamma_{12}}{ds_1} + q_0'^2 \frac{d\Gamma_{22}}{ds_1}$$
$$= -\tfrac{2}{3}K(p_0'q_1' - q_0'p_1')(Hp_0' + Bq_0') + (\Gamma_{111} \between p_1', q_1' \between p_0', q_0')^2$$
$$+ p_1'(2\alpha_0^2 + 2\beta_0\epsilon_0) + q_1'(2\alpha_0\beta_0 + 2\beta_0\eta_0).$$

Hence, writing

$$S = (\Gamma_{111} \between p_1', q_1' \between p_0', q_0')^2 - \tfrac{2}{3}(Hp_0' + Bq_0')(p_0'q_1' - q_0'p_1')K,$$

we have

$$z^2 p_5'' = t^2(p_0'' - uS).$$

Further, accurately,

$$t^2 p_0'' = -t^2(\Gamma_{11} p_0'^2 + 2\Gamma_{12} p_0'q_0' + \Gamma_{22} q_0'^2)$$
$$= v^2 p_2'' + 2uv\gamma_{12} + u^2 p_1'',$$

using the symbol γ_{12} to denote $\Gamma_{11}p_1'p_2' + \Gamma_{12}(p_1'q_2' + q_1'p_2') + \Gamma_{22}q_1'q_2'$; so that

$$t^2 p_0'' = v^2 p_2'' - u^2 p_1'' + 2ut(\alpha_1 p_0' + \beta_1 q_0').$$

Consequently, up to the necessary third order, inclusive,

$$\tfrac{1}{2}(z^2 p_5'' - v^2 p_2'' + u^2 p_1'') = ut(\alpha_1 p_0' + \beta_1 q_0') - \tfrac{1}{2}ut^2 S.$$

Again, as $z^3 = t^3$ up to the third order, inclusive, we have

$$z^3 p_5''' = t^3 p_5''',$$

where now on the right-hand side only the finite terms (that is, no terms involving small quantities) need be retained. Thus

$$p_5''' = p_0'''$$
$$= -(\Gamma_{111} \between p_0', q_0')^3,$$

and therefore

$$z^3 p_5''' = -t^3(\Gamma_{111} \between p_0', q_0')^3$$
$$= -(\Gamma_{111} \between vp_2' - up_1', vq_2' - uq_1')^3$$
$$= v^3 p_2''' - u^3 p_1''' + 3uvt(\Gamma_{111} \between p_1', q_1' \between p_2', q_2' \between p_0', q_0').$$

Consequently,

$$\tfrac{1}{6}(z^3 p_5''' - v^3 p_2''' + u^3 p_1''') = \tfrac{1}{2}uvt(\Gamma_{111} \between p_1', q_1' \between p_2', q_2' \between p_0', q_0').$$

Finally, among the contributory terms in the p-parametric equation, we have, up to the third order inclusive,

$$zp_5' = tp_0' - ut(\alpha_1 p_0' + \beta_1 q_0') + Zp_0' + t\mathbf{P}_2,$$

so that

$$zp_5' - vp_2' + up_1' = -ut(\alpha_1 p_0' + \beta_1 q_0') + Zp_0' + t\mathbf{P}_2.$$

When these various aggregates are substituted in the p-parametric equation, the terms of the first order and also the terms of the second order cancel, as is to be expected after the earlier use of those terms; and, as a surviving condition which involves magnitudes of the third order, we find

$$Zp_0' + t\mathbf{P}_2 = \tfrac{1}{2}ut^2 S - \tfrac{1}{2}uvt(\Gamma_{111} \between p_1', q_1' \between p_2', q_2' \between p_0', q_0').$$

But

$$tS - v(\Gamma_{111} \between p_1', q_1' \between p_2', q_2' \between p_0', q_0')$$
$$= -u(\Gamma_{111} \between p_0', q_0' \between p_1', q_1')^2 - \tfrac{2}{3}t(Hp_0' + Bq_0')(p_0'q_1' - q_0'p_1')K;$$

and so the surviving p-condition becomes

$$Zp_0' + t\mathbf{P}_2 = -\tfrac{1}{2}u^2 t(\Gamma_{111} \between p_0', q_0' \between p_1', q_1')^2 - \tfrac{1}{3}ut^2(Hp_0' + Bq_0')(p_0'q_1' - q_0'p_1')K.$$

Proceeding similarly from the q-parametric equation, when terms up to the third order of small quantities are retained, we find the surviving q-condition to be

$$Zq_0' + t\mathbf{Q}_2 = -\tfrac{1}{2}u^2 t(\Delta_{111} \between p_0', q_0' \between p_1', q_1')^2 + \tfrac{1}{3}ut^2(Ap_0' + Hq_0')(p_0'q_1' - q_0'p_1')K.$$

As a partial verification, we note that interchange of p and q (with the associated interchange of relevant magnitudes, such as Γ_{111} and Δ_{222} into one another), changes the p-condition and the q-condition into one another.

The permanent arc-relation

$$A^{(R)}p_5'^2 + 2H^{(R)}p_5'q_5' + B^{(R)}q_5'^2 = 1$$

at R has to be taken to one degree of approximation further than before (p. 293): we therefore must retain small quantities up to the second order inclusive. We have

$$p_5' = p_0' - u(\alpha_1 p_0' + \beta_1 q_0') + \mathbf{P}_2,$$
$$q_5' = q_0' - u(\epsilon_1 p_0' + \eta_1 q_0') + \mathbf{Q}_2,$$
$$A^{(R)} = A + u\frac{dA}{ds_1} + \tfrac{1}{2}u^2\frac{d^2A}{ds_1^2},$$
$$H^{(R)} = H + u\frac{dH}{ds_1} + \tfrac{1}{2}u^2\frac{d^2H}{ds_1^2},$$
$$B^{(R)} = B + u\frac{dB}{ds_1} + \tfrac{1}{2}u^2\frac{d^2B}{ds_1^2},$$

the values of the derivatives of A, H, B, being already obtained in § 98.

The terms on the left-hand side, free from small quantities, being

$$Ap_0'^2 + 2Hp_0'q_0' + Bq_0'^2,$$

balance the right-hand side.

The aggregate of terms on that left-hand side, which are of the first order of small quantities, are equal to uW, where

$$W = p_0'^2 \frac{dA}{ds_1} + 2p_0'q_0' \frac{dH}{ds_1} + q_0'^2 \frac{dB}{ds_1}$$
$$- 2(Ap_0' + Hq_0')(\alpha_1 p_0' + \beta_1 q_0') - 2(Hp_0' + Bq_0')(\epsilon_1 p_0' + \eta_1 q_0').$$

On the substitution of the values of these first derivatives of A, H, B, it is found that W vanishes identically. Hence there are no surviving terms of the first order, to constitute a residual condition—a result to be expected, owing to the earlier approximation.

The aggregate of terms of the second order must vanish; and they provide a residual condition, initially in the form

$$0 = 2(Ap_0' + Hq_0')\mathbf{P}_2 + 2(Hp_0' + Bq_0')\mathbf{Q}_2$$
$$+ u^2\{A(\alpha_1 p_0' + \beta_1 q_0')^2 + 2H(\alpha_1 p_0' + \beta_1 q_0')(\epsilon_1 p_0' + \eta_1 q_0') + B(\epsilon_1 p_0' + \eta_1 q_0')^2\}$$
$$+ \tfrac{1}{2}u^2 \left(p_0'^2 \frac{d^2A}{ds_1^2} + 2\beta_0' q_0' \frac{d^2H}{ds_1^2} + q_0'^2 \frac{d^2B}{ds_1^2}\right)$$
$$- 2u^2 p_0'(\alpha_1 p_0' + \beta_1 q_0') \frac{dA}{ds_1}$$
$$- 2u^2\{p_0'(\epsilon_1 p_0' + \eta_1 q_0') + q_0'(\alpha_1 p_0' + \beta_1 q_0')\} \frac{dH}{ds_1}$$
$$- 2u^2 q_0'(\epsilon_1 p_0' + \eta_1 q_0') \frac{dB}{ds_1}.$$

On the right-hand side, we take the aggregate of terms having $\frac{1}{2}u^2$ as their coefficient.

The part of this coefficient, provided by the third line, has its value given by the expression on p. 262 when we take $j = k = 0$, $i = 1$. One portion of this value, involving K,

$$= -\tfrac{2}{3}V^2 K(p_0'q_1' - q_0'p_1')^2 = -\tfrac{2}{3}\frac{v^2}{t^2} K \sin^2\epsilon;$$

and another portion, involving the magnitudes Γ_{ijk} and Δ_{ijk},

$$= 2(Ap_0' + Hq_0')(\Gamma_{111} \between p_0', q_0' \between p_1', q_1')^2 + 2(Hp_0' + Bq_0')(\Delta_{111} \between p_0', q_0' \between p_1', q_1')^2.$$

The remainder of this value consists of an aggregate of terms, similar to those contained in the last three lines on the right-hand side.

When this remainder aggregate is combined with the aggregate of the second line and the last three lines, after substitution is made for the first derivatives of A, H, B, the corporate aggregate is found to be equal to zero.

Hence the residual condition, arising from the second-order terms in the permanent arc-relation at R, becomes

$$(Ap_0' + Hq_0')\{2\mathbf{P}_2 + u^2(\Gamma_{111} \between p_0', q_0' \between p_1', q_1')^2\}$$
$$+ (Hp_0' + Bq_0')\{2\mathbf{Q}_2 + u^2(\Delta_{111} \between p_0', q_0' \between p_1', q_1')^2\} = \tfrac{1}{3}\frac{v^2}{t^2} K \sin^2\epsilon.$$

We thus have three equations involving the magnitudes $\mathbf{P}_2$, $\mathbf{Q}_2$, Z, viz. this residual condition, and the two conditions, emerging from the p-parametric equation and the q-parametric equation when the corresponding approximation is taken (p. 296).

When the two latter conditions are multiplied by $Ap_0' + Hq_0'$ and $Hp_0' + Bq_0'$ respectively, and the results are substituted in the residual condition just obtained, we find, after slight reduction,

$$Z = -\tfrac{1}{6}K\frac{u^2v^2}{t}\sin^2\epsilon.$$

The right-hand side is, of course, not the complete value of Z; it provides the most important term in the approximation and shews that Z can definitely be regarded as of the third order in the small quantities u, v, t.

Let this value of Z be substituted in the residual third-order conditions surviving from the p-parametric equation and the q-parametric equation. As (p. 292)

$$p_0'q_1' - q_0'p_1' = -\frac{1}{V}\sin R_0 = -\frac{v}{tV}\sin\epsilon,$$

we find

$$\left.\begin{aligned}\mathbf{P}_2 &= -\tfrac{1}{2}u^2(\Gamma_{111}\between p_0', q_0'\between p_1', q_1')^2\\ &\qquad + \frac{1}{3V}(Hp_0' + Bq_0')Kuv\sin\epsilon + \tfrac{1}{6}p_0'K\frac{u^2v^2}{t^2}\sin^2\epsilon\\ \mathbf{Q}_2 &= -\tfrac{1}{2}u^2(\Delta_{111}\between p_0', q_0'\between p_1', q_1')^2\\ &\qquad - \frac{1}{3V}(Ap_0' + Hq_0')Kuv\sin\epsilon + \tfrac{1}{6}q_0'K\frac{u^2v^2}{t^2}\sin^2\epsilon\end{aligned}\right\},$$

where

$$tp_0' = vp_2' - up_1',\quad tq_0' = vq_2' - uq_1',\quad t^2 = u^2 + v^2 - 2uv\cos\epsilon,$$
$$\cos\epsilon = Ap_1'p_2' + H(p_1'q_2' + q_1'p_2') + Bq_1'q_2'.$$

We thus have the approximate length of the geodesic arc RT, and approximate values of the direction-variables of that geodesic at R in the direction RT. The direction-variables of that geodesic at T in the direction TR can be deduced by relevant interchange of magnitudes.

Angles of the geodesic triangle.

111. Next, we require a value for the angle ORT denoted by R, for which R_0 can be regarded as a first approximation. The direction-variables of the geodesic OR at R in the direction RV, that continues through R onwards from O, are

$$p_1' + up_1'' + \tfrac{1}{2}u^2p_1''',\quad q_1' + uq_1'' + \tfrac{1}{2}u^2q_1''',$$

up to the second order of small quantities inclusive. The direction-variables of

the geodesic RT at R in the direction RT are p_5', q_5'; and their values have been obtained also up to the second order of small quantities inclusive. Hence

$$\begin{aligned} -\cos R = \cos TRU \\ &= \{A^{(R)}p_5' + H^{(R)}q_5'\}(p_1' + up_1'' + \tfrac{1}{2}u^2p_1''') \\ &\quad + \{H^{(R)}p_5' + B^{(R)}q_5'\}(q_1' + uq_1'' + \tfrac{1}{2}u^2q_1''').\end{aligned}$$

We shall require terms up to the second order inclusive; the necessary values of $A^{(R)}$, $H^{(R)}$, $B^{(R)}$, the values of A, H, B, at R, have already been used on p. 293; and

$$p_5' = p_0' - u(\alpha_1 p_0' + \beta_1 q_0') + \mathbf{P}_2, \quad q_5' = q_0' - u(\epsilon_1 p_0' + \eta_1 q_0') + \mathbf{Q}_2,$$

the values of $\mathbf{P}_2$ and $\mathbf{Q}_2$ being known.

On the right-hand side, the aggregate of finite terms (that is, of terms free from small quantities)

$$\begin{aligned} &= (Ap_0' + Hq_0')p_1' + (Hp_0' + Bq_0')q_1' \\ &= \frac{1}{t}[(Ap_1' + Hq_1')(vp_2' - up_1') + (Hp_1' + Bq_1')(vq_2' - uq_1')] \\ &= \frac{1}{t}(v\cos\epsilon - u) = -\cos R_0,\end{aligned}$$

thus justifying the declaration that R_0 is a first approximation to R, vanishing when all the small quantities vanish.

The aggregate of terms in the expression for $-\cos R$, which involve the first order of small quantities, is uW_1, where

$$\begin{aligned} W_1 = (Ap_0' + Hq_0')p_1'' + (Hp_0' + Bq_0')q_1'' \\ - (\alpha_1 p_0' + \beta_1 q_0')(Ap_1' + Hq_1') - (\epsilon_1 p_0' + \eta_1 q_0')(Hp_1' + Bq_1') \\ + p_0'p_1'\frac{dA}{ds_1} + (p_0'q_1' + q_0'p_1')\frac{dH}{ds_1} + q_0'q_1'\frac{dB}{ds_1}.\end{aligned}$$

Now

$$p_1'' = -(\alpha_1 p_1' + \beta_1 q_1'), \quad q_1'' = -(\epsilon_1 p_1' + \eta_1 q_1'),$$

and the values of the arc-derivatives of A, H, B, are as given in § 98; when all these values are substituted, and reduction is effected, we find

$$W_1 = 0,$$

or the aggregate of first-order terms in the approximation to $-\cos R$ is zero. Accordingly, if we take

$$R = R_0 + \bar{w},$$

the quantity $\bar{w}$ is of the second order (including terms of higher orders) of small quantities.

The aggregate of terms in the expression for $-\cos R$, which involve small quantities of the second order, is

$$= (Ap_1' + Hq_1')\mathbf{P}_2 + (Hp_1' + Bq_1')\mathbf{Q}_2 + u^2Z,$$

where Z temporarily is given by the expression

$$
\begin{aligned}
Z=\tfrac{1}{2}(Ap_0'+Hq_0')p_1'''&+\tfrac{1}{2}(Hp_0'+Bq_0')q_1''' \\
&+\tfrac{1}{2}\left\{p_0'p_1'\frac{d^2A}{ds_1^2}+(p_0'q_1'+q_0'p_1')\frac{d^2H}{ds_1^2}+q_0'q_1'\frac{d^2B}{ds_1^2}\right\} \\
&-2(A\alpha_1+H\epsilon_1)p_1'(\alpha_1p_0'+\beta_1q_0') \\
&-(A\beta_1+H\eta_1+H\alpha_1+B\epsilon_1)\{q_1'(\alpha_1p_0'+\beta_1q_0')+p_1'(\epsilon_1p_0'+\eta_1q_0')\} \\
&-2(H\beta_1+B\eta_1)q_1'(\epsilon_1p_0'+\eta_1q_0') \\
&+p_1''\{2(A\alpha_1+H\epsilon_1)p_1'+(A\beta_1+H\eta_1+H\alpha_1+B\epsilon_1)q_1' \\
&\qquad\qquad -A(\alpha_1p_0'+\beta_1q_0')-H(\epsilon_1p_0'+\eta_1q_0')\} \\
&+q_1''\{(A\beta_1+H\eta_1+H\alpha_1+B\epsilon_1)p_1'+2(H\beta_1+B\eta_1)q_1' \\
&\qquad\qquad -H(\alpha_1p_0'+\beta_1q_0')-B(\epsilon_1p_0'+\eta_1q_0')\}.
\end{aligned}
$$

Now

$$p_1'''=-(\Gamma_{111}\between p_1',q_1')^3,\quad q_1'''=-(\Delta_{111}\between p_1',q_1')^3;$$

and an expression for the second line in Z has been obtained in § 98 by taking $i=1$, $j=0$, $k=1$. When all the values are substituted and reduction is effected, we find

$$
\begin{aligned}
Z=&-\tfrac{1}{2}(Ap_0'+Hq_0')\{(\Gamma_{111}\between p_1',q_1')^3\}-\tfrac{1}{2}(Hp_0'+Bq_0')\{(\Delta_{111}\between p_1',q_1')^3\} \\
&+\tfrac{1}{2}[(Ap_0'+Hq_0')\{(\Gamma_{111}\between p_1',q_1')^3\}+(Ap_1'+Hq_1')\{(\Gamma_{111}\between p_0',q_0'\between p_1',q_1')^2\} \\
&\qquad+(Hp_0'+Bq_0')\{(\Delta_{111}\between p_1',q_1')^3\}+(Hp_1'+Bq_1')\{(\Delta_{111}\between p_0',q_0'\between p_1',q_1')^2\}] \\
=&\tfrac{1}{2}(Ap_1'+Hq_1')\{(\Gamma_{111}\between p_0',q_0'\between p_1',q_1')^2\}+\tfrac{1}{2}(Hp_1'+Bq_1')\{(\Delta_{111}\between p_0',q_0'\between p_1',q_1')^2\}.
\end{aligned}
$$

Consequently, the aggregate of terms of the second order in the expression for $-\cos R$

$$
\begin{aligned}
&=(Ap_1'+Hq_1')\{\mathbf{P}_2+\tfrac{1}{2}u^2(\Gamma_{111}\between p_0',q_0'\between p_1',q_1')^2\} \\
&\qquad+(Hp_1'+Bq_1')\{\mathbf{Q}_2+\tfrac{1}{2}u^2(\Delta_{111}\between p_0',q_0'\between p_1',q_1')^2\}.
\end{aligned}
$$

In this expression, we substitute the values of $\mathbf{P}_2$ and $\mathbf{Q}_2$, as obtained in § 110, with the result that there are two sets of terms involving K. One of these sets

$$
\begin{aligned}
&=\frac{1}{3V}\{(Ap_1'+Hq_1')(Hp_0'+Bq_0')-(Hp_1'+Bq_1')(Ap_0'+Hq_0')\}Kuv\sin\epsilon \\
&=\frac{1}{3V}V^2(p_1'q_0'-q_1'p_0')Kuv\sin\epsilon=\tfrac{1}{3}Kuv\sin\epsilon\sin R_0\,;
\end{aligned}
$$

and the other of the sets

$$
\begin{aligned}
&=\tfrac{1}{6}\{(Ap_1'+Hq_1')p_0'+(Hp_1'+Bq_1')q_0'\}K\frac{u^2v^2}{t^2}\sin^2\epsilon \\
&=-\tfrac{1}{6}K\frac{u^2v^2}{t^2}\sin^2\epsilon\cos R_0=-\tfrac{1}{6}Kuv\sin T_0\cos R_0\sin R_0.
\end{aligned}
$$

Accordingly, the equation for $-\cos R$ now gives

$$-\cos R = -\cos R_0 + \tfrac{1}{3}Kuv\sin\epsilon\sin R_0 - \tfrac{1}{6}Kuv\sin T_0\cos R_0\sin R_0,$$

accurately up to the second order of small quantities inclusive; and therefore, also accurately to the same degree,

$$R - R_0 = \tfrac{1}{3}Kuv\sin\epsilon - \tfrac{1}{6}Kuv\sin T_0\cos R_0.$$

Similarly, for the angle OTR denoted by T, we find

$$T - T_0 = \tfrac{1}{3}Kuv\sin\epsilon - \tfrac{1}{6}Kuv\sin R_0\cos T_0.$$

The length of the side RT, and values of the angles at R and T, of the geodesic triangle ORT have thus been obtained, as well as the direction-variables at R of the geodesic side RT in the direction RT, the latter being

$$p_5' = p_0' - u(\alpha_1 p_0' + \beta_1 q_0') + \mathbf{P}_2, \quad q_5' = q_0' - u(\epsilon_1 p_0' + \eta_1 q_0') + \mathbf{Q}_2.$$

Either by similar calculations, or by relevant transpositions of variables and associated magnitudes, we can obtain the direction-variables at T of the same geodesic side TR in the direction TR, in the form

$$p_6' = p_3' - v(\alpha_2 p_3' + \beta_2 q_3') + \mathbf{S}_2, \quad q_6' = q_3' - v(\epsilon_2 p_3' + \eta_2 q_3') + \mathbf{T}_2,$$

where

$$tp_3' = up_1' - vp_2', \quad tq_3' = uq_1' - vq_2',$$

(that is, $p_3' = -p_0'$, $q_3' = -q_0'$), while

$$\left.\begin{aligned} \mathbf{S}_2 = &-\tfrac{1}{2}v^2(\Gamma_{111} \between p_3', q_3' \between p_2', q_2')^2 \\ &\qquad - \frac{1}{3V}(Hp_3' + Bq_3')Kuv\sin\epsilon + \tfrac{1}{6}p_3'K\frac{u^2v^2}{t^2}\sin^2\epsilon \\ \mathbf{T}_2 = &-\tfrac{1}{2}v^2(\Delta_{111} \between p_3', q_3' \between p_2', q_2')^2 \\ &\qquad + \frac{1}{3V}(Ap_3' + Hq_3')Kuv\sin\epsilon + \tfrac{1}{6}q_3'K\frac{u^2v^2}{t^2}\sin^2\epsilon \end{aligned}\right\},$$

the length of the arc RT being

$$z = t - \tfrac{1}{6}\frac{u^2v^2}{t}K\sin^2\epsilon.$$

As the place T is on the geodesic RT passing through R, we have the relations

$$\left.\begin{aligned} -p_6' &= p_5' + zp_5'' + \tfrac{1}{2}z^2p_5''' \\ -q_6' &= q_5' + zq_5'' + \tfrac{1}{2}z^2q_5''' \end{aligned}\right\},$$

because of the two modes of expressing the direction-variables at T: the verification of these relations is left as an exercise.

Ex. It is not without interest, at this stage, to compare the results with the corresponding results of ordinary spherical trigonometry. For the purpose, we take the radius of the sphere to be unity, so that the arcs OR, OT, are u, v, respectively; and we have

$$\cos z = \cos u\cos v + \sin u\sin v\cos\epsilon.$$

In this equation u, v, ϵ, are exact magnitudes, while

$$z = t + Z$$

approximately, the approximation for z being accurate up to the third power inclusive, so that

$$z^2 = t^2 + 2tZ, \quad z^4 = t^4,$$

each accurately up to the fourth power inclusive of the small magnitudes u and v. Hence expanding the quantities $\cos z$, $\cos u$, $\cos v$, $\sin u$, $\sin v$, so as to retain fourth powers of small quantities throughout, we have

$$1 - \tfrac{1}{2}(t^2 + 2tZ) + \tfrac{1}{24}t^4 = 1 - \tfrac{1}{2}(u^2 + v^2) + \tfrac{1}{24}(u^4 + v^4 + 6u^2v^2) + uv\{1 - \tfrac{1}{6}(u^2 + v^2)\}\cos\epsilon.$$

Accordingly

$$\begin{aligned} t^2 &= u^2 + v^2 - 2uv\cos\epsilon, \\ tZ &= \tfrac{1}{24}(-t^4 - u^4 - v^4 + 2t^2u^2 + 2t^2v^2 + 2u^2v^2) \\ &= -\tfrac{1}{6}u^2v^2\sin^2\epsilon, \end{aligned}$$

thus verifying the results for t and Z, on the assumption of unit radius for the sphere.

Again, in the same spherical triangle, we have

$$\sin u \sin z \cos R = \cos v - \cos u \sin z.$$

When the right-hand side is expanded, so as to retain the permissible fourth powers connected with the expansion of $\cos z$, it becomes

$$\tfrac{1}{2}(t^2 + u^2 - v^2) - \tfrac{1}{24}(t^4 + u^4 + 6t^2u^2 - v^4) - \tfrac{1}{6}u^2v^2\sin^2\epsilon,$$

on substituting the value of tZ which has just been verified. But

$$\begin{aligned} t^2 + u^2 - v^2 &= 2tu\cos R_0, \\ t^4 + 6t^2u^2 + u^4 - v^4 &= 4t^2u^2\sin^2 R_0 + 4tu(t^2 + u^2)\cos R_0 \\ &= 4u^2v^2\sin^2\epsilon + 4tu(t^2 + u^2)\cos R_0\,; \end{aligned}$$

and so the right-hand side becomes

$$tu\{1 - \tfrac{1}{6}(t^2 + u^2)\}\cos R_0 - \tfrac{1}{3}u^2v^2\sin^2\epsilon.$$

The left-hand side, to the same order of small quantities

$$\begin{aligned} &= uz\{1 - \tfrac{1}{6}(u^2 + z^2)\}\cos R \\ &= ut\{1 - \tfrac{1}{6}(u^2 + t^2)\}\cos R + uZ\cos R. \end{aligned}$$

If then we take $R - R_0 = \bar{w}$, so that

$$\cos R = \cos R_0 - \bar{w}\sin R_0,$$

the left-hand side, up to the specified order

$$= ut\{1 - \tfrac{1}{6}(u^2 + t^2)\}\cos R_0 - \bar{w}ut\sin R_0 + uZ\cos R_0.$$

We thus have

$$\bar{w}ut\sin R_0 - uZ\cos R_0 = \tfrac{1}{3}u^2v^2\sin^2\epsilon,$$

which, on the substitution of Z and by means of the relations

$$\frac{\sin R_0}{v} = \frac{\sin T_0}{u} = \frac{\sin\epsilon}{t},$$

leads to the value

$$\bar{w} = \tfrac{1}{3}uv \sin \epsilon - \tfrac{1}{6}uv \sin T_0 \cos R_0,$$

thus verifying the result for the particular instance.

This property of the particular spherical triangle is distinct from the well-known property relating to the difference between the angles of a plane triangle with the sides u, v, t, and the angles of a spherical triangle with the sides u, v, t: each such difference is one-third of the spherical excess of the triangle. In the present instance, the third side of the spherical triangle is $t+Z$, not t; and the angle ϵ is kept the same for the spherical triangle and the plane triangle.

The Riemann measure: sphericity of the surface.

112. Now the angles of the geodesic triangle ORT are ϵ, R, T; and ϵ, R_0, T_0, are the angles of a plane rectilinear triangle with sides of lengths u, v, t, not u, v, z. Thus $\epsilon + R_0 + T_0 = \pi$. But, from the approximate values obtained for R and T, we have

$$\begin{aligned} R + T - R_0 - T_0 &= \tfrac{2}{3}Kuv \sin \epsilon - \tfrac{1}{6}Kuv \sin (R_0 + T_0) \\ &= \tfrac{1}{2}Kuv \sin \epsilon\,; \end{aligned}$$

and therefore

$$\epsilon + R + T - \pi = \tfrac{1}{2}Kuv \sin \epsilon.$$

The magnitude $\epsilon + R + T - \pi$, the excess of the sum of the three angles of the geodesic triangle over the sum of the angles of a plane rectilinear triangle, may be called the *angular excess* of the geodesic triangle. Also, $\tfrac{1}{2}uv \sin \epsilon$ is the area of the geodesic triangle when the adjacent sides u and v, containing the angle ϵ, are very small. Hence the area of a small triangle, bounded by geodesics of the surface, is equal to

$$\frac{1}{K} \,.\, \text{angular excess of the triangle},$$

where K denotes the Riemann measure of curvature of the surface. An inspection of the analytical expression of K shews that, in geometrical linear dimensions, K is of order minus two, being the reciprocal of the square of a line. The result is valid for all surfaces existing freely in homaloidal space, whatever be the number of dimensions in the plenary space.

Recalling the corresponding theorem for a spherical surface (being of constant curvature throughout its range) in triple homaloidal space, which declares the area of a spherical triangle bounded by arcs of great circles (that is, by geodesics on the sphere) to be the afore-defined angular excess multiplied by the square of the radius of the sphere, we describe the Riemann measure of curvature of the surface at any point as the *sphericity* of the surface at the point. Thus when a surface has a sphericity K at any point, the implication is that the area of any

small triangle at the point, bounded by three small geodesic arcs and having an angular excess E, is

$$\frac{E}{K}.$$

The sphericity at a point of a non-special surface is usually a variable magnitude, changing from point to point, so that the specified property is of the nature of a limiting result; if δS denote the area of the small geodesic triangle, and if $\delta\chi$ denote its small angular excess, we have

$$K = \text{limit of } \frac{\delta\chi}{\delta S},$$

in whatever way the area of the small triangle decreases towards zero. Should the sphericity of a surface be constant, the area of any geodesic triangle on the surface (that is, a triangle bounded by three geodesic arcs) is the quotient of the angular excess of the triangle by the sphericity of the surface *.

When the surface exists in homaloidal triple space, this sphericity K is the Gauss measure of curvature of the surface, being the product of its two principal circular curvatures; and these are defined as the circular curvatures of superficial geodesics touching the curves of circular curvature through the point. But, as already (§ 74) stated, such an interpretation is not possible when the plenary space is of more than three dimensions. In such a plenitude, a surface possesses four curves of circular curvature at a point; and even in a quadruple plenary space, there is no relation † between the Riemann measure and the measures of linear curvature on the surface; all of these are independent of one another when the plenary homaloidal space of the surface is of more than three dimensions.

It will be noted that the foregoing interpretation of the Riemann measure K for a free surface has been obtained solely from the immediate march of the geodesics on the surface. Apparently the earliest explicit statement of the result is due to Levi-Civita ‡, who derives it by considerations of a cyclic displacement (originally due to Pérès) round an infinitesimal concave circuit; and these **, in turn, are connected with his theory of parallel geodesics, soon to be considered.

* Interesting examples occur for surfaces, in triple homaloidal space and of constant sphericity, especially for such surfaces as are of revolution, whether spheres (when the measure is positive) or pseudo-spheres (when the measure is negative). Reference may be made, upon this topic, to my *Lectures on Differential Geometry*, §§ 154, 155, 211-214.

† The general discussion of this matter will come later (§ 137). For these features of surfaces in a quadruple plenary homaloidal space, reference may be made to my *G.F.D.*, vol. i, chap. xiii.

‡ In his treatise *Lezioni di calcolo differenziale assoluto* (1925); see p. 198 of the English translation, *The absolute differential calculus* (1928).

** It will suffice, here, to refer to the memoirs by Levi-Civita himself, by Severi, Schouten, Pérès, and Bompiani, which Levi-Civita cites, pp. 171, 172, 198, of the English translation quoted in the preceding note.

113. As an inference from the result in the text, we can deduce an earlier result, similar in form, due to Pérès.*

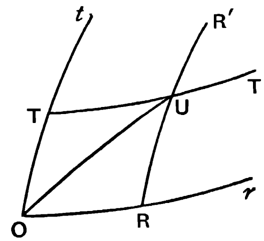

FIG. 6.

Let OR, OT be two geodesics on the surface, both OR and OT being small arcs; through T, let a unique geodesic TUT' be drawn making the angle tTU equal to TOR, denoted by ϵ; through R, let a unique geodesic RUR' be drawn making the angle rRU equal to TOR; and let U be their point of intersection on the surface. The angle $R'UT'$ will differ from ROT by a small magnitude: we represent it by $\epsilon + \delta\epsilon$.

Now taking K as the sphericity at O, we have

$$\begin{aligned} K \,.\, \text{area } ORU &= \text{angular excess of } ORU \\ &= ROU + OUR + ORU - \pi \\ &= ROU + OUR - ROT, \end{aligned}$$

and

$$\begin{aligned} K \,.\, \text{area } OTU &= \text{angular excess of } OTU \\ &= TOU + OUT - ROT\,; \end{aligned}$$

hence, adding,

$$K \,.\, \text{area } ORUT = ROT + TUR - 2ROT = \delta\epsilon.$$

The area $ORUT$ (later, styled a geodesic parallelogram), with the arcs u (denoting OR) and v (denoting OT), is equal to $uv \sin \epsilon$, to the second order of small quantities; and so we have

$$uv \sin \epsilon = \frac{1}{K}\, \delta\epsilon,$$

which is the result due to Pérès.

But for this variation $\delta\epsilon$ of the angle ROT in passing from O to U, the special geodesics RUR' and TUT' have to be drawn, through R and T respectively, making the angles rRU, tTU, each equal to ROT.

There is, however, an inference more general than the preceding Pérès result, which is cited here as being apparently the earliest instance of the relation between the sphericity of a surface and the angles of a figure bounded by small geodesic arcs.

Consider a figure, for simplicity taken to be convex, on the surface bounded by a number of small geodesic arcs; and let the angular excess of such a geodesically bounded figure be defined as the excess of the sum of all its internal angles over the sum of the internal angles of a plane convex polygon, which has as many sides as the geodesic polygon and each side of which is a line†. Let A denote the

* *Rendiconti dei Lincei*, ser. 5, t. xxviii(1), (1919), pp. 425-428.

† The actual sum for the plane polygon is $(n-2)\pi$, where n is the number of its sides; the form in the text propounds the angular excess as a relation between the superficial geodesic figure and a plane linear polygon with the same number of sides.

area of the geodesic figure, necessarily a small quantity ; let E denote the angular excess of the figure, and let K denote the measure of the sphericity of the surface at any point of the figure ; then, in the limit as the figure becomes very small,

$$KA = E.$$

The theorem is an immediate corollary from the proposition (§ 112) relating to a small geodesic triangle.

114. In this connection, it is proper to refer to a theorem due to Gauss * concerning the *total curvature* (as defined by him) of a geodesic triangle upon a surface. His investigations relate to surfaces in triple homaloidal space ; the proof, however, of the particular theorem is unaffected by the dimensionality of the plenary space.

Let the general free surface be referred to geodesic polar coordinates as in § 96, so that its arc-element is represented by the relation

$$ds^2 = dp^2 + T^2 dq^2.$$

The sphericity K is given (*l.c. Ex.* 1) by

$$K = -\frac{1}{T}\frac{\partial^2 T}{\partial p^2},$$

where p is a geodesic variable, which denotes the length of the arc p measured along a radial geodesic through O.

We take two consecutive points P and Q ; through P we draw the curve PL orthogonal to the geodesic OP, meeting the consecutive radial geodesic curve OQ in L ; from Q we draw a line QM parallel to LP, and a line LN parallel to PM meeting QM in N. Let ϕ denote the angle OPA ; then $\phi + d\phi$ denotes the angle LQP. We have

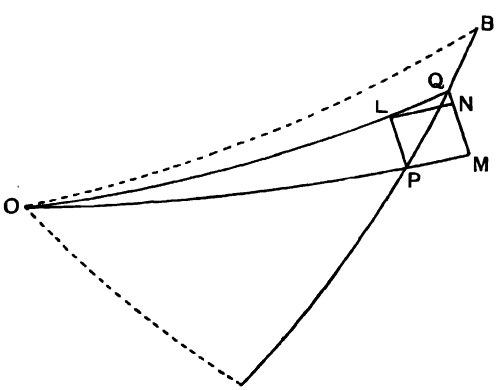

FIG. 7.

$$LPQ = \tfrac{1}{2}\pi - \phi, \quad LQP = \phi + d\phi,$$

and therefore

$$QLP = \tfrac{1}{2}\pi - d\phi,$$

the sum of the angles of the plane triangle PLQ being π ; hence, as LN is parallel to PM so that $NLP = \tfrac{1}{2}\pi$, it follows that

$$QLN = -d\phi.$$

Next, as PQ is taken to represent the arc ds, so that (after the explanations in § 96) we have $LQ = dp = LN$, to the first order of small quantities, we have

* In his classical memoir *Disquisitiones generales circa superficies curvas, Ges. Werke*, vol. iv, p. 245.

$LP = Tdq$. We denote the angle POQ by $d\theta$; and therefore $d\theta$ and dq vanish together, whatever be the value of p. Hence we may take $dq = \mu\, d\theta$; or, absorbing μ into T as from the beginning, we may take $dq = d\theta$ without loss of generality, and now we have

$$LP = T\, d\theta.$$

Consequently $QM = (T + dT)d\theta$, and therefore

$$QN = dT\, d\theta.$$

Now QLN, being a small angle (estimated in circular measure), we have

$$QLN = \frac{QN}{LN} = \frac{\partial T}{\partial p}\, d\theta :$$

and therefore

$$d\phi = -\frac{\partial T}{\partial p}\, d\theta.$$

The quantity, entitled by Gauss the total curvature of the geodesic triangle AOB, is taken to be the integral of the quantity

$$K(dp\,.\,T\,d\theta),$$

(where K denotes the measure of curvature at P), taken over the whole triangle AOB: that is, it is

$$\iint\left(-\frac{1}{T}\frac{\partial^2 T}{\partial p^2}\right) T\, dp\, d\theta$$

between proper limits. When integration with regard to p along OP from O to P is effected, we have

$$\int\left(a - \frac{\partial T}{\partial p}\right) d\theta,$$

where a is the value of $\dfrac{\partial T}{\partial p}$ at O. But when P is near O, the geodesic OP is a small rectilinear arc, and the most important term in $T = p$: that is, at O, we have $\dfrac{\partial T}{\partial p} = 1$, so that $a = 1$. Thus the total curvature

$$= \int\left(1 - \frac{\partial T}{\partial p}\right) d\theta$$

$$= \int (d\theta + d\phi)$$

$$\begin{aligned} &= BOA, \text{ for the integral of } d\theta, \\ &\qquad\qquad + \{OBA - (\pi - OAB)\}, \text{ for the integral of } \phi, \\ &= BOA + OBA + OAB - \pi \\ &= E, \end{aligned}$$

where E denotes the angular excess of the whole triangle.

This result is Gauss's theorem for the "total curvature" of the geodesic triangle, originally stated for a surface in a triple homaloidal space. In essence, it is quite distinct from the theorem of § 112, which interprets the Riemann measure K (the sphericity) as given by

$$\text{area of a small geodesic triangle} = KE,$$

where E is the angular excess of the small triangle.

115. As a measurement in a geodesic triangle on a surface which may be useful later, we draw any other geodesic OP through a vertex of the geodesic triangle ORT already (§§ 109-111) considered. We take the direction-variables of the geodesic OP at O to be p_3', q_3' accurately; and we shall require approximations* to the lengths of the geodesic arcs OP, RP.

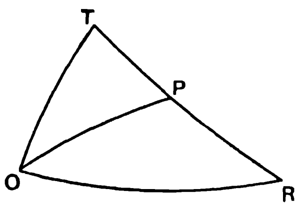

FIG. 8.

We take the direction-variables at R of the geodesic RP in the direction RP to be p_5', q_5', as before, so that

$$p_5' = p_0' - u(\alpha_1 p_0' + \beta_1 q_0') + \mathbf{P}_2, \quad q_5' = q_0' - u(\epsilon_1 p_0' + \eta_1 q_0') + \mathbf{Q}_2.$$

The angle ROP is denoted by θ; thus

$$\sin\theta = V(p_1' q_3' - q_1' p_3'), \quad \sin R_0 = V(p_1' q_0' - q_1' p_0'),$$

and we shall take

$$\sin P_0 = V(p_3' q_0' - q_0' p_3'),$$

where P_0 will be found to be a first approximation to the angle OPR. Hence

$$p_1' \sin P_0 + p_0' \sin\theta - p_3' \sin R_0 = 0, \quad q_1' \sin P_0 + q_0' \sin\theta - q_3' \sin R_0 = 0.$$

Also let the arc OP be denoted by λ, and the arc RP be denoted by μ.

At P, the p-parameter must have the same value, whether approached by the path ORP or by the path OP, and likewise for the q-parameter. The first of these requirements imposes the condition

$$p + up_1' + \tfrac{1}{2}u^2 p_1'' + \tfrac{1}{6}u^3 p_1''' + \mu p_5' + \tfrac{1}{2}\mu^2 p_5'' + \tfrac{1}{6}\mu^3 p_5'''$$
$$= p + \lambda p_3' + \tfrac{1}{2}\lambda^2 p_3'' + \tfrac{1}{6}\lambda^3 p_3''',$$

accurate up to the third order of small quantities inclusive. For approximation, let

$$\lambda = l + L, \quad \mu = m + M,$$

where l and m denote small quantities of the first order, chosen so that the p-condition and the similar q-condition shall be satisfied exactly in first-order terms. The p-condition yields the relation

$$up_1' + mp_0' = lp_3',$$

* The approximations can be deduced directly from preceding results; see a Note on p. 317, due to Prof. E. H. Neville.

while the relation

$$uq_1' + mq_0' = lq_3',$$

is similarly yielded by the q-condition. We have, at once,

$$\frac{l}{\sin R_0} = \frac{m}{\sin\theta} = \frac{u}{\sin P_0},$$

so that there are relations similar to those for a plane triangle with sides l, m, n, and angles R_0, θ, P_0.

For terms of the second order, we can take

$$\lambda^2 = l^2, \quad \mu^2 = m^2,$$

and

$$\mu^2 p_5'' = m^2 p_5'' = m^2 p_0'',$$

up to the second order inclusive. Hence as regards second-order terms, the p-condition yields the relation

$$Lp_3' + \tfrac{1}{2}l^2 p_3'' = \tfrac{1}{2}u^2 p_1'' + Mp_0' - mu(\alpha_1 p_0' + \beta_1 q_0') + \tfrac{1}{2}m^2 p_0''.$$

Now

$$\begin{aligned} u^2 p_1'' - l^2 p_3'' - 2mu(\alpha_1 p_0' + \beta_1 q_0') + m^2 p_0'' &\\ = -l^2 p_3'' - (\Gamma_{11}, \Gamma_{12}, \Gamma_{22}\between up_1' + mp_0', uq_1' + mq_0')^2 &\\ = -l^2 p_3'' - (\Gamma_{11}, \Gamma_{12}, \Gamma_{22}\between lp_3', lq_3')^2 = 0\,; & \end{aligned}$$

and therefore

$$Lp_3' = Mp_0'.$$

Similarly the second-order terms in the q-condition give

$$Lq_3' = Mq_0'.$$

Consequently, up to the second-order inclusive,

$$L = 0, \quad M = 0:$$

thus, the magnitudes L and M are of the third order of small quantities.

For terms of the third order, we have

$$\lambda^2 = l^2, \quad \lambda^3 = l^3, \quad \mu^2 = m^2, \quad \mu^3 = m^3.$$

Also we have had (§ 110)

$$p_5'' = p_0'' - uS,$$

where

$$S = (\Gamma_{111}\between p_1', q_1'\between p_0', q_0')^2 - \tfrac{2}{3}(Hp_0' + Bq_0')(p_0'q_1' - q_0'p_1')K\,;$$

and therefore the third-order terms in the p-relation yield the condition

$$Lp_3' + \tfrac{1}{6}l^3 p_3''' = \tfrac{1}{6}u^3 p_1''' + Mp_0' + m\mathbf{P}_2 - \tfrac{1}{2}um^2 S + \tfrac{1}{6}m^3 p_0'''.$$

The value of $\mathbf{P}_2$, by § 110, is given by

$$\mathbf{P}_2 = -\tfrac{1}{2}u^2(\Gamma_{111}\between p_0', q_0'\between p_1', q_1')^2 + \frac{1}{3V}(Hp_0' + Bq_0')Kuv\sin\epsilon + \tfrac{1}{6}p_0'K\frac{u^2v^2}{t^2}\sin^2\epsilon.$$

The aggregate of terms on the right-hand side of this condition, involving the magnitudes Γ_{ijk},

$$\begin{aligned}
&= \tfrac{1}{6}u^3 p_1''' - \tfrac{1}{2}u^2 m (\Gamma_{111} \between p_0', q_0' \between p_1', q_1')^2 - \tfrac{1}{2}um^2(\Gamma_{111} \between p_1', q_1' \between p_0', q_0')^2 + \tfrac{1}{6}m^3 p_0''' \\
&= -\tfrac{1}{6}(\Gamma_{111} \between up_1' + mp_0', uq_1' + mq_0')^3 \\
&= -\tfrac{1}{6}(\Gamma_{111} \between lp_3', lq_3')^3 = \tfrac{1}{6}l^3 p_3''',
\end{aligned}$$

thus cancelling the corresponding term on the left-hand side. Consequently, the condition becomes

$$\begin{aligned}
Lp_3' - Mp_0' &= \tfrac{1}{6}p_0' Km \frac{u^2v^2}{t^2}\sin^2\epsilon \\
&\qquad + \frac{1}{3V}(Hp_0' + Bq_0')K(muv\sin\epsilon - m^2u\sin R_0) \\
&= \tfrac{1}{6}p_0' Km \frac{u^2v^2}{t^2}\sin^2\epsilon + \frac{1}{3V}(Hp_0' + Bq_0')Kmu(t-m)\sin R_0.
\end{aligned}$$

Corresponding analysis leads, from the third-order terms in the q-parameter relation, to the condition

$$Lq_3' - Mq_0' = \tfrac{1}{6}q_0' Km \frac{u^2v^2}{t^2}\sin^2\epsilon - \frac{1}{3V}(Ap_0' + Hq_0')Kmu(t-m)\sin R_0.$$

Now

$$Ap_0'^2 + 2Hp_0'q_0' + Bq_0'^2 = 1, \quad Ap_0'p_3' + H(p_0'q_3' + q_0'p_3') + Bq_0'q_3' = \cos P_0,$$

where P_0 is the angle between OP produced and RP produced, that is, P_0 is the internal angle of the plane triangle with sides u, l, m. Hence

$$\begin{aligned}
L - M\cos P_0 &= \tfrac{1}{6}Km\frac{u^2v^2}{t^2}\sin^2\epsilon\cos P_0 + \tfrac{1}{3}Kmu(t-m)\sin R_0\sin P_0, \\
L\cos P_0 - M &= \tfrac{1}{6}Km\frac{u^2v^2}{t^2}\sin^2\epsilon;
\end{aligned}$$

and therefore, after a slight reduction,

$$L = \tfrac{1}{3}Klm(t-m), \quad M = \tfrac{1}{3}Klm(t-m)\cos P_0 - \tfrac{1}{6}K\frac{m}{t^2}u^2v^2\sin^2\epsilon.$$

Hence the length of the arc OP, up to the third order of small quantities inclusive, is

$$l\{1 + \tfrac{1}{3}Km(t-m)\};$$

the length of the arc RP, to the same order, is

$$m\left\{1 + \tfrac{1}{3}Kl(t-m)\cos P_0 - \tfrac{1}{6}K\frac{u^2v^2}{t^2}\sin^2\epsilon\right\};$$

and as (§ 111) the length of RT is $t - \frac{1}{6}K\frac{u^2v^2}{t}\sin^2\epsilon$ to the same order, it follows that the length of the arc TP to that same order is

$$(t-m)\left\{1 - \tfrac{1}{3}Klm\cos P_0 - \tfrac{1}{6}K\frac{u^2v^2}{t^2}\sin^2\epsilon\right\}.$$

The magnitude of the angle OPR can be at once deduced from the angular excess of the triangle OPR: we have

$$\begin{aligned}\tfrac{1}{2}Kum\sin R_0 &= \theta + P + R - \pi\\ &= P - P_0 + R - R_0,\end{aligned}$$

and the value of $R - R_0$ has been obtained in § 111.

Geodesic parallelograms on a surface.

116. When the theory of parallels on a surface comes to be discussed, there will arise the necessity of considering figures, bounded by four geodesic arcs and analogous to parallelograms in a plane: they will be called geodesic parallelograms. It will appear, however, that there are various types of such figures, each type associated with some specific property of a plane parallelogram; and though the plane properties always lead to the same unique plane parallelogram, it will be found that the corresponding geodesic properties do not lead to a geodesic parallelogram, which is the same for all.

Thus, in a plane, we can have a line $T'R$; two lines $T'R'$, RT, parallel to one another and equal in length; when the line $R'T$ is drawn, we have a plane parallelogram. Assuming temporarily the characteristic property of geodesic parallelism on a surface (thus RT is declared parallel to $T'R'$ if the angle between $T'R$ produced and RT is equal to $RT'R'$), we make the corresponding construction with geodesics on a surface: we take any geodesic $T'R$; we draw a geodesic $T'R'$, and another geodesic RT parallel to $T'R'$; we measure equal lengths $T'R'$ and RT along these parallel geodesics; and we draw the superficial geodesic $R'T$. We thus have a geodesic parallelogram: it is a Levi-Civita parallelogram.

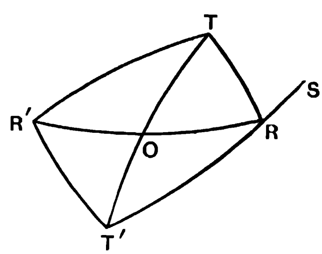

FIG. 9.

Again, in the plane, we can have two lines $T'R$ and $T'R'$; when $R'T$ is drawn parallel to $T'R$, and RT is drawn parallel to TR', these lines meeting in T, we have the same plane parallelogram as before. We make the corresponding construction on a surface with geodesics; we take any two geodesics $T'R$, $T'R'$, through T'; through R' we draw a geodesic $R'T$ parallel to $T'R$, and through R we draw a geodesic RT parallel to $T'R'$, these new geodesics meeting in T. We thus have a

geodesic parallelogram: it is a Pérès parallelogram; and it is distinct from a Levi-Civita parallelogram of the last paragraph.

Once more, in the plane, we can have two lines RR' and TT' bisecting one another at O; when the lines RT, TR', $R'T'$, $T'R$, are drawn, we have the same plane parallelogram as before. We make the corresponding construction on a surface with geodesics; the geodesics RO and TO are produced through O to R' and to T' respectively, where $OR' = OR$, and $OT' = OT$; and the geodesics RT, TR', $R'T'$, $T'R$, are drawn. We thus have a geodesic parallelogram; it is distinct from both the Levi-Civita parallelogram and the Pérès parallelogram.

What usually is required is a knowledge of geodesics drawn from points in one geodesic through O parallel to another geodesic through O. Thus the properties of the last parallelogram are less useful, in this respect, than those of the other two, because the originating point of the parallelogram is its centre and not one of its angular points; accordingly, we shall deal briefly with its properties. These, moreover, can be derived from the analysis of §§ 109-111 without reference to the theory of parallels.

As before, we shall assume that geodesic arcs are small, on a tacit assumption that the sphericity of the surface is not constant over its range. It is easy to see that, up to the third order of small quantities inclusive in the measurement of lengths and up to the second order of small quantities inclusive in the measurement of inclinations, the length $R'T'$ is equal to RT, the angle $OR'T'$ is equal to the angle ORT, and the angle $OT'R'$ to the angle OTR: that is, up to the specified order, the geodesic triangles ORT and $OR'T'$ are equal in all respects. Likewise the geodesic triangles ORT' and $OR'T$ are equal in all respects. The geodesics TR' and $T'R$ are parallel, making equal angles with TT' at T and T'; and the geodesics $T'R'$, TR, also are parallel, for a like reason at T and T' relative to TT'. But, in angles,

$$\begin{aligned} TRS &= \pi - ORT - ORT' \\ &= \pi - OR'T' - ORT' \\ &= RT'R' + \pi - (OR'T' + ORT' + RTR') \\ &= RT'R' - E, \end{aligned}$$

where E is the angular excess of the spherical triangle $T'RT$: that is, the geodesics RT and $T'R'$ are not parallel, estimated with respect to RT'.

We proceed to find the remaining elements of the geodesic triangle $R'OT$, arising in connection with the third type of parallelogram indicated; they can be deduced from the earlier analysis by simple modifications. We retain R and T to denote the angles ORT and OTR, with R_0 and T_0 for approximations, as before. We denote the angles $OR'T$ and OTR' by R' and $\bar{T}$, with R_0' and $\bar{T}_0$ as corresponding approximations. Let p_7', q_7', denote the direction-cosines of $R'T$ at R' in the

direction $R'T$; and let y denote the length of $R'T$. The direction-variables of OT are p_2', q_2', as before; those of OR', at O in the direction OR', are $-p_1', -q_1'$; and we take quantities p_4', q_4', w, such that

$$wp_4' = up_1' + vp_2', \quad wq_4' = uq_1' + vq_2', \quad Ap_4'^2 + 2Hp_4'q_4' + Bq_4'^2 = 1,$$

so that

$$w^2 = u^2 + v^2 + 2uv\cos\epsilon,$$

where, as before,

$$\cos\epsilon = Ap_1'p_2' + H(p_1'q_2' + p_2'q_1') + Bq_1'q_2'.$$

Then we have

$$\sin R_0' = V(p_1'q_4' - p_4'q_2'), \quad \sin \overline{T}_0 = V(p_4'q_2' - p_2'q_4'),$$

so that

$$\frac{\sin R_0'}{v} = \frac{\sin \overline{T}_0}{u} = \frac{\sin\epsilon}{w},$$

$$w\cos R_0' = u + v\cos\epsilon, \quad w\cos\overline{T}_0 = v + u\cos\epsilon.$$

The length y of the geodesic arc $R'T$, up to the third order of small quantities inclusive, is

$$y = w - \tfrac{1}{6}\frac{u^2v^2}{w}K\sin^2\epsilon,$$

and the direction-variables p_7', q_7', of $R'T$ at R' in the direction $R'T$, are given by

$$\left.\begin{aligned} p_7' &= p_4' + u(\alpha_1 p_4' + \beta_1 q_4') - \tfrac{1}{2}u^2(\Gamma_{111}\between p_4', q_4'\between p_1', q_1')^2 \\ &\qquad + \frac{1}{3V}(Hp_4' + Bq_4')Kuv\sin\epsilon + \tfrac{1}{6}p_4'K\frac{u^2v^2}{w^2}\sin^2\epsilon \\ q_7' &= q_4' + u(\epsilon_1 p_4' + \eta_1 q_4') - \tfrac{1}{2}u^2(\Delta_{111}\between p_4', q_4'\between p_1', q_1')^2 \\ &\qquad - \frac{1}{3V}(Ap_4' + Hq_4')Kuv\sin\epsilon + \tfrac{1}{6}q_4'K\frac{u^2v^2}{w^2}\sin^2\epsilon \end{aligned}\right\}.$$

The angles at R' and T of the geodesic triangle $OR'T$, up to the second order of small quantities inclusive, are given by

$$\left.\begin{aligned} R' - R_0' &= \tfrac{1}{3}Kuv\sin\epsilon - \tfrac{1}{6}Kuv\sin\overline{T}_0\cos R_0' \\ \overline{T} - \overline{T}_0 &= \tfrac{1}{3}Kuv\sin\epsilon - \tfrac{1}{6}Kuv\sin R_0'\cos\overline{T}_0 \end{aligned}\right\};$$

and modifications of these expressions can be obtained by substituting the values of $\sin\overline{T}_0$, $\cos\overline{T}_0$, $\sin R_0'$, $\cos R_0'$, in terms of u, v, ϵ.

Sphericity of an amplitude in any superficial orientation; geodesic surface through the orientation.

117. In his original statement as to the measure of curvature, Riemann propounded a measure of curvature of an amplitude by means of geodesics of the n-fold amplitude originating in a superficial orientation. For this purpose, he postulated * two geodesics in directions $u_1', \dots, u_n'$, and $v_1', \dots, v_n'$, and the consequent surface generated by the geodesics in all directions $\alpha u_1' + \beta v_1', \dots, \alpha u_n' + \beta v_n'$, for parametric values of α and β; such a surface now is usually termed a *geodesic surface* of the amplitude. He postulated the measure of curvature of the amplitude by means of an expression which is, in fact, the sphericity of this geodesic surface at the point; so that Riemann's measure of curvature of an amplitude, being expressed by magnitudes which are orientation-variables, is the sphericity of the amplitude in that orientation.

The propounded expression of the measure can be constructed as follows.

Manifestly the amplitudinal geodesics through the point are themselves superficial geodesics: if they were not, shorter superficial arcs on the surface through the point would exist and these, being on the surface and therefore in the amplitude, would have to be shorter than the amplitudinal geodesic. Accordingly, the direction and the magnitude of the radius of circular curvature of a superficial geodesic through the point are the same, respectively, as the direction and the magnitude of the radius of circular curvature of the amplitudinal geodesic originating in that direction: the two curves are one and the same, when the surface is geodesic. Consequently, the spatial direction-cosines of the prime normal, and the circular curvature of a superficial geodesic through the point, are given by equations of the type

$$\frac{Y}{\rho} = \sum_i \sum_j \eta_{ij} x_i' x_j'.$$

To keep the originating directions of all geodesics considered, so that they shall lie within the orientation and shall give rise to a geodesic surface, one mode is to suppose all the parametric variables $x_1, \dots, x_n$, of the amplitude expressed in terms of two selected parametric variables, the forms of expression being such as to ensure the representation of the surface at the point. The element of arc is the same; it merely appertains to the geodesic, which belongs to the amplitude and to the surface. These selected variables may be, either new quantities p and q, in terms of which all the magnitudes $x_1, \dots, x_n$, are to be expressed: or (what is the analytical equivalent) they could be any two of the amplitudinal parameters, the surface being determined by the assignment of appropriate expressions for the remaining $n-2$ parameters in terms of those two.

* *Ges. math. Werke*, p. 403.

We take two superficial parameters p and q; for our purpose, each of the amplitudinal parameters x is a function of p and q, and we have

$$x_m' = \frac{\partial x_m}{\partial p} p' + \frac{\partial x_m}{\partial q} q',$$

for $m=1, \ldots, n$. The arc-relation becomes

$$Ap'^2 + 2Hp'q' + Bq'^2 = 1,$$

where

$$A = \sum_i \sum_j A_{ij} \frac{\partial x_i}{\partial p} \frac{\partial x_j}{\partial p}, \quad H = \sum_i \sum_j A_{ij} \frac{\partial x_i}{\partial p} \frac{\partial x_j}{\partial q}, \quad B = \sum_i \sum_j A_{ij} \frac{\partial x_i}{\partial q} \frac{\partial x_j}{\partial q}.$$

Hence

$$AB - H^2 = \sum_i \sum_j \sum_k \sum_l (A_{ik}A_{jl} - A_{il}A_{jk}) \frac{\partial(x_i, x_j)}{\partial(p, q)} \frac{\partial(x_k, x_l)}{\partial(p, q)},$$

with the customary summation for all values $i, j, k, l, =1, \ldots, n$, taken independently of one another.

The equations, determining the circular curvature of the amplitudinal geodesic, are represented by the typical equation

$$\frac{Y}{\rho} = \sum_i \sum_\eta \eta_{ij} x_i' x_j'.$$

This circular curvature, as regards both the magnitude and the direction of the radius, is the same for the curve considered as a superficial geodesic; thus ρ and each of the direction-cosines Y are unaltered. But the modified expression of the right-hand side is required. Let it be

$$\xi_{11} p'^2 + 2\xi_{12} p'q' + \xi_{22} q'^2,$$

where the parametric relation of the quantities ξ to the typical variable y is similar to that borne by the quantities $\eta_{\alpha\beta}$; then, by substitution of the quantities x', we find

$$\xi_{11} = \sum_\alpha \sum_\beta \eta_{\alpha\beta} \frac{\partial x_\alpha}{\partial p} \frac{\partial x_\beta}{\partial p}, \quad \xi_{12} = \sum_\alpha \sum_\beta \eta_{\alpha\beta} \frac{\partial x_\alpha}{\partial p} \frac{\partial x_\beta}{\partial q}, \quad \xi_{22} = \sum_\alpha \sum_\beta \eta_{\alpha\beta} \frac{\partial x_\alpha}{\partial q} \frac{\partial x_\beta}{\partial q}.$$

Hence

$$\xi_{11}\xi_{22} - \xi_{12}^2 = \sum_i \sum_j \sum_k \sum_l (\eta_{ik}\eta_{jl} - \eta_{jk}\eta_{il}) \frac{\partial(x_i, x_j)}{\partial(p, q)} \frac{\partial(x_k, x_l)}{\partial(p, q)}.$$

When the sum is taken over all the equations corresponding to the space-variables, we have

$$\sum_m \left[\eta_{ik}^{(m)}\eta_{jl}^{(m)} - \eta_{jk}^{(m)}\eta_{il}^{(m)}\right] = (ij, kl),$$

the Riemann four-index symbol; and therefore

$$\sum \xi_{11}\xi_{22} - \sum \xi_{12}^2 = \sum_i \sum_j \sum_k \sum_l (ij, kl) \frac{\partial(x_i, x_j)}{\partial(p, q)} \frac{\partial(x_k, x_l)}{\partial(p, q)}.$$

Now let any two directions $x_1', \ldots, x_n'$, and $z_1', \ldots, z_n'$, be taken in the amplitude, uniquely determining amplitudinal geodesics for each direction. These

determine a geodesic surface in the region ; and we denote by p_1', q_1', and p_2', q_2', the superficial direction-variables of these two geodesics, which belong also to the surface and define all superficial geodesics through the point uniquely as they define all amplitudinal geodesics originating in the orientation of the surface. Owing to the relations

$$x_\alpha' = \frac{\partial x_\alpha}{\partial p} p_1' + \frac{\partial x_\alpha}{\partial q} q_1', \quad z_\alpha' = \frac{\partial x_\alpha}{\partial p} p_2' + \frac{\partial x_\alpha}{\partial q} q_2',$$

$$x_\beta' = \frac{\partial x_\beta}{\partial p} p_1' + \frac{\partial x_\beta}{\partial q} q_1', \quad z_\beta' = \frac{\partial x_\beta}{\partial p} p_2' + \frac{\partial x_\beta}{\partial q} q_2',$$

we have

$$\begin{vmatrix} x_\alpha', & x_\beta' \\ z_\alpha', & z_\beta' \end{vmatrix} = \begin{vmatrix} p_1', & q_1' \\ p_2', & q_2' \end{vmatrix} \frac{\partial(x_\alpha, x_\beta)}{\partial(p, q)};$$

and therefore

$$(p_1'q_2' - p_2'q_1')^2(\sum\xi_{11}\xi_{22} - \sum\xi_{12}^2) = \sum_i \sum_j \sum_k \sum_l (ij, kl) \begin{vmatrix} x_i', & x_j' \\ z_i', & z_j' \end{vmatrix} \begin{vmatrix} x_k', & x_l' \\ z_k', & z_l' \end{vmatrix}.$$

Similarly

$$(p_1'q_2' - p_2'q_1')^2(AB - H^2) = \sum_i \sum_j \sum_k \sum_l (A_{ik}A_{jl} - A_{il}A_{jk}) \begin{vmatrix} x_i', & x_j' \\ z_i', & z_j' \end{vmatrix} \begin{vmatrix} x_k', & x_l' \\ z_k', & z_l' \end{vmatrix}.$$

Consequently, the value of the quantity

$$\frac{\sum_i \sum_j \sum_k \sum_l (ij, kl)(x_i'z_j' - x_j'z_i')(x_k'z_l' - x_l'z_k')}{\sum_i \sum_j \sum_k \sum_l (A_{ik}A_{jl} - A_{il}A_{jk})(x_i'z_j' - x_j'z_i')(x_k'z_l' - x_l'z_k')}$$

is equal to

$$\frac{\sum\xi_{11}\xi_{22} - \sum\xi_{12}^2}{AB - H^2}.$$

This last quantity is (§ 112) the Riemann measure for the geodesic surface. The former quantity is the measure propounded by Riemann * for the geodesic surface so constituted, and is taken as the measure of the amplitude in the orientation of the surface. Accordingly, we can declare the measure to be the *sphericity of the amplitude* in the orientation defined by the two directions $x_1', \ldots, x_n'$, and $z_1', \ldots, z_n'$, in the amplitude.

Two remarks may be made in passing. The denominator in the general Riemann measure can be expressed in the form

$$\left(\sum_i \sum_j A_{ij}x_i'x_j'\right)\left(\sum_i \sum_j A_{ij}z_i'z_j'\right) - \left(\sum_i \sum_j A_{ij}x_i'z_j'\right)^2,$$

which also is equal to $\sin^2 \epsilon$, where ϵ is the angle between the two directions determining the orientation : it is this last form which actually occurs in the statement of the measure made by Riemann.

* *Ges. math. Werke*, p. 403.

Also, in the simpler expression arising from the transformation to the parameters p and q of the geodesic surface, the numerator is equal to the four-index symbol (12, 12) appertaining specially to that surface. As the angle ϵ is the same for the surface as for the amplitude, and as

$$\sin^2\epsilon = (AB - H^2)(p_1'q_2' - p_2'q_1')^2,$$

we deduce the law of transformation of the invariantive magnitude represented by the sphericity.

NOTE *on* § 115, *by Prof. E. H. Neville.*

Construct a plane rectilinear figure, having O_0R_0 and O_0T_0 equal in length to the small geodesic arcs OR and OT in Fig. 8, p. 308, and the angle $R_0O_0T_0$ equal to the angle ROT. Take P_0 in R_0T_0 so that the angles $R_0O_0P_0$ and ROP are equal, and produce O_0P_0 to P_λ so that $O_0P_\lambda = \lambda$; join R_0P_λ, T_0P_λ. Then

$$O_0R_0 = u, \quad O_0T_0 = v, \quad O_0P_0 = l, \quad R_0P_0 = m, \quad R_0O_0P_0 = \theta.$$

Let $P_0T_0 = n$, $R_0P_\lambda = \bar{m}$, $T_0P_\lambda = \bar{n}$, $P_0O_0T_0 = \epsilon - \theta = \phi$. By § 111, we have, to the third order,

$$OPR - O_0P_\lambda R_0 = \tfrac{1}{3}Ku\lambda(\sin\theta - \tfrac{1}{2}\sin R_0 \cos O_0P_\lambda R_0),$$
$$OPT - O_0P_\lambda T_0 = \tfrac{1}{3}Kv\lambda(\sin\phi - \tfrac{1}{2}\sin T_0 \cos O_0P_\lambda T_0),$$

the addition of which shews that $\pi - P_\lambda$ is at least of the second order.

In the plane triangle $T_0R_0P_\lambda$, we have

$$\begin{aligned}(\lambda - l)t\sin P_0 &= \bar{m}\bar{n}\sin P_\lambda \\ &= \tfrac{1}{3}K\lambda\bar{m}\bar{n}\{(u\sin\theta + v\sin\phi) - \tfrac{1}{2}(u\sin R_0\cos O_0P_\lambda R_0 + v\sin T_0\cos O_0P_\lambda T_0)\}.\end{aligned}$$

Now $u\sin R_0 = v\sin T_0$; and $\cos O_0P_\lambda T_0 = -\cos O_0P_\lambda R_0$, to the first order inclusive; so that $u\sin R_0\cos O_0P_\lambda R_0 + v\sin T_0\cos O_0P_\lambda T_0$ is of the third order. Accordingly, $\lambda - l$ is of the third order; and, to this order,

$$\bar{m} - m = (\lambda - l)\cos O_0P_0R_0 = -(\bar{n} - n).$$

Therefore, to the third order,

$$(\lambda - l)\sin P_0 = \tfrac{1}{3}K\frac{mn}{t}(ul\sin\theta + vl\sin\phi) = \tfrac{1}{3}Kmnl\sin P_0,$$

that is, to the third order,

$$\lambda - l = \tfrac{1}{3}Kmnl = \tfrac{1}{3}Klm(t - m).$$

Also, by the result of § 110,

$$\mu = \bar{m} - \tfrac{1}{6}K\frac{u^2\lambda^2}{\bar{m}}\sin^2\theta = m + \tfrac{1}{3}Klmn\cos P_0 - \tfrac{1}{6}K\frac{u^2l^2}{m}\sin^2\theta,$$

to the third order: or, since

$$\frac{m}{t} = \frac{l\sin\theta}{v\sin\epsilon},$$

we have

$$\mu = m\left(1 + \tfrac{1}{3}Kln\cos P_0 - \tfrac{1}{6}K\frac{u^2v^2}{t^2}\sin^2\epsilon\right).$$

The values of λ and μ agree with the results on p. 310.

CHAPTER X

Parallel Geodesics on Free Surfaces

Direction-variables of geodesics drawn under assigned conditions.

118. As a preliminary to investigations concerning parallel geodesics on a surface, consider any two geodesics OG_1, OG_2, through O a central point $y_1, y_2, \ldots$, with direction-variables p_1', q_1', and p_2', q_2', $\left(\text{that is, } \frac{dp}{ds_1}, \frac{dq}{ds_1}, \text{ and } \frac{dp}{ds_2}, \frac{dq}{ds_2}\right)$, their inclination ϵ being given by the equation

$$\cos\epsilon = Ap_1'p_2' + H(p_1'q_2' + q_1'p_2') + Bq_1'q_2'.$$

Along the geodesic OG_1, taken temporarily as a base, let geodesics on the surface be drawn at successive points, their inclinations (a varying ϵ) to the basic geodesic following some assigned law along the base; then p_2', q_2', must change so as to admit this assigned law. Accordingly, we must have

$$-\sin\epsilon\frac{d\epsilon}{ds_1} = (Ap_1' + Hq_1')\frac{dp_2'}{ds_1} + (Hp_1' + Bq_1')\frac{dq_2'}{ds_1}$$
$$+ p_2'\frac{d}{ds_1}(Ap_1' + Hq_1') + q_2'\frac{d}{ds_1}(Hp_1' + Bq_1').$$

Now we have

$$\frac{d}{ds_1}(Ap_1' + Hq_1') = Ap_1'' + Hq_1'' + p_1'^2\frac{\partial A}{\partial p} + p_1'q_1'\left(\frac{\partial A}{\partial q} + \frac{\partial H}{\partial p}\right) + q_1'^2\frac{\partial H}{\partial q}$$
$$= (Ap_1' + Hq_1')\alpha_1 + (Hp_1' + Bq_1')\epsilon_1,$$
$$\frac{d}{ds}(Hp_1' + Bq_1') = (Ap_1' + Hq_1')\beta_1 + (Hp_1' + \beta q_1')\eta_1,$$

with the significance for $\alpha_1, \beta_1, \epsilon_1, \eta_1$, as defined on p. 260; and therefore the foregoing equation becomes

$$-\sin\epsilon\frac{d\epsilon}{ds_1} = (Ap_1' + Hq_1')\left(\frac{dp_2'}{ds_1} + \alpha_1 p_2' + \beta_1 q_2'\right) + (Hp_1' + Bq_1')\left(\frac{dq_2'}{ds_1} + \epsilon_1 p_2' + \eta_1 q_2'\right).$$

Also, because p_2', q_2', are variables of a superficial direction through O, we have

$$Ap_2'^2 + 2Hp_2'q_2' + Bq_2'^2 = 1;$$

and therefore

$$2(Ap_2' + Hq_2')\frac{dp_2'}{ds_1} + 2(Hp_2' + Bq_2')\frac{dq_2'}{ds_1}$$
$$= -\left(p_2'^2\frac{dA}{ds_1} + 2p_2'q_2'\frac{dH}{ds_1} + q_2'^2\frac{dB}{ds_1}\right)$$
$$= -2p_2'^2(A\alpha_1 + H\epsilon_1) - 2p_2'q_2'(A\beta_1 + H\eta_1 + H\alpha_1 + B\epsilon_1) - 2q_2'^2(H\beta_1 + B\eta_1),$$

that is,

$$(Ap_2' + Hq_2')\left(\frac{dp_2'}{ds_1} + \alpha_1 p_2' + \beta_1 q_2'\right) + (Hp_2' + Bq_2')\left(\frac{dq_2'}{ds_1} + \epsilon_1 p_2' + \eta_1 q_2'\right) = 0.$$

Now
$$\sin\epsilon = V(p_1'q_2' - q_1'p_2');$$
and so, when the two equations are resolved, we find

$$\frac{dp_2'}{ds_1} + \alpha_1 p_2' + \beta_1 q_2' = -\frac{1}{V}\frac{d\epsilon}{ds_1}(Hp_2' + Bq_2'),$$
$$\frac{dq_2'}{ds_1} + \epsilon_1 p_2' + \eta_1 q_2' = \frac{1}{V}\frac{d\epsilon}{ds_1}(Ap_2' + Hq_2').$$

But we have

$$\frac{1}{V}(Hp_2' + Bq_2')\sin\epsilon = p_1' - p_2'\cos\epsilon,$$
$$\frac{1}{V}(Ap_2' + Hq_2')\sin\epsilon = -(q_1' - q_2'\cos\epsilon);$$

and therefore the equations, which determine the arc-variations of p_2', q_2', become

$$\left.\begin{aligned}\frac{dp_2'}{ds_1} + \alpha_1 p_2' + \beta_1 q_2' &= -\frac{p_1' - p_2'\cos\epsilon}{\sin\epsilon}\frac{d\epsilon}{ds_1}\\ \frac{dq_2'}{ds_1} + \epsilon_1 p_2' + \eta_1 q_2' &= -\frac{q_1' - q_2'\cos\epsilon}{\sin\epsilon}\frac{d\epsilon}{ds_1}\end{aligned}\right\}.$$

If, similarly, geodesics be drawn at successive points on OG_2, taken as a temporary base, with inclinations (again a varying ϵ) to that basic geodesic following some assigned law along the base, then the direction-variables p_1', q_1', of the geodesics thus drawn must vary according to the conditions

$$\left.\begin{aligned}\frac{dp_1'}{ds_2} + \alpha_2 p_1' + \beta_2 q_1' &= -\frac{p_2' - p_1'\cos\epsilon}{\sin\epsilon}\frac{d\epsilon}{ds_2}\\ \frac{dq_1'}{ds_2} + \epsilon_2 p_1' + \eta_2 q_1' &= -\frac{q_2' - q_1'\cos\epsilon}{\sin\epsilon}\frac{d\epsilon}{ds_2}\end{aligned}\right\}.$$

We note that

$$\alpha_1 p_2' + \beta_1 q_2' = \alpha_2 p_1' + \beta_2 q_1' = \Gamma_{11} p_1' p_2' + \Gamma_{12}(p_1'q_2' + q_1'p_2') + \Gamma_{22} q_1' q_2',$$
$$\epsilon_1 p_2' + \eta_1 q_2' = \epsilon_2 p_1' + \eta_2 q_1' = \Delta_{11} p_1' p_2' + \Delta_{12}(p_1'q_2' + q_1'p_2') + \Delta_{22} q_1' q_2';$$

also that for general laws of variation of the inclination ϵ along the respective basic geodesics, the relations

$$\frac{dp_1'}{ds_2} = \frac{d}{ds_2}\left(\frac{dp}{ds_1}\right) = \frac{d}{ds_1}\left(\frac{dp}{ds_2}\right) = \frac{dp_2'}{ds_1},$$
$$\frac{dq_1'}{ds_2} = \frac{d}{ds_2}\left(\frac{dq}{ds_1}\right) = \frac{d}{ds_1}\left(\frac{dq}{ds_2}\right) = \frac{dq_2'}{ds_1},$$

are not satisfied.

Manifestly the simplest law for the construction of the successive geodesics in each set is, in both cases, that of requiring a constant inclination ϵ. As will be

seen, such geodesics drawn at points along OG_1 are said to be *parallel* to OG_2, and such geodesics drawn at points along OG_2 are said to be *parallel* to OG_1. The respective laws for p_2', q_2', in the former set, and for p_1', q_1', in the later set, are

$$\left.\begin{aligned} -\frac{dp_2'}{ds_1} &= \alpha_1 p_2' + \beta_1 q_2' = \Gamma_{11} p_1' p_2' + \Gamma_{12}(p_1' q_2' + q_1' p_2') + \Gamma_{22} q_1' q_2' \\ -\frac{dq_2'}{ds_1} &= \epsilon_1 p_2' + \eta_1 q_2' = \Delta_{11} q_1' q_2' + \Delta_{12}(p_1' q_2' + q_1' p_2') + \Delta_{22} q_1' q_2' \end{aligned}\right\},$$

$$\left.\begin{aligned} -\frac{dp_1'}{ds_2} &= \alpha_2 p_1' + \beta_2 q_1' = \Gamma_{11} p_1' p_2' + \Gamma_{12}(p_1' q_2' + q_1' p_2') + \Gamma_{22} q_1' q_2' \\ -\frac{dq_1'}{ds_2} &= \epsilon_2 p_1' + \eta_2 q_1' = \Delta_{11} q_1' q_2' + \Delta_{12}(p_1' q_2' + q_1' p_2') + \Delta_{22} q_1' q_2' \end{aligned}\right\}.$$

Moreover, for such a succession of geodesics drawn at points along OG_1 as the basic geodesic, we find

$$\left.\begin{aligned} \frac{d}{ds_1}(Ap_2' + Hq_2') &= \alpha_1(Ap_2' + Hq_2') + \epsilon_1(Hp_2' + Bq_2') \\ \frac{d}{ds_1}(Hp_1' + Bq_1') &= \beta_1(Ap_2' + Hq_2') + \eta_1(Hp_2' + Bq_2') \end{aligned}\right\},$$

and, for such geodesics drawn at points along OG_2 as basic,

$$\left.\begin{aligned} \frac{d}{ds_2}(Ap_1' + Hq_1') &= \alpha_2(Ap_1' + Hq_1') + \epsilon_2(Hp_1' + Bq_1') \\ \frac{d}{ds_2}(Hp_1' + Bq_1') &= \beta_2(Ap_1' + Hq_1') + \eta_2(Hp_1' + Bq_1') \end{aligned}\right\}.$$

Parallel geodesics.

119. The notion of parallel geodesics on a surface, existing freely in multiple space, can be presented as follows.

At a point O on a surface, let ORE be any geodesic *; and consider the tangent planes of the surface at successive points along ORE. The envelope of these tangent planes is a developable surface; the direction of any geodesic OT on the original surface gives a direction on the developable touching OT; the direction of any other geodesic RU on the original surface gives another direction on the developable touching RU, where R is any point along ORE. Let the

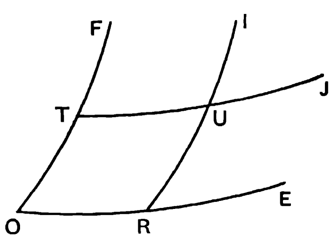

FIG. 10.

* In his original account, the guiding curve ORE was taken by Levi-Civita to be any definite curve on the surface: and the account, given here in the text, is obviously as applicable to any curve as to a geodesic. But the march of a geodesic on a surface is certainly simple, and all the applications are to be associated with a geodesic as a base; accordingly, from the beginning, the basic curve ORE is taken to be a geodesic.

developable surface be developed into a plane, without any stretching or tearing. In that developed plane, the line corresponding to the geodesic *ORE* becomes a straight line; the direction touching the geodesic *OT* on the original surface becomes a straight line, making an angle equal to *ROT* with the development of *OR*; and the direction touching the geodesic *RU* on the original surface becomes a straight line, making an angle equal to *ERU* with the development of *RU* which is only the continuation of the development of *OR*. In that plane, the developed directions *OT* and *RU* are parallel when the angles *ROT* and *ERU* are equal; and so the geodesics *OTF* and *RUI* on the original surface are said to be *parallel* when the angles *ROT* and *ERU* are equal. Manifestly, we can have a set of geodesics parallel to *OT* by drawing on the surface geodesics through successive points of *ORE*, each making the same angle equal to *ORT* with the varying direction along the geodesic *ORE* at the successive points.

Thus the geodesics parallel to *OT* are defined by reference to a basic geodesic *ORE*. A different choice of basic geodesic through *O* would provide a different set of geodesics parallel to *OT*—as, indeed, appears from the figure in § 116; consequently, the composition of the set of geodesics parallel to *OT* depends not only upon the determining geodesic *OT* but also upon the selection of the basic line.

Similarly, by taking successive points along *OTF* and drawing through these points superficial geodesics making, with the varying direction along the geodesic *OTF*, an angle equal to *TOR*, we obtain a set of superficial geodesics parallel to *ORE*; but, as before, the composition of this set of geodesics parallel to *OR* is governed not only by the determining geodesic *OR* but also by the selection of *OT* as the basic line.

Obviously geodesics on a surface, like straight lines in a plane, can be described as having the same direction along their course: or, as in the Euclidean definition of a line, " lying evenly between two points ".

This association, of the property of parallelism in a plane, with geodesics on a surface was first effected by Levi-Civita *; it was initiated by him for the construction of parallel geodesics in any amplitude, not solely in a surface. Through Levi-Civita's initial investigations, the conditions of parallelism were established by reference to the plenary homaloidal space containing the surface and the amplitude, though the actual dimensionality of this plenary space had neither significance nor influence in relation to the result. Another parallelism of geodesics in any n-fold amplitude was established by Severi, by the use of geodesic

* *Rend. Circ. Mat. di Palermo*, t. xlii (1917), pp. 172-205; see also his treatise, in the English translation (1927), *The absolute differential calculus*, chap. v, §§ 10, 26.

In relation to the theory of parallelism of geodesics in any amplitude, the memoir by Severi, *Rend. Circ. Mat. di Palermo*, t. xlii (1917), pp. 227-259, should be consulted, as well as memoirs by Bompiani, *Atti d. R. Ist. Veneto*, t. lxxx, Parte 2ª (1921), pp. 355-386, 839-859. Other references will be found in these authorities.

surfaces of the amplitude as originally due to Riemann; thus in the preceding diagram (p. 320), the geodesics ORE and OTF are taken as defining a geodesic surface (§ 117) of the amplitude at O in the orientation defined by the directions of the tangent lines at O to OR and OT. The geodesics in this geodesic surface, drawn parallel to OT at successive points along ORE as a basic curve, are taken to be the geodesics of the amplitude parallel to the amplitudinal geodesic OTF, constructed by reference to the basic amplitudinal geodesic ORE; and likewise for the geodesics in the surface, drawn parallel to ORE through successive points along OTF as a basic curve.

The analytical developments for parallel geodesics in amplitudes, which are more extensive than surfaces, will be reserved for later stages. Here we shall discuss some of the properties connected with parallel geodesics, purely as belonging to surfaces, without any regard to intrinsic relations which the surfaces may bear to enclosing amplitudes.

Superficial parallelograms, after Levi-Civita, Pérès.

120. It has already been pointed out that, when any two geodesics ORE and OTF through a point O on a surface are arbitrarily selected, and when parallels are drawn with either ORE or OTF as basic lines, varieties of geodesic parallelograms can be drawn.

Thus we have a Levi-Civita parallelogram, by measuring equal arcs OT and RU along parallel geodesics OT and RU, and by drawing the superficial geodesic TU. Levi-Civita proves that, if $\square$ denote the area of the geodesic parallelogram $OTUR$, then

$$K\square^2 = OR^2 - TU^2,$$

in the limit as the arcs OR, OT, diminish indefinitely, where K denotes the Riemann measure of curvature of the surface *. The fourth side TU of the Levi-Civita geodesic parallelogram is not parallel to the opposite side OR; nor are their lengths equal.

There is a Pérès parallelogram obtained, by drawing a geodesic RUI through R parallel to OT, and by drawing a geodesic TUJ through T parallel to OR, these two new geodesics intersecting in U. Pérès proves that

$$K\square = \text{angle } JUI - \text{angle } ROT,$$

also in the limit as the arcs OR, OT, diminish indefinitely, with the preceding significance † for K and $\square$; so that, if we take $JUI = \epsilon + \delta\epsilon$, and $\square = \delta S$ an

* This result was regarded by Levi-Civita as giving an interpretation of the Riemann measure of curvature of the surface in the orientation at O; see § 18 of his memoir, cited p. 321, *note*.

† At the date of publication of the memoirs of Levi-Civita and of Pérès, the interpretation of K as the sphericity of the surface in connection with the interpretation of the result in § 112 was not enunciated.

element of area at O specially constructed, we have

$$\text{limit of } \frac{\delta\epsilon}{\delta S} = K,$$

an expression formally similar to that of the circular curvature of a curve. In the Pérès parallelogram, the length of the geodesic side TU is not equal to the length of its opposite side OR, nor is the length of the geodesic side RU equal to the length of its opposite side OT.

Ex. Both these cited results can be illustrated by birectangular quadrilaterals on a sphere in triple space. On a spherical surface, the Riemann measure K (being the same, for any surface in triple space, as the Gauss measure) is $1/a^2$, where a is the radius of the sphere; for simplicity, we take $a=1$.

Let AB, AC, be small arcs of two great circles meeting at right angles in A; and let N, M, be the respective poles of AB, AC. Then the great circle BN is geodesically parallel to AN. Let $AB=\alpha$, $AC=\beta$, where α and β are small.

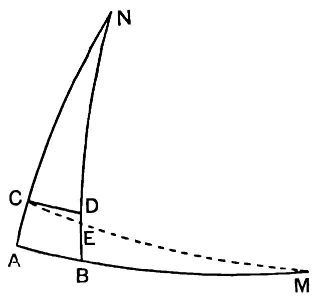

FIG. 11.

(i) The Levi-Civita parallelogram is obtained by taking BD equal to AC and drawing the great circle CD. The angle $ANB=\alpha$, while the arcs CN and DN are each equal to $\frac{1}{2}\pi-\beta$; hence

$$\begin{aligned} \cos CD &= \cos^2(\tfrac{1}{2}\pi-\beta) + \sin^2(\tfrac{1}{2}\pi-\beta)\cos\alpha \\ &= 1 - \tfrac{1}{2}\alpha^2 + \tfrac{1}{24}\alpha^4 + \tfrac{1}{2}\alpha^2\beta^2, \end{aligned}$$

accurately up to the fourth order of small quantities inclusive. Thus, accurately to the same order,

$$CD^2 = \alpha^2 - \alpha^2\beta^2;$$

and so the limit of

$$\frac{AB^2 - CD^2}{\alpha^2\beta^2},$$

as α and β diminish indefinitely, is unity: which is the Levi-Civita theorem for the present instance.

To estimate the deviation of CD from parallelism to AB, we have

$$\begin{aligned} \cos NCD &= \frac{\cos ND - \cos NC \cos CD}{\sin NC \sin CD} \\ &= \frac{\sin\beta}{\cos\beta} \tan\tfrac{1}{2}CD = \tfrac{1}{2}\alpha\beta, \end{aligned}$$

when the most important term is retained; thus

$$NCD = \tfrac{1}{2}\pi - \tfrac{1}{2}\alpha\beta.$$

A geodesic at C, parallel to AB, is at right angles to AC and passes through M; thus the deviation from parallelism is

$$\tfrac{1}{2}\alpha\beta,$$

the arcs α, β, being measured on a sphere of radius unity.

(ii) The Pérès parallelogram is obtained by drawing the great circle CM; let it meet BN in E. In the particular parallelogram thus drawn, the angles BAC, ACE, ABE, are right angles; let $CE=x$, $BE=y$. We have

$$\sin y=\sin\beta\cos x, \quad \sin x=\sin\alpha\cos y,$$
$$\sin E\cos y=\cos\beta;$$

thus, when α and β are small, we easily find

$$y=\beta-\tfrac{1}{2}\alpha^2\beta, \quad x=\alpha-\tfrac{1}{2}\alpha\beta^2,$$
$$E-\tfrac{1}{2}\pi=\alpha\beta,$$

accurately up to the retained order of small quantities in each measure. This last result, in effect, is the Pérès theorem, the quantities α and β being arcs measured on a spherical surface.

The result can be inferred also from the customary theorem concerning the area of a convex spherical figure bounded by arcs of great circles. The spherical excess of the quadrilateral $ABEC$

$$=BAC+ACE+ABE+E-2\pi$$
$$=E-\tfrac{1}{2}\pi;$$

and the area of the parallelogram, when α and β are small, is equal to $\alpha\beta$, up to the second order. The radius of the sphere is unity: hence the result.

In the Pérès parallelogram, the side BE is not equal to AC; we there have

$$AC-DB=\tfrac{1}{2}\alpha^2\beta,$$

a quantity of the third order. The quantity $AC-DB$, for the Levi-Civita parallelogram, is actually zero, by the construction of the parallelogram.

We note that, in the Pérès parallelogram,

$$\frac{AB^2-CE^2}{\alpha^2\beta^2}$$

tends to the limit unity with indefinite decrease of α and β, as in the Levi-Civita parallelogram; but this equality of limits is specially due to the right angle at A. Even so, the equality occurs only in the limit, the quantities being unequal when further powers of α and β are retained in approximations.

121. The course of the analysis, that belongs to the range of a surface in the immediate vicinity of a point, whether concerned with parallels or investigations similar to those in §§ 109-111, requires approximations in powers of the small quantities involved, these usually being lengths of arcs. The approximations here used must proceed to second powers inclusive when they relate to magnitudes such as inclinations, and to third powers inclusive when they relate to magnitudes such as lengths. Occasional exceptions to this statement, when they occur, will be duly noted.

The symbols, of § 98 defined by

$$\left.\begin{aligned}\alpha_i&=\Gamma_{11}p_i'+\Gamma_{12}q_i'\\ \beta_i&=\Gamma_{12}p_i'+\Gamma_{22}q_i'\end{aligned}\right\}, \qquad \left.\begin{aligned}\epsilon_i&=\Delta_{11}p_i'+\Delta_{12}q_i'\\ \eta_i&=\Delta_{12}p_i'+\Delta_{22}q_i'\end{aligned}\right\},$$

where p_i', q_i', are direction-variables for any direction ds_i in the surface, are of frequent recurrence. The values of $\dfrac{dA}{ds_i}$, $\dfrac{dH}{ds_i}$, $\dfrac{dB}{ds_i}$, and the values of the second derivatives of A, H, B, taken along the geodesic in that direction, have been obtained (§ 98).

We shall require, up to the second order of small quantities inclusive, the expressions for the direction-variables at R (in figure on p. 320) of the geodesic RUI in the direction RU, drawn parallel to OT. Let these direction-variables be denoted by p_4', q_4', and the length of OR by u; as p_4', q_4', coincide with p_2', q_2', respectively when u is zero, and as they are functions of position along the basic geodesic ORE, we have

$$p_4' = p_2' + u\frac{dp_2'}{ds_1} + \tfrac{1}{2}u^2\frac{d^2p_2'}{ds_1^2},$$

$$q_4' = q_2' + u\frac{dq_2'}{ds_1} + \tfrac{1}{2}u^2\frac{d^2q_2'}{ds_1^2},$$

accurately up to the second order inclusive, the coefficients of u and of u^2 being determined in accordance with the assigned parallelism. Now we have already (§ 118) obtained the two coefficients of u, in the forms

$$\left.\begin{aligned} -\frac{dp_2'}{ds_1} &= \alpha_1 p_2' + \beta_1 q_2' = \alpha_2 p_1' + \beta_2 q_1' = \gamma_{12} = \Gamma_{11}p_1'p_2' + \Gamma_{12}(p_1'q_2' + q_1'p_2') + \Gamma_{22}q_1'q_2' \\ -\frac{dq_2'}{ds_1} &= \epsilon_1 p_2' + \eta_1 q_2' = \epsilon_2 p_1' + \eta_2 q_1' = \delta_{12} = \Delta_{11}p_1'p_2' + \Delta_{12}(p_1'q_2' + q_1'p_2') + \Delta_{22}q_1'q_2' \end{aligned}\right\}.$$

By direct differentiation, we have

$$\begin{aligned} \frac{d^2p_2'}{ds_1^2} = (\alpha_1^2 + \beta_1\epsilon_1)p_2' + (\alpha_1\beta_1 + \beta_1\eta_1)q_2' - \alpha_2 p_1'' - \beta_2 q_1'' \\ - p_1'p_2'\frac{d\Gamma_{11}}{ds_1} - (p_1'q_2' + q_1'p_2')\frac{d\Gamma_{12}}{ds_1} - q_1'q_2'\frac{d\Gamma_{22}}{ds_1}. \end{aligned}$$

But

$$p_1'' = -\alpha_1 p_1' - \beta_1 q_1', \quad q_1'' = -\epsilon_1 p_1' - \eta_1 q_1';$$

and the value of the second line can be deduced from the general result in § 98 by taking $i=1$, $j=2$, $k=1$, so that it becomes

$$\begin{aligned} = -\tfrac{1}{3}K(Hp_1' + Bq_1')(p_1'q_2' - q_1'p_2') - (\Gamma_{111} \between p_2', q_2' \between p_1', q_1')^2 \\ - p_1'(2\alpha_1\alpha_2 + \beta_1\epsilon_2 + \beta_2\epsilon_1) - q_1'(\alpha_1\beta_2 + \alpha_2\beta_1 + \beta_1\eta_2 + \beta_2\eta_1). \end{aligned}$$

When these values are substituted and reduction is effected, we find

$$\frac{d^2p_2'}{ds_1^2} = -\tfrac{1}{3}K(Hp_1' + Bq_1')(p_1'q_2' - q_1'p_2') - (\Gamma_{111} \between p_2', q_2' \between p_1', q_1')^2.$$

Similarly, we find

$$\frac{d^2q_2'}{ds_1^2} = \tfrac{1}{3}K(Ap_1' + Hq_1')(p_1'q_2' - q_1'p_2') - (\Delta_{111} \between p_2', q_2' \between p_1', q_1')^2.$$

The results can also be inferred directly from the arc-relation at R and from the equation, which expresses the parallelism of the geodesic RUI at R to OT in the form

$$\cos\epsilon = A^{(R)}p_4'P_1' + H^{(R)}(p_4'Q_1' + q_4'P_1') + B^{(R)}q_4'Q_1',$$

where P_1', Q_1', denote the direction-variables of the geodesic ORE at R in the direction RE, being given in value by

$$P_1' = p_1' + up_1'' + \tfrac{1}{2}u^2p_1''', \quad Q_1' = q_1' + uq_1'' + \tfrac{1}{2}u^2q_1''',$$

while $A^{(R)}$, $H^{(R)}$, $B^{(R)}$, denote the values of A, H, B, at R; and with this equation there must be combined the permanent arc-relation, which at R for the geodesic RUI has the form

$$A^{(R)}p_4'^2 + 2H^{(R)}p_4'q_4' + B^{(R)}q_4'^2 = 1.$$

We have

$$A^{(R)} = A + u\frac{dA}{ds_1} + \tfrac{1}{2}u^2\frac{d^2A}{ds_1^2},$$

with corresponding values for $H^{(R)}$ and $B^{(R)}$. To make the necessary approximations, we take

$$p_4' = p_2' + uP_1 + \tfrac{1}{2}u^2P,$$
$$q_4' = q_2' + uQ_1 + \tfrac{1}{2}u^2Q,$$

where P_1, Q_1, P, Q, are finite magnitudes independent of u.

When finite terms are taken in the two equations, these cancel without leaving any condition.

When terms of the first order are taken in the two equations, the equation in $\cos\epsilon$ leads to the condition

$$(Ap_1' + Hq_1')(P_1 + \alpha_1p_2' + \beta_1q_2') + (Hp_1' + Bq_1')(Q_1 + \epsilon_1p_2' + \eta_1q_2') = 0;$$

and the permanent arc-relation at R for the geodesic RUI leads to the condition

$$(Ap_2' + Hq_2')(P_1 + \alpha_1p_2' + \beta_1q_2') + (Hp_2' + Bq_2')(Q_1 + \epsilon_1p_2' + \eta_1q_2') = 0.$$

Now

$$(Ap_1' + Hq_1')(Hp_2' + Bq_2') - (Ap_2' + Hq_2')(Hp_1' + Bq_1') = V^2(p_1'q_2 - q_1'p_2') = V\sin\epsilon,$$

which is not zero; and therefore

$$P_1 + \alpha_1p_2' + \beta_1q_2' = 0, \quad Q_1 + \epsilon_1p_2' + \eta_1q_2' = 0.$$

When terms of the second order are taken in the two equations, the equation in $\cos\epsilon$ leads to the condition

$$(Ap_1' + Hq_1')\{P + (\Gamma_{111}\between p_2', q_2'\between p_1', q_1')^2\}$$
$$+ (Hp_1' + Bq_1')\{Q + (\Delta_{111}\between p_2', q_2'\between p_1', q_1'^2)\} = 0;$$

and the permanent arc-relation at R for the geodesic RUI leads to the condition

$$(Ap_2'+Hq_2')\{P+(\Gamma_{111}\between p_2', q_2'\between p_1', q_1')^2\} \\ +(Hp_2'+Bq_2')\{Q+(\Delta_{111}\between p_2', q_2'\between p_1', q_1')^2\}=\tfrac{1}{3}V^2K(p_1'q_2'-q_1'p_2')^2.$$

When these two equations are resolved, they yield the relations

$$P+(\Gamma_{111}\between p_2', q_2'\between p_1', q_1')^2=-\tfrac{1}{3}K(Hp_1'+Bq_1')(p_1'q_2'-q_1'p_2'),$$
$$Q+(\Delta_{111}\between p_2', q_2'\between p_1', q_1')^2=\tfrac{1}{3}K(Ap_1'+Hq_1')(p_1'q_2'-q_1'p_2').$$

With these values of P_1 and P, in p_4'; and with these values of Q_1 and Q, in q_4'; the forms of p_4', q_4', thus obtained, agree with the earlier forms.

Similarly, we shall require, up to the second order of small quantities inclusive, the expressions for the direction-variables at T of the geodesic TUJ in the direction TU. Let these direction-variables be denoted by p_3', q_3', and the arc OT by v; in the same way as for p_4', q_4', we find

$$p_3'=p_1'+v\frac{dp_1'}{ds_2}+\tfrac{1}{2}v^2\frac{d^2p_1'}{ds_2^2},\quad q_3'=q_1'+v\frac{dq_1'}{ds_2}+\tfrac{1}{2}v^2\frac{d^2q_1'}{ds_2^2},$$

where

$$\left.\begin{aligned}\frac{dp_1'}{ds_2}&=-\alpha_2p_1'-\beta_2q_1'=-\alpha_1p_2'-\beta_1q_2'=-\Gamma_{11}p_1'p_2'-\Gamma_{12}(p_1'q_2'+q_1'p_2')-\Gamma_{22}q_1'q_2'\\ \frac{dq_1'}{ds_2}&=-\epsilon_2p_1'-\eta_2q_1'=-\epsilon_1p_2'-\eta_1q_2'=-\Delta_{11}p_1'p_2'-\Delta_{12}(p_1'q_2'+q_1'p_2')-\Delta_{22}q_1'q_2'\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{d^2p_1'}{ds_2^2}&=-\tfrac{1}{3}K(Hp_2'+Bq_2')(p_2'q_1'-q_2'p_1')-(\Gamma_{111}\between p_1', q_1'\between p_2', q_2')^2\\ \frac{d^2q_1'}{ds_2^2}&=\tfrac{1}{3}K(Ap_2'+Hq_2')(p_2'q_1'-q_2'p_1')-(\Delta_{111}\between p_1', q_1'\between p_2', q_2')^2\end{aligned}\right\}.$$

It will be noticed that we have

$$\frac{d}{ds_2}\left(\frac{dp}{ds_1}\right)=\frac{dp_1'}{ds_2}=\frac{dp_2'}{ds_1}=\frac{d}{ds_1}\left(\frac{dp}{ds_2}\right),$$
$$\frac{d}{ds_2}\left(\frac{dq}{ds_1}\right)=\frac{dq_1'}{ds_2}=\frac{dq_2'}{ds_1}=\frac{d}{ds_1}\left(\frac{dq}{ds_2}\right);$$

these relations hold for systems of parallel geodesics but not in general (§ 118).

NOTE. The preceding forms can be modified by transforming the terms which involve the sphericity K.

There are identical relations

$$p_2'-p_1'\cos\epsilon=-(Hp_1'+Bq_1')(p_1'q_2'-q_1'p_2'),$$
$$q_2'-q_1'\cos\epsilon=(Ap_1'+Hq_1')(p_1'q_2'-q_1'p_2'),$$
$$p_1'-p_2'\cos\epsilon=(Hp_2'+Bq_2')(p_1'q_2'-q_1'p_2'),$$
$$q_1'-q_2'\cos\epsilon=-(Ap_2'+Hq_2')(p_1'q_2'-q_1'p_2');$$

and therefore we have

$$\left.\begin{aligned}
\frac{d^2 p_1'}{ds_2^{\,2}} &= \tfrac{1}{3}K(p_1' - p_2' \cos\epsilon) - (\Gamma_{111})(p_1',\, q_1')(p_2',\, q_2')^2 \\
\frac{d^2 q_1'}{ds_2^{\,2}} &= \tfrac{1}{3}K(q_1' - q_2' \cos\epsilon) - (\Delta_{111})(p_1',\, q_1')(p_2',\, q_2')^2 \\
\frac{dp_1'}{ds_2} &= -\gamma_{12}, \quad \frac{dq_1'}{ds_2} = -\delta_{12}
\end{aligned}\right\}.$$

$$\left.\begin{aligned}
\frac{d^2 p_2'}{ds_1^{\,2}} &= \tfrac{1}{3}K(p_2' - p_1' \cos\epsilon) - (\Gamma_{111})(p_2',\, q_2')(p_1',\, q_1')^2 \\
\frac{d^2 q_2'}{ds_1^{\,2}} &= \tfrac{1}{3}K(q_2' - q_1' \cos\epsilon) - (\Delta_{111})(p_2',\, q_2')(p_1',\, q_1')^2 \\
\frac{dp_2'}{ds_1} &= -\gamma_{12}, \quad \frac{dq_2'}{ds_1} = -\delta_{12}
\end{aligned}\right\}.$$

These forms are commodious for applications specially associated with a surface; the earlier forms are convenient for comparison, when parallel geodesics in a region and in more extensive amplitudes are under consideration.

The Levi-Civita parallelogram.

122. We begin with the Levi-Civita geodesic parallelogram. When this is represented in the figure (§ 119), the geodesic RU is parallel to OT, and the arcs RU, OT are equal; the first requirements are the magnitude of the geodesic arc TU, and the direction-variables at T of the geodesic TU in the direction TU. As before, we take $OR=u$, $OT=v$. Let the length of the geodesic arc TU be x, so that we shall expect to find $x-u$ to be a small quantity of order higher than the first (it will be found to be of the third order) which would vanish with v: we write

$$x - u = M.$$

We denote by p', q', the direction-variables at T of the geodesic TU in the direction TU: thus p' will be p_1' and q' will be q_1' when $v=0$, and we therefore take, initially,

$$p' = p_1' + vP_1, \quad q' = q_1' + vQ_1,$$

so that M, P_1, Q_1, are three quantities to be determined.

Three conditions are at our disposal * for their determination. The values of p and of q at U must respectively be the same for the approach by the geodesic path ORU as by the geodesic path OTU; and there is the permanent arc-relation at T

$$A^{(T)}p'^2 + 2H^{(T)}p'q' + B^{(T)}q'^2 = 1.$$

It is convenient to obtain the first-order approximations for P_1, Q_1, M: the first-order approximation for M is found to be zero.

* The method of the following investigations differs from the methods respectively due to Levi-Civita, Severi, Pérès, Bompiani, in their memoirs cited p. 321, *note*.

Up to the first order of small quantities, we have

$$A^{(T)}=A+2v(A\alpha_2+H\epsilon_2),$$
$$H^{(T)}=H+v(A\beta_2+H\eta_2+H\alpha_2+B\epsilon_2),$$
$$B^{(T)}=B+2v(H\beta_2+B\eta_2).$$

When these quantities, and the foregoing assumed expressions for p', q', are substituted in the arc-relation, the finite terms (that is, the terms free from small quantities) cancel. In order that the aggregate of terms involving the first power of v may disappear from the equation, there remains the condition

$$2\{(Ap_1'+Hq_1')P_1+(Hp_1'+Bq_1')Q_1\}$$
$$+2(A\alpha_2+H\epsilon_2)p_1'^2+2(A\beta_2+H\eta_2+H\alpha_2+B\epsilon_2)p_1'q_1'+2(H\beta_2+\beta\eta_2)q_1'^2=0,$$

which, by re-arrangement, can be changed to the form

$$(Ap_1'+Hq_1')(P_1+\alpha_2p_1'+\beta_2q_1')+(Hp_1'+Bq_1')(Q_1+\epsilon_2p_1'+\eta_2q_1')=0.$$

When the direction-variables at R of the geodesic RU in the direction RU are denoted by p_4', q_4', we have (§ 121)

$$p_4'=p_2'+u\frac{dp_2'}{ds_1}+\tfrac{1}{2}u^2\frac{d^2p_2'}{ds_1^2},\quad q_4'=q_2'+u\frac{dq_2'}{ds_1}+\tfrac{1}{2}u^2\frac{d^2q_2'}{ds_1^2},$$

up to the retained order, the coefficients of u and of u^2 being known. Then the value of the parameter p at U by the geodesic approach ORU

$$=p_R+vp_4'+\tfrac{1}{2}v^2p_4''+\tfrac{1}{6}v^3p_4'''$$
$$=p\ +up_1'+\tfrac{1}{2}u^2p_1''+\tfrac{1}{6}u^3p_1'''+vp_4'+\tfrac{1}{2}v^2p_4''+\tfrac{1}{6}v^3p_4''',$$

accurately up to the third order inclusive: while its value at the same place by the geodesic approach OTU

$$=p_T+xp'+\tfrac{1}{2}x^2p''+\tfrac{1}{6}x^3p'''$$
$$=p\ +vp_2'+\tfrac{1}{2}v^2p_2''+\tfrac{1}{6}v^3p_2'''+xp'+\tfrac{1}{2}x^2p''+\tfrac{1}{6}x^3p''',$$

accurately up to the same order. Hence, up to the retained third order, the p-relation becomes

$$xp'+\tfrac{1}{2}x^2p''+\tfrac{1}{6}x^3p'''=up_1'+\tfrac{1}{2}u^2p_1''+\tfrac{1}{6}u^3p_1'''$$
$$+v(p_4'-p_2')+\tfrac{1}{2}v^2(p_4''-p_2'')+\tfrac{1}{6}v^3(p_4'''-p_2''').$$

Now

$$xp'=up_1'+Mp_1'+uvP_1+MvP_1.$$

Up to the second order inclusive, and only up to that order, the quantities MvP_1, x^3p_1''', u^3p_1''', $v^3(p_4'''-p_2''')$, are negligible; as $p_4'-p_2'$, $q_4'-q_2'$, are at least of the first order, the quantity $v^2(p_4''-p_2'')$, being

$$=-v^2\{\Gamma_{11}(p_4'^2-p_2'^2)+2\Gamma_{12}(p_4'q_4'-p_2'q_2')+\Gamma_{22}(q_4'^2-q_2'^2)\},$$

also is of the third order and so, in relation to second-order terms, is negligible. Again, up to this second order,

$$\tfrac{1}{2}x^2p''=\tfrac{1}{2}u^2p_1'',$$
$$v(p_4'-p_2')=uv\frac{dp_2'}{ds_1};$$

and therefore, up to the second order, and only up to that order, the p-relation gives the equation

$$Mp_1'+uvP_1=uv\frac{dp_2'}{ds_1}=-uv(\alpha_2p_1'+\beta_2q_1'),$$

or, what is the same thing,

$$uv(P_1+\alpha_2p_1'+\beta_2q_1')=-Mp_1'.$$

Similarly, the consideration of second-order terms in the q-relation at the point U gives the equation

$$uv(Q_1+\epsilon_2p_1'+\eta_2q_1')=-Mq_1'.$$

Multiply this p-equation by $Ap_1'+Hq_1'$ and this q-equation by $Hp_1'+Bq_1'$, add the products, and use the earlier equation deduced from the arc-relation at R; then

$$M=0,$$

that is, up to the second order inclusive, M is zero. Hence M is a magnitude of the third order of small quantities at least. Moreover, the p-relation and q-relation, now that M is known to be of at least the third order, give the quantities

$$P_1+\alpha_2p_1'+\beta_2q_1',\quad Q_1+\epsilon_2p_1'+\eta_2q_1',$$

as of the first order at least; so we take

$$\left.\begin{aligned}p'&=p_1'+vP_1=p_1'-v(\alpha_2p_1'+\beta_2q_1')+vP=p_1'-v\gamma_{12}+vP\\ q'&=q_1'+vQ_1=q_1'-v(\epsilon_2p_1'+\eta_2q_1')+vQ=q_1'-v\delta_{12}+vQ\end{aligned}\right\},$$

where the terms vP, vQ, are of the second order at least, while γ_{12}, δ_{12}, have their former significance (§ 121).

123. The determination of the second-order magnitudes vP, vQ, and of the third-order magnitude M, can be made by considering the approximation in third-order terms in the p-relation and the q-relation at U, and the approximation in second-order terms in the arc-relation at R.

The arc-relation is

$$A^{(T)}p'^2+2H^{(T)}p'q'+B^{(T)}q'^2=1;$$

also

$$A^{(T)}=A+v\frac{dA}{ds_2}+\tfrac{1}{2}v^2\frac{d^2A}{ds_2^2},$$

with like values for $H^{(T)}$ and $B^{(T)}$, while

$$p'^2 = p_1'^2 - 2vp_1'\gamma_{12} + (v^2\gamma_{12}^2 + 2vPp_1'),$$

with like values for $p'q'$, q'^2. The terms of the first order now disappear, for they had been made to balance; so the first form of the condition, surviving from the second-order terms, is

$$\begin{aligned}2(Ap_1' + Hq_1')vP + 2(Hp_1' + Bq_1')vQ \\ + \tfrac{1}{2}v^2\Big[p_1'^2\frac{d^2A}{ds_2^2} + 2p_1'q_1'\frac{d^2H}{ds_2^2} + q_1'^2\frac{d^2B}{ds_2^2} + 2(A\gamma_{12}^2 + 2H\gamma_{12}\delta_{12} + B\delta_{12}^2) \\ - 4\gamma_{12}\left(p_1'\frac{dA}{ds_2} + q_1'\frac{dH}{ds_2}\right) - 4\delta_{12}\left(p_1'\frac{dH}{ds_2} + q_1'\frac{dB}{ds_2}\right)\Big] = 0.\end{aligned}$$

The coefficient of $\frac{1}{2}v^2$ is, by the results in § 98,

$$\begin{aligned}= &-\tfrac{2}{3}V^2K(p_1'q_2' - q_1'p_2')^2 + 2(A\gamma_{12}^2 + 2H\gamma_{12}\delta_{12} + B\delta_{12}^2) \\ &+ 2(Ap_1' + Hq_1')(\Gamma_{111}\between p_1', q_1'\between p_2', q_2')^2 + 2(Hp_1' + Bq_1')(\Delta_{111}\between p_1', q_1'\between p_2', q_2')^2 \\ &+ 4(Ap_1' + Hq_1')(\alpha_2\gamma_{12} + \beta_2\delta_{12}) + 4(Hp_1' + Bq_1')(\epsilon_2\gamma_{12} + \eta_2\delta_{12}) \\ &+ 2(A\gamma_{12}^2 + 2H\gamma_{12}\delta_{12} + B\delta_{12}^2) \\ &- 4\gamma_{12}\{A\gamma_{12} + H\delta_{12} + \alpha_2(Ap_1' + Hq_1') + \epsilon_2(Hp_1' + Bq_1')\} \\ &- 4\delta_{12}\{H\gamma_{12} + B\delta_{12} + \beta_2(Ap_1' + Hq_1') + \eta_2(Hp_1' + Bq_1')\} \\ = &-\tfrac{2}{3}V^2K(p_1'q_2' - q_1'p_2')^2 \\ &+ 2(Ap_1' + Hq_1')(\Gamma_{111}\between p_1', q_1'\between p_2', q_2')^2 + 2(Hp_1' + Bq_1')(\Delta_{111}\between p_1', q_1'\between p_2', q_2')^2.\end{aligned}$$

Hence the condition, surviving from the second-order terms in the arc-relation, has the form

$$\begin{aligned}(Ap_1' + Hq_1')\{2P + v(\Gamma_{111}\between p_1', q_1'\between p_2', q_2')^2\} \\ + (Hp_1' + Bq_1')\{2Q + v(\Delta_{111}\between p_1', q_1'\between p_2', q_2')^2\} = \tfrac{1}{3}vV^2K(p_1'q_2' - q_1'p_2')^2.\end{aligned}$$

The p-relation for U, accurate up to the third order, is

$$\begin{aligned}xp' + \tfrac{1}{2}x^2p'' + \tfrac{1}{6}x^3p''' = up_1' + \tfrac{1}{2}u^2p_1'' + \tfrac{1}{6}u^3p_1''' \\ + v(p_4' - p_2') + \tfrac{1}{2}v^2(p_4'' - p_2'') + \tfrac{1}{6}v^3(p_4''' - p_2''').\end{aligned}$$

where p'' and p''' are the magnitudes at T and p_4'' and p_4''' are the magnitudes at R. The terms must be taken in turn.

(i) We have

$$xp' = (u + M)(p_1' - v\gamma_{12} + vP),$$

so that the third-order contribution is

$$Mp_1' + uvP.$$

(ii) Because, up to the third order inclusive, $x^2 = u^2$, we have

$$\tfrac{1}{2}x^2p'' = \tfrac{1}{2}u^2p'';$$

and so we require first-order terms in p''. Now, taken at T,

$$\begin{aligned} p'' &= -\Gamma_{11}^{(T)}p'^2 - 2\Gamma_{12}^{(T)}p'q' - \Gamma_{22}^{(T)}q'^2 \\ &= -\left(\Gamma_{11} + v\frac{d\Gamma_{11}}{ds_2}\right)(p_1'^2 - 2vp_1'\gamma_{12}) \\ &\quad -2\left(\Gamma_{12} + v\frac{d\Gamma_{12}}{ds_2}\right)(p_1'q_1' - vp_1'\delta_{12} - vq_1'\gamma_{12}) \\ &\quad -\left(\Gamma_{22} + v\frac{d\Gamma_{22}}{ds_2}\right)(q_1'^2 - 2vq_1'\delta_{12}), \end{aligned}$$

so that the first-order terms in p''

$$\begin{aligned} &= 2v(\alpha_1\gamma_{12} + \beta_1\delta_{12}) - v\left(p_1'^2\frac{d\Gamma_{11}}{ds_2} + 2p_1'q_1'\frac{d\Gamma_{12}}{ds_2} + q_1'^2\frac{d\Gamma_{22}}{ds_2}\right) \\ &= v\{\tfrac{2}{3}K(Hp_1' + Bq_1')(p_1'q_2' - q_1'p_2') - (\Gamma_{111}\between p_2', q_2'\between p_1', q_1')^2\}, \end{aligned}$$

by the result of p. 261. Thus the third-order contribution from the term $\frac{1}{2}x^2p''$ is

$$\tfrac{1}{3}u^2vK(Hp_1' + Bq_1')(p_1'q_2' - q_1'p_2') - \tfrac{1}{2}u^2v(\Gamma_{111}\between p_2', q_2'\between p_1', q_1')^2.$$

(iii) Up to the third order inclusive, $x^3 = u^3$; so

$$\tfrac{1}{6}x^3p''' = \tfrac{1}{6}u^3p''' = \tfrac{1}{6}u^3p_1''',$$

up to that retained order: the terms cancel.

(iv) We have

$$p_4' = p_2' - u\gamma_{12} + \tfrac{1}{2}u^2\frac{d^2p_2}{ds_1^2},$$

so that the third-order contribution from the term $v(p_4' - p_2')$

$$\begin{aligned} &= \tfrac{1}{2}u^2v\frac{d^2p_2}{ds_1^2} \\ &= -\tfrac{1}{6}u^2vK(Hp_1' + Bq_1')(p_1'q_2' - q_1'p_2') - \tfrac{1}{2}u^2v(\Gamma_{111}\between p_2', q_2'\between p_1', q_1')^2 \end{aligned}$$

(v) The value of p_4'' must be taken at R, so that

$$\begin{aligned} p_4'' &= -\Gamma_{11}^{(R)}p_4'^2 - 2\Gamma_{12}^{(R)}p_4'q_4' - \Gamma_{22}^{(R)}q_4'^2 \\ &= -\left(\Gamma_{11} + u\frac{d\Gamma_{11}}{ds_1}\right)(p_2'^2 - 2up_2'\gamma_{12}) \\ &\quad -2\left(\Gamma_{12} + u\frac{d\Gamma_{12}}{ds_1}\right)(p_2'q_2' - up_2'\delta_{12} - uq_2'\gamma_{12}) \\ &\quad -\left(\Gamma_{22} + u\frac{d\Gamma_{22}}{ds_1}\right)(q_2'^2 - 2uq_2'\delta_{12}); \end{aligned}$$

hence

$$\begin{aligned} p_4'' - p_2'' &= 2u(\alpha_2\gamma_{12} + \beta_2\delta_{12}) \\ &\quad - u\left(p_2'^2\frac{d\Gamma_{11}}{ds_1} + 2p_2'q_2'\frac{d\Gamma_{12}}{ds_1} + q_2'^2\frac{d\Gamma_{22}}{ds_1}\right) \\ &= u\{-\tfrac{2}{3}K(Hp_2' + Bq_2')(p_1'q_2' - q_1'p_2') - (\Gamma_{111}\between p_1', q_1'\between p_2', q_2')^2\}, \end{aligned}$$

again by the result of p. 261. Thus the third-order contribution from the term $\frac{1}{2}v^2(p_4'' - p_2'')$

$$= -\tfrac{1}{3}uv^2K(Hp_2' + Bq_2')(p_1'q_2' - q_1'p_2') - \tfrac{1}{2}uv^2(\Gamma_{111}\between p_1', q_1'\between p_2', q_2')^2.$$

(vi) The third-order contribution from the term $\frac{1}{6}v^3(p_4''' - p_2''')$ is at once seen to be zero.

Consequently the condition to be satisfied, in order that the third-order approximation in the p-relation may be effected, becomes

$$\begin{aligned} Mp_1' + uvP + \tfrac{1}{3}u^2vK(Hp_1' + Bq_1')(p_1'q_2' - q_1'p_2') \\ = -\tfrac{1}{6}u^2vK(Hp_1' + Bq_1')(p_1'q_2' - q_1'p_2') \\ - \tfrac{1}{3}uv^2K(Hp_2' + Bq_2')(p_1'q_2' - q_1'p_2') - \tfrac{1}{2}uv^2(\Gamma_{111}\between p_1', q_1'\between p_2', q_2')^2, \end{aligned}$$

that is,

$$\begin{aligned} Mp_1' + uv\{P + \tfrac{1}{2}v(\Gamma_{111}\between p_1', q_1'\between p_2', q_2')^2\} \\ = -uvK(p_1'q_2' - q_1'p_2')\{\tfrac{1}{2}u(Hp_1' + Bq_1') + \tfrac{1}{3}v(Hp_2' + Bq_2')\}. \end{aligned}$$

The corresponding third-order condition, arising from the q-relation, is

$$\begin{aligned} Mq_1' + uv\{Q + \tfrac{1}{2}v(\Delta_{111}\between p_1', q_1'\between p_2', q_2')^2\} \\ = uvK(p_1'q_2' - q_1'p_2')\{\tfrac{1}{2}u(Ap_1' + Hq_1') + \tfrac{1}{3}v(Ap_2' + Hq_2')\}. \end{aligned}$$

Thus there are three equations for the determination of M, P, Q. Let these last two conditions be multiplied by $Ap_1' + Hq_1'$ and $Hp_1' + Bq_1'$ respectively; then, when the earlier condition surviving from the arc-relation is used, we have

$$M + \tfrac{1}{6}uv^2KV^2(p_1'q_2' - q_1'p_2')^2 = -\tfrac{1}{3}uv^2KV^2(p_1'q_2' - q_1'p_2')^2,$$

and therefore

$$M = -\tfrac{1}{2}uv^2KV^2(p_1'q_2' - q_1'p_2')^2 = -\tfrac{1}{2}uv^2K\sin^2\epsilon.$$

Accordingly,

$$\begin{aligned} x &= u + M \\ &= u - \tfrac{1}{2}uv^2K\sin^2\epsilon; \end{aligned}$$

hence

$$u^2 - x^2 = u^2v^2K\sin^2\epsilon = K\square^2,$$

where $\square$ denotes the area of the small parallelogram $ORUT$. This result is Levi-Civita's theorem.

When the value of M is substituted, the values of P and Q are derived from the conditions arising from the p-relation and the q-relation respectively. It is convenient to modify the forms. We have (§ 121)

$$\frac{d^2p_1'}{ds_2^2} = -(\Gamma_{111}\between p_1', q_1'\between p_2', q_2')^2 - \tfrac{1}{3}(Hp_2' + Bq_2')(p_2'q_1' - p_1'q_2');$$

and therefore the condition from the p-relation can be expressed in the form

$$Mp_1' + uv\left(P - \tfrac{1}{2}v\frac{d^2p_1'}{ds_2^2}\right) = -\tfrac{1}{2}uvK(p_1'q_2' - q_1'p_2')\{u(Hp_1' + Bq_1') + v(Hp_2' + Bq_2')\};$$

while the corresponding condition from the q-relation becomes

$$Mq_1' + uv\left(Q - \tfrac{1}{2}v\frac{d^2q_1'}{ds_2^2}\right) = \tfrac{1}{2}uvK(p_1'q_2' - q_1'p_2')\{u(Ap_1' + Hq_1') + v(Ap_2' + Hq_2')\}.$$

When the value of M is substituted, we find, after a simple reduction,

$$\left.\begin{aligned} P - \tfrac{1}{2}v\frac{d^2p_1'}{ds_2^2} &= -\tfrac{1}{2}K(p_1'q_2' - q_1'p_2')(Hp_1' + Bq_1')(u + v\cos\epsilon) \\ Q - \tfrac{1}{2}v\frac{d^2q_1'}{ds_2^2} &= \tfrac{1}{2}K(p_1'q_2' - q_1'p_2')(Ap_1' + Hq_1')(u + v\cos\epsilon) \end{aligned}\right\},$$

thus completing the second approximation to the direction-variables of the geodesic TU at T in the direction TU, as given by

$$p' = p_1' - v\gamma_{12} + vP, \quad q' = q_1' - v\delta_{12} + vQ.$$

The foregoing form shews that, in the Levi-Civita parallelogram, the geodesic arc TU is not parallel to the geodesic OR. If δ denote the small deviation at T between this arc TU and the geodesic through T actually parallel to OR, we have

$$\begin{aligned} \frac{1}{V}\sin\delta &= \begin{vmatrix} p_1' - v\gamma_{12} + \tfrac{1}{2}v^2\dfrac{d^2p_1'}{ds_2^2}, & q_1' - v\delta_{12} + \tfrac{1}{2}v^2\dfrac{d^2q_1'}{ds_2^2} \\ p_1' - v\gamma_{12} + vP, & q_1' - v\delta_{12} + vQ \end{vmatrix} \\ &= \tfrac{1}{2}Kv(u + v\cos\epsilon)(p_1'q_2' - q_1'p_2'), \end{aligned}$$

to the retained order of approximation; and therefore the deviation δ is approximately equal to

$$\tfrac{1}{2}K(u + v\cos\epsilon)v\sin\epsilon.$$

The Pérès parallelogram.

124. We now pass to the consideration of a Pérès geodesic parallelogram. When this is represented in the figure (§ 119), RU is the geodesic parallel to OT and TU is the geodesic parallel to OR. The direction-variables of the geodesic TU at T in the direction TU are denoted by p_3', q_3'; those of the geodesic RU at R in the direction RU by p_4', q_4'; and we have

$$\left.\begin{aligned} p_3' &= p_1' - v\gamma_{12} + \tfrac{1}{2}v^2\frac{d^2p_1'}{ds_2^2} \\ q_3' &= q_1' - v\delta_{12} + \tfrac{1}{2}v^2\frac{d^2q_1'}{ds_2^2} \end{aligned}\right\}, \qquad \left.\begin{aligned} p_4' &= p_2' - u\gamma_{12} + \tfrac{1}{2}u^2\frac{d^2p_2'}{ds_1^2} \\ q_4' &= q_2' - u\delta_{12} + \tfrac{1}{2}u^2\frac{d^2q_2'}{ds_1^2} \end{aligned}\right\},$$

where the arc $OR = u$ and the arc $OT = v$. The lengths of TU and RU are denoted by x and y respectively; and we write

$$x = u + L, \quad y = v + M,$$

where L and M are certainly of the second order of small quantities. The initial requirement is the determination of L and M.

We proceed as before. The same values of the parameters at U must be attained, when we pass from O to U by the geodesic path ORU as when we pass by the geodesic path OTU. Now, along OTU, we have

$$\begin{aligned} p_U &= p_T + xp_3' + \tfrac{1}{2}x^2 p_3'' + \tfrac{1}{6}x^3 p_3''' \\ &= p + vp_2' + \tfrac{1}{2}v^2 p_2'' + \tfrac{1}{6}v^3 p_2''' + xp_3' + \tfrac{1}{2}x^2 p_3'' + \tfrac{1}{6}x^3 p_3''' ; \end{aligned}$$

along ORU, we similarly have

$$p_U = p + up_1' + \tfrac{1}{2}u^2 p_1'' + \tfrac{1}{6}u^3 p_1''' + yp_4' + \tfrac{1}{2}y^2 p_4'' + \tfrac{1}{6}y^3 p_4''' ;$$

in each instance, accurately up to the third order inclusive, while the values of p_4'' and p_4''' must be taken at R, and those of p_3'' and p_3''' must be taken at T. Hence

$$\begin{aligned} xp_3' - up_1' + \tfrac{1}{2}(x^2 p_3'' - u^2 p_1'') + \tfrac{1}{6}(x^3 p_3''' - u^3 p_1''') \\ = yp_4' - vp_2' + \tfrac{1}{2}(y^2 p_4'' - v^2 p_2'') + \tfrac{1}{6}(y^3 p_4''' - v^3 p_2'''), \end{aligned}$$

accurate to the retained third order. There is a similar relation arising out of the q-parameter at U.

When approximation only to the second order of small quantities inclusive is made, we have

$$\begin{aligned} xp_3' &= (u + L)(p_1' - v\gamma_{12}) = up_1' - uv\gamma_{12} + Lp_1', \\ yp_4' &= (v + M)(p_2' - u\gamma_{12}) = vp_2' - uv\gamma_{12} + Mp_2', \end{aligned}$$

so that

$$xp_3' - up_1' = -uv\gamma_{12} + Lp_1', \quad yp_4' - vp_2' = -uv\gamma_{12} + Mp_2',$$

the omitted terms in Lv and Mu being of order higher than L and M. Again,

$$\begin{aligned} p_3'' &= -\Gamma_{11}^{(T)} p_3'^2 - 2\Gamma_{12}^{(T)} p_3' q_3' - \Gamma_{22}^{(T)} q_3'^2 \\ &= -\left(\Gamma_{11} + v\frac{d\Gamma_{11}}{ds_2}\right)(p_1'^2 - 2vp_1'\gamma_{12}) \\ &\quad -2\left(\Gamma_{12} + v\frac{d\Gamma_{12}}{ds_2}\right)(p_1' q_1' - vp_1'\delta_{12} - vq_1'\gamma_{12}) \\ &\quad -\left(\Gamma_{22} + v\frac{d\Gamma_{22}}{ds_2}\right)(q_1'^2 - 2vq_1'\delta_{12}) \\ &= p_1'' + 2v(\alpha_1\gamma_{12} + \beta_1\delta_{12}) \\ &\quad - v\left(p_1'^2\frac{d\Gamma_{11}}{ds_2} + 2p_1' q_1'\frac{d\Gamma_{12}}{ds_2} + q_1'^2\frac{d\Gamma_{22}}{ds_2}\right) \\ &= p_1'' + v\{\tfrac{2}{3}K(Hp_1' + Bq_1')(p_1' q_2' - q_1' p_2') - (\Gamma_{111}\,)\!(\,p_2', q_2'\,)\!(\,p_1', q_1')^2\}, \end{aligned}$$

accurately up to the first order of small quantities inclusive, on using the formula of p. 261. Similarly,

$$p_4''=p_2''+u\{\tfrac{2}{3}K(Hp_2'+Bq_2')(p_2'q_1'-q_2'p_1')-(\Gamma_{111}\between p_1',\, q_1'\between p_2',\, q_2')^2\}.$$

Hence, for approximation up to the second order (and only to the second order) inclusive,

$$\tfrac{1}{2}x^2p_3''=\tfrac{1}{2}x^2p_1''=\tfrac{1}{2}u^2p_1'', \quad \tfrac{1}{2}y^2p_4''=\tfrac{1}{2}y^2p_2''=\tfrac{1}{2}v^2p_2''.$$

The terms involving p_3''', p_1''', p_4''', p_2''', are of the third order. Hence, up to a retained second order only, we have

$$Lp_1'-Mp_2'=0,$$

from the p-relation; and similarly, the q-relation gives

$$Lq_1'-Mq_2'=0.$$

Consequently, up to that retained second order, $L=0$, $M=0$: that is, L and M are magnitudes of the third order, at least.

We now take approximations to the third order (but only to the third order) of small quantities. Then

$$\begin{aligned} xp_3' &= (u+L)\left(p_1'-v\gamma_{12}+\tfrac{1}{2}v^2\frac{d^2p_1'}{ds_2^2}\right) \\ &= up_1'-uv\gamma_{12}+Lp_1'+\tfrac{1}{2}uv^2\frac{d^2p_1'}{ds_2^2}, \end{aligned}$$

so that

$$xp_3'-up_1'=-uv\gamma_{12}+Lp_1'+\tfrac{1}{2}uv^2\frac{d^2p_1'}{ds_2^2};$$

and similarly

$$yp_4'-vp_2'=-uv\gamma_{12}+Mp_2'+\tfrac{1}{2}u^2v\frac{d^2p_2'}{ds_1^2},$$

both accurately up to the retained third order.

Again, up to this order, we have

$$x^2=u^2, \quad y^2=v^2,$$

and the values of p_3'', p_4'', have been obtained, each accurately up to the first order. Hence

$$\begin{aligned} x^2p_3'' &= u^2p_3'' \\ &= u^2p_1''+u^2v\{\tfrac{2}{3}K(Hp_1'+Bq_1')(p_1'q_2'-q_1'p_2')-(\Gamma_{111}\between p_2',\, q_2'\between p_1',\, q_1')^2\}, \end{aligned}$$

so that

$$\tfrac{1}{2}(x^2p_3''-u^2p_1'')=\tfrac{1}{3}Ku^2v(Hp_1'+Bq_1')(p_1'q_2'-q_1'p_2')-\tfrac{1}{2}u^2v(\Gamma_{111}\between p_2',\, q_2'\between p_1',\, q_1')^2;$$

and similarly

$$\tfrac{1}{2}(y^2p_4''-v^2p_2'')=-\tfrac{1}{3}Kuv^2(Hp_2'+Bq_2')(p_1'q_2'-q_1'p_2')-\tfrac{1}{2}uv^2(\Gamma_{111}\between p_1',\, q_1'\between p_2',\, q_2')^2;$$

in each instance, up to the retained third order.

Further, up to that order,

$$x^3 p_3''' = u^3 p_1''', \quad y^3 p_4''' = v^3 p_2'''.$$

Hence the third-order condition, which survives from the p-relation, becomes

$$\begin{aligned} Lp_1' + \tfrac{1}{2}uv^2 \frac{d^2 p_1'}{ds_2^2} - \left(Mp_2' + \tfrac{1}{2}u^2 v \frac{d^2 p_2'}{ds_1^2}\right) \\ = -\tfrac{1}{3}Kuv(p_1'q_2' - q_1'p_2')\{u(Hp_1' + Bq_1') + v(Hp_2' + Bq_2')\} \\ + \tfrac{1}{2}u^2 v(\Gamma_{111} \between p_2', q_2' \between p_1', q_1')^2 - \tfrac{1}{2}uv^2(\Gamma_{111} \between p_1', q_1' \between p_2', q_2')^2. \end{aligned}$$

The values of $\dfrac{d^2 p_1'}{ds_2^2}$ and $\dfrac{d^2 p_2'}{ds_1^2}$ are given in § 121; when these values are substituted, the condition acquires the form

$$Lp_1' - Mp_2' = -\tfrac{1}{2}Kuv(p_1'q_2' - q_1'p_2')\{u(Hp_1' + Bq_1') + v(Hp_2' + Bq_2')\}.$$

Similarly the third-order condition, which survives from the q-relation, acquires the form

$$Lq_1' - Mq_2' = \tfrac{1}{2}Kuv(p_1'q_2' - q_1'p_2')\{u(Ap_1' + Hq_1') + v(Ap_2' + Hq_2')\}.$$

When these two conditions are resolved, so as to give the values of L and M, we find

$$L = -\tfrac{1}{2}Kuv(v + u\cos\epsilon), \quad M = -\tfrac{1}{2}Kuv(u + v\cos\epsilon).$$

Accordingly, the lengths of the two sides in the Pérès parallelogram, opposite to OR and OT respectively, are

$$\left.\begin{aligned} x &= u + L = u - \tfrac{1}{2}Kuv(v + u\cos\epsilon) \\ y &= v + M = v - \tfrac{1}{2}Kuv(u + v\cos\epsilon) \end{aligned}\right\}.$$

We note for future use that, if we take a quantity r, and direction-variables p_8', q_8', such that

$$rp_8' = up_1' + vp_2', \quad rq_8' = uq_1' + vq_2', \quad r^2 = u^2 + v^2 + 2uv\cos\epsilon,$$

so that p_8', q_8', is a direction through O, we have

$$\left.\begin{aligned} Lp_1' - Mp_2' &= -\tfrac{1}{2}Kruv(p_1'q_2' - q_1'p_2')(Hp_8' + Bq_8') \\ Lq_1' - Mq_2' &= \tfrac{1}{2}Kruv(p_1'q_2' - q_1'p_2')(Ap_8' + Hq_8') \end{aligned}\right\}.$$

125. The length of the geodesic diagonal OU of the Pérès parallelogram, and its direction-variables at O in the direction OU, can be obtained by similar analysis.

We first construct the values of the parameters p and q at U, up to the third order of small quantities inclusive. We have, by proceeding from O along the geodesic path ORU to U,

$$p_U = p + up_1' + \tfrac{1}{2}u^2 p_1'' + \tfrac{1}{6}u^3 p_1''' + yp_4' + \tfrac{1}{2}y^2 p_4'' + \tfrac{1}{6}y^3 p_4'''.$$

Up to the retained order, we have

$$yp_4' = (v+M)\left(p_2' - u\gamma_{12} + \tfrac{1}{2}u^2\frac{d^2p_2'}{ds_1^2}\right)$$
$$= vp_2' - uv\gamma_{12} + Mp_2' + \tfrac{1}{2}u^2v\frac{d^2p_2'}{ds_1^2}.$$

In $\frac{1}{2}y^2p_4''$, we can take $y^2 = v^2$ up to the required order; and then the value of p_4'', already (p. 332) obtained, is

$$p_4'' = p_2'' - u\{\tfrac{2}{3}K(Hp_2' + Bq_2')(p_1'q_2' - q_1'p_2') - (\Gamma_{111} \between p_1', q_1' \between p_2', q_2')^2\}.$$

In $\frac{1}{6}y^3p_4'''$, we can take $y^3 = v^3$ up to the required order; and then it is sufficient to take $p_4''' = p_2'''$. Also

$$\frac{d^2p_2'}{ds_1^2} = -\tfrac{1}{3}K(Hp_1' + Bq_1')(p_1'q_2' - q_1'p_2') - (\Gamma_{111} \between p_2', q_2' \between p_1', q_1')^2.$$

Hence we have

$$\begin{aligned} p_U - p = {} & up_1' + vp_2' \\ & + \tfrac{1}{2}(u^2p_1'' - 2uv\gamma_{12} + v^2p_2'') \\ & + \tfrac{1}{6}u^3p_1''' - \tfrac{1}{2}u^2v(\Gamma_{111} \between p_2', q_2' \between p_1', q_1')^2 - \tfrac{1}{2}uv^2(\Gamma_{111} \between p_1', q_1' \between p_2', q_2')^2 + \tfrac{1}{6}v^3p_2''' \\ & + Mp_2' - \tfrac{1}{6}uvK(p_1'q_2' - q_1'p_2')\{u(Hp_1' + Bq_1') + 2v(Hp_2' + Bq_2')\}. \end{aligned}$$

With the direction-variables p_8', q_8', defined (p. 337) by the relations

$$rp_8' = up_1' + vp_2', \quad rq_8' = uq_1' + vq_2', \quad r^2 = u^2 + v^2 + 2uv\cos\epsilon,$$

we have, for the second line in the expression of $p_U - p$,

$$-\tfrac{1}{2}(\Gamma_{11}, \Gamma_{12}, \Gamma_{22} \between up_1' + vp_2', uq_1' + vq_2')^2, \quad = \tfrac{1}{2}r^2p_8'';$$

and the third line in the same expression

$$= -\tfrac{1}{6}(\Gamma_{111}, \Gamma_{112}, \Gamma_{122}, \Gamma_{222} \between up_1' + vp_2', uq_1' + vq_2')^3, \quad = \tfrac{1}{6}r^3p_8'''.$$

When the value of M is substituted in the fourth line, the aggregate in the fourth line is equal to $-Kuv\Theta$, where

$$\begin{aligned} \Theta &= \tfrac{1}{2}p_2'(u + v\cos\epsilon) + \tfrac{1}{6}(p_1'q_2' - q_1'p_2')\{u(Hp_1' + Bq_1') + 2v(Hp_2' + Bq_2')\} \\ &= \tfrac{1}{2}p_2'(u + v\cos\epsilon) + \tfrac{1}{6}u(-p_2' + p_1'\cos\epsilon) + \tfrac{1}{3}v(p_1' - p_2'\cos\epsilon) \\ &= \tfrac{1}{3}(up_2' + vp_1') + \tfrac{1}{6}rp_8'\cos\epsilon, \end{aligned}$$

by the use of the relations on p. 327 (Note). Thus

$$p_U - p = rp_8' + \tfrac{1}{2}r^2p_8'' + \tfrac{1}{6}r^3p_8''' - \tfrac{1}{3}Kuv(up_2' + vp_1' + \tfrac{1}{2}rp_8'\cos\epsilon),$$

giving the value of the p-parameter at U.

Similarly the value of the q-parameter at U is given by

$$q_U - q = rq_8' + \tfrac{1}{2}r^2 q_8'' + \tfrac{1}{6}r^3 q_8''' - \tfrac{1}{3}Kuv(uq_2' + vq_1' + \tfrac{1}{2}rq_8' \cos\epsilon).$$

Each of the values is accurate up to the third order of small quantities inclusive.

These results enable us to determine the geodesic diagonal OU of the Pérès parallelogram, in magnitude and by direction-variables. The length of the geodesic arc we denote by w; as the quantity r is easily seen to be the first-order approximation to w, we write

$$w = r + W.$$

The direction-variables at O, of the geodesic OU in the direction OU, we denote by p_6', q_6'; as p_8', q_8', are easily seen to be the finite approximations to p_6', q_6', respectively, we write

$$p_6' = p_8' + P, \quad q_6' = q_8' + Q.$$

All the quantities P, Q, W, are of order smaller than the magnitudes of which they are the small increments.

As U is the extremity of this geodesic diagonal OU, we have

$$p_U - p = wp_6' + \tfrac{1}{2}w^2 p_6'' + \tfrac{1}{6}w^3 p_6''',$$

accurately up to the third order inclusive. Now, up to the second order,

$$wp_6' = rp_8' + Wp_8' + rP,$$
$$w^2 p_6'' = r^2 p_8'',$$

the omitted terms being of higher order. Hence, equating the new value of $p_U - p$ with the earlier value, we find that the first-order terms cancel: also that, to balance the second-order terms, the condition

$$Wp_8' + rP = 0$$

must be satisfied. Similarly, the condition

$$Wq_8' + rQ = 0$$

survives from the second-order terms in the equality of the second-order terms in $q_U - q$.

Further, we have the arc-relation

$$Ap_6'^2 + 2Hp_6'q_6' + Bq_6'^2 = 1$$

at O. When we substitute for p_6' and q_6', the finite terms cancel; and for the next approximation, we have

$$(Ap_8' + Hq_8')P + (Hp_8' + Bq_8')Q = 0.$$

When these three relations are combined, we find

$$W = 0, \quad P = 0, \quad Q = 0:$$

that is, W is of the third order at least, while P and Q are of the second order at least.

With this knowledge, we take the third-order terms in the relations which arise by equating the two values of $p_U - p$ and the two values of $q_U - q$ respectively. Up to the third order of small quantities inclusive, we now have

$$wp_6' = rp_8' + (Wp_8' + rP)$$

as before, the bracketed quantity being of the third order. Also

$$\begin{aligned} -p_6'' &= \Gamma_{11} p_6'^2 + 2\Gamma_{12} p_6' q_6' + \Gamma_{22} q_6'^2 \\ &= \Gamma_{11} p_8'^2 + 2\Gamma_{12} p_8' q_8' + \Gamma_{22} q_8'^2 = -p_8'', \end{aligned}$$

accurately up to the first order inclusive, as P and Q are of the second order; also $w^2 = r^2$, accurately up to the third order; and therefore, up to the retained third order inclusive,

$$w^2 p_6'' = r^2 p_8''.$$

Likewise, up to this same order,

$$w^3 p_6''' = r^3 p_8'''.$$

Hence, as arising out of third-order terms in the equality of the values of $p_U - p$, the condition

$$Wp_8' + rP = -\tfrac{1}{3} Kuv (up_2' + vp_1' + \tfrac{1}{2} rp_8' \cos \epsilon)$$

must be satisfied.

Similarly, as arising out of the third-order terms in the equality of the values $q_U - q$, the condition

$$Wq_8' + rQ = -\tfrac{1}{3} Kuv (uq_2' + vp_1' + \tfrac{1}{2} rq_8' \cos \epsilon)$$

must be satisfied.

The condition arising out of the permanent arc-relation at O is

$$(Ap_8' + Hq_8')P + (Hp_8' + Bq_8')Q = 0,$$

the same as before, P and Q now being known to be of the second order of small quantities.

When the p-condition is multiplied by $Ap_8' + Hq_8'$, the q-condition by $Hp_8' + Bq_8'$, and the results are added, then using the condition from the arc-relation, we find, after some simple reductions

$$W = -K \frac{uv}{r} (\tfrac{1}{2} r^2 \cos \epsilon + \tfrac{2}{3} uv \sin^2 \epsilon),$$

accurately up to the third order inclusive, while r is given by

$$r^2 = u^2 + v^2 + 2uv \cos \epsilon.$$

Again, we have

$$r(Pq_8' - Qp_8') = -\tfrac{1}{3} Kuv \{u(p_2' q_8' - q_2' p_8') + v(p_1' q_8' - q_1' p_8')\};$$

but

$$r\{u(p_2'q_3'-q_2'p_3')+v(p_1'q_3'-q_1'p_3')\}$$
$$=(up_2'+vp_1')(uq_1'+vq_2')-(uq_2'+vq_1')(up_1'+vp_2')$$
$$=(v^2-u^2)(p_1'q_2'-q_1'p_2'),$$

so that

$$Pq_3'-Qp_3'=\frac{1}{3r^2}Kuv(u^2-v^2)(p_1'q_2'-q_1'p_2').$$

Consequently

$$P=\tfrac{1}{3}K\frac{uv(u^2-v^2)}{u^2+v^2+2uv\cos\epsilon}(p_1'q_2'-q_1'p_2')(Hp_3'+Bq_3'),$$

$$Q=\tfrac{1}{3}K\frac{uv(v^2-u^2)}{u^2+v^2+2uv\cos\epsilon}(p_1'q_2'-q_1'p_2')(Ap_3'+Hq_3');$$

and by use of the formulæ in § 121, Note, these can be expressed in the forms

$$\left.\begin{aligned}P&=\tfrac{1}{3}K\frac{uv(u^2-v^2)}{(u^2+v^2+2uv\cos\epsilon)^{\frac{3}{2}}}\{(v+u\cos\epsilon)p_1'-(u+v\cos\epsilon)p_2'\}\\ Q&=\tfrac{1}{3}K\frac{uv(u^2-v^2)}{(u^2+v^2+2uv\cos\epsilon)^{\frac{3}{2}}}\{(v+u\cos\epsilon)q_1'-(u+v\cos\epsilon)q_2'\}\end{aligned}\right\}.$$

The length of the diagonal geodesic arc OU is

$$r+W;$$

and the direction-variables of that geodesic at O in the direction OU are

$$\frac{1}{r}(up_1'+vp_2')+P,\quad \frac{1}{r}(uq_1'+vq_2')+Q.$$

Ex. It is easy to verify the result, as regards the approximate length of the diagonal of the Pérès parallelogram, in a simple instance for the sphere.

The sphere is taken to be of radius unity, so that $K=1$. The angles at O, R, T, equal to one another, are taken to be $\tfrac{1}{2}\pi$: and so

$$\cos OU=\cos OR\cos RU.$$

Now

$$OR=u,\quad RU=v-\tfrac{1}{2}u^2v,$$

the value of OR being accurate, and that of UR approximately accurate (up to the third order of small quantities). Hence, accurately up to the fourth order inclusive, if $OU=w$,

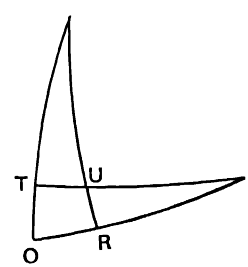

FIG. 12.

$$1-\tfrac{1}{2}w^2+\tfrac{1}{24}w^4=(1-\tfrac{1}{2}u^2+\tfrac{1}{24}u^4)(1-\tfrac{1}{2}v^2+\tfrac{1}{2}u^2v^2+\tfrac{1}{24}v^4).$$

Then, taking

$$r^2=u^2+v^2,\quad w=r+W,$$

we have, accurately up to the fourth order,

$$1-\tfrac{1}{2}r^2-rW+\tfrac{1}{24}r^4=1-\tfrac{1}{2}u^2-\tfrac{1}{2}v^2+\tfrac{3}{4}u^2v^2+\tfrac{1}{24}u^4+\tfrac{1}{24}v^4,$$

that is,
$$-rW=\tfrac{2}{3}u^2v^2,$$
or
$$w=r-\tfrac{2}{3}\frac{u^2v^2}{r},$$
which is in accordance with the general result when $\epsilon=\tfrac{1}{2}\pi$.

126. Next, we require the angle of the Pérès parallelogram at U. It is approximately equal to ϵ, so that $U-\epsilon$ is small. This small quantity will contain no term in u, because $\frac{d\epsilon}{ds_1}$ is zero when geodesics parallel to OT are drawn (§ 118): likewise it will contain no term in v, because $\frac{d\epsilon}{ds_2}$ is zero when geodesics parallel to OR are drawn. Thus $U-\epsilon$ will be a small quantity of the second order at least. Further, with a set of parallel geodesics along ORE as a base, ϵ is unchanged, so that the coefficient of u^2 in $U-\epsilon$ will be zero; and with a set of parallel geodesics along OTF as a base, ϵ is unchanged, so that the coefficient of v^2 in $U-\epsilon$ will be zero. Consequently the second-order approximation will contain uv as a factor; to find its actual expression, we proceed to the second-order approximations.

The values of p and q at U are known (§ 125) up to the third order of small quantities; but only the terms up to the second order inclusive are wanted for the required estimates of A, H, B, at U. With the notation of § 121, we have
$$p_U=p+rp_8'+\tfrac{1}{2}r^2p_8'',\quad q_U=q+rq_8'+\tfrac{1}{2}r^2q_8'';$$
and therefore
$$A^{(U)}=A+r\frac{dA}{ds_8}+\tfrac{1}{2}r^2\frac{d^2A}{ds_8^{\,2}},$$
$$H^{(U)}=H+r\frac{dH}{ds_8}+\tfrac{1}{2}r^2\frac{d^2H}{ds_8^{\,2}},$$
$$B^{(U)}=B+r\frac{dB}{ds_8}+\tfrac{1}{2}r^2\frac{d^2B}{ds_8^{\,2}},$$
up to the second order inclusive, the values of the coefficients of r and of $\tfrac{1}{2}r^2$ being those in § 98 when $i=8$.

The direction-variables at U of the geodesics TUJ and RUI in the respective directions UJ and UI are required. For the geodesic direction UJ, the value at U of the p'-variable is
$$p_J'=p_3'+xp_3''+\tfrac{1}{2}x^2p_3''',$$
second-order terms being retained. Now (§ 124)
$$p_3'=p_1'-v\gamma_{12}+\tfrac{1}{2}v^2\frac{d^2p_1'}{ds_2^{\,2}},$$
$$p_3''=p_1''+v\{\tfrac{2}{3}K(Hp_1'+Bq_1')(p_1'q_2'-q_1'p_2')-(\Gamma_{111}\between p_2',q_2'\between p_1',q_1')^2\},$$
and, up to this order, $x=u$; hence
$$p_J'=p_1'+P_J+R_J,$$

where P_J and R_J are aggregates of the first order and the second order respectively, their values being

$$P_J = up_1'' - v\gamma_{12},$$

$$R_J = \tfrac{1}{2}u^2 p_1''' + uv\{\tfrac{2}{3}K(Hp_1' + Bq_1')(p_1'q_2' - q_1'p_2') - (\Gamma_{111}\between p_2', q_2'\between p_1', q_1')^2\} + \tfrac{1}{2}v^2\frac{d^2p_1'}{ds_2^2}.$$

Similarly, up to the same retained second order, we have

$$q_J' = q_1' + Q_J + S_J,$$

where

$$Q_J = uq_1'' - v\delta_{12},$$

$$S_J = \tfrac{1}{2}u^2 q_1''' + uv\{\tfrac{2}{3}K(Ap_1' + Hq_1')(p_2'q_1' - q_2'p_1') - (\Delta_{111}\between p_2', q_2'\between p_1', q_1')^2\} + \tfrac{1}{2}v^2\frac{d^2q_1'}{ds_2^2}.$$

Modifications in the forms of R_J and S_J can be made: they are deferred until the magnitudes come into combinations with other like magnitudes.

For the geodesic direction UI, the values at U of the p'-variable and the q'-variable are

$$p_I' = p_2' + P_I + R_I,$$

$$q_I' = q_2' + Q_I + S_I,$$

where

$$P_I = vp_2'' - u\gamma_{12},$$

$$R_I = \tfrac{1}{2}v^2 p_2''' + uv\{\tfrac{2}{3}K(Hp_2' + Bq_2')(q_1'p_2' - q_2'p_1') - (\Gamma_{111}\between p_1', q_1'\between p_2', q_2')^2\} + \tfrac{1}{2}u^2\frac{d^2p_2'}{ds_1^2},$$

$$Q_I = vq_2'' - u\delta_{12},$$

$$S_I = \tfrac{1}{2}v^2 q_2''' + uv\{\tfrac{2}{3}K(Ap_2' + Hq_2')(p_1'q_2' - q_1'p_2') - (\Delta_{111}\between p_1', q_1'\between p_2', q_2')^2\} + \tfrac{1}{2}u^2\frac{d^2q_2'}{ds_1^2}.$$

Denoting the angle JUI by U, we have

$$\cos U = A^{(U)}p_I'p_J' + H^{(U)}(p_I'q_J' + p_J'q_I') + B^{(U)}q_I'q_J';$$

and we are to expand the right-hand side in powers of the small quantities u and v, retaining terms up to the second order inclusive.

The finite terms in the expression for $\cos U$

$$= Ap_1'p_2' + H(p_1'q_2' + q_1'p_2') + Bq_1'q_2' = \cos\epsilon.$$

The aggregate of all the terms of the first order of small quantities in that expression

$$= r\left\{p_1'p_2'\frac{dA}{ds_8} + (p_1'q_2' + q_1'p_2')\frac{dH}{ds_8} + q_1'q_2'\frac{dB}{ds_8}\right\}$$
$$+ A(p_1'P_I + p_2'P_J) + H(p_1'Q_I + p_2'Q_J + q_1'P_I + q_2'P_J) + B(q_1'Q_I + q_2'Q_J).$$

With the values of the first derivatives of A, H, B, as given in § 98, the coefficient of A in the first line

$$= r\{2\alpha_8 p_1'p_2' + \beta_8(p_1'q_2' + q_1'p_2')\}$$
$$= 2p_1'p_2'(u\alpha_1 + v\alpha_2) + (p_1'q_2' + q_1'p_2')(u\beta_1 + v\beta_2)$$
$$= u(p_1'\gamma_{12} - p_2'p_1'') + v(p_2'\gamma_{12} - p_1'p_2'')$$
$$= -p_1'P_I - p_2'P_J;$$

and similarly for the coefficients of H and B in that first line. The whole expression vanishes; and therefore there are no terms of the first order of small quantities in the expression for $\cos U$.

The aggregate of all the terms of the second order of small quantities in the expression for $\cos U$ is Ω, where

$$\begin{aligned}\Omega=\tfrac{1}{2}r^2&\left\{p_1'p_2'\frac{d^2A}{ds_8{}^2}+(p_1'q_2'+q_1'p_2')\frac{d^2H}{ds_8{}^2}+q_1'q_2'\frac{d^2B}{ds_8{}^2}\right\}\\&+(Ap_1'+Hq_1')R_I+(Hp_1'+Bq_1')S_I+(Ap_2'+Hq_2')R_J+(Hp_2'+Bq_2')S_J\\&+r\left\{\frac{dA}{ds_8}(p_1'P_I+p_2'P_J)+\frac{dH}{ds_8}(p_1'Q_I+p_2'Q_J+q_1'P_I+q_2'P_J)+\frac{dB}{ds_8}(q_1'Q_I+q_2'Q_J)\right\}\\&+AP_IP_J+H(P_IQ_J+Q_IP_J)+BQ_IQ_J.\end{aligned}$$

In Ω, the terms can be arranged in three sets after substitution for the various coefficients is effected, (i) those involving K, (ii) those involving the quantities Γ_{ijk} and Δ_{ijk}, and (iii) those free from K, Γ_{ijk}, Δ_{ijk}. We assume the expression for the first line as given in § 98 when we take $j=k=1$, $i=8$.

(i). The first set is composed from terms arising out of the first and second lines in Ω. Their aggregate

$$=\tfrac{1}{2}r^2(-\tfrac{2}{3}V^2K)(p_1'q_8'-q_1'p_8')(p_2'q_8'-q_2'p_8')+K(p_1'q_2'-q_1'p_2')\Phi,$$

where

$$\begin{aligned}\Phi=(Ap_1'+Hq_1')&\{-\tfrac{2}{3}uv(Hp_2'+Bq_2')-\tfrac{1}{2}u^2(Hp_1'+Bq_1')\}\\&+(Ap_2'+Hq_2')\{\quad\tfrac{2}{3}uv(Hp_1'+Bq_1')+\tfrac{1}{2}v^2(Hp_2'+Bq_2')\}\\&+(Hp_1'+Bq_1')\{\quad\tfrac{2}{3}uv(Ap_2'+Hq_2')+\tfrac{1}{2}u^2(Ap_1'+Hq_1')\}\\&+(Hp_2'+Bq_2')\{-\tfrac{2}{3}uv(Ap_1'+Hq_1')-\tfrac{1}{2}v^2(Ap_2'+Hq_2')\}\\=-\tfrac{4}{3}uvV^2&(p_1'q_2'-q_1'p_2').\end{aligned}$$

Also

$$r(p_1'q_8'-q_1'p_8')=v(p_1'q_2'-q_1'p_2'),\quad r(p_2'q_8'-q_2'p_8')=-u(p_1'q_2'-q_1'p_2').$$

Hence the aggregate

$$\begin{aligned}&=\ \tfrac{1}{3}uvV^2K(p_1'q_2'-q_1'p_2')^2-\tfrac{4}{3}uvV^2K(p_1'q_2'-q_1'p_2')^2\\&=-uvV^2K(p_1'q_2'-q_1'p_2')^2=-Kuv\sin^2\epsilon.\end{aligned}$$

(ii). The second set of terms, being those which involve Γ_{ijk} and Δ_{ijk}, is composed by selections from the first and second lines in Ω. The aggregate of terms, which involve the magnitudes Γ_{ijk},

$$\begin{aligned}=\tfrac{1}{2}r^2\{&(Ap_1'+Hq_1')(\Gamma_{111}\between p_2',q_2'\between p_8',q_8')^2+(Ap_2'+Hq_2')(\Gamma_{111}\between p_1',q_1'\between p_8',q_8')^2\}\\&+(Ap_1'+Hq_1')\{-uv(\Gamma_{111}\between p_1',q_1'\between p_2',q_2')^2-\tfrac{1}{2}u^2(\Gamma_{111}\between p_2',q_2'\between p_1',q_1')^2+\tfrac{1}{2}v^2p_2'''\}\\&+(Ap_2'+Hq_2')\{-uv(\Gamma_{111}\between p_2',q_2'\between p_1',q_1')^2-\tfrac{1}{2}v^2(\Gamma_{111}\between p_1',q_1'\between p_2',q_2')^2+\tfrac{1}{2}u^2p_1'''\}.\end{aligned}$$

The coefficient of $Ap_1' + Hq_1'$ in the second line

$$= -\tfrac{1}{2}r^2(\Gamma_{111} \between p_2', q_2' \between p_8', q_8')^2;$$

and the coefficient of $Ap_2' + Hq_2'$ in the third line

$$= -\tfrac{1}{2}r^2(\Gamma_{111} \between p_1', q_1' \between p_8', q_8')^2.$$

Consequently the aggregate of terms, involving the quantities Γ_{ijk}, is zero.

Similarly the aggregate of terms, involving the quantities Δ_{ijk}, is zero.

Hence the contribution of the second set of terms to Ω, involving the quantities Γ_{ijk} and Δ_{ijk}, is zero.

(iii). The remaining terms in Ω, being those which are independent of K and of the quantities Γ_{ijk}, Δ_{ijk},

$$\begin{aligned}=r^2[(Ap_1' + Hq_1')(\alpha_8\gamma_{28} + \beta_8\delta_{28}) + (Ap_2' + Hq_2')(\alpha_8\gamma_{18} + \beta_8\delta_{18}) \\ + (Hp_1' + Bq_1')(\epsilon_8\gamma_{28} + \eta_8\delta_{28}) + (Hp_2' + Bq_2')(\epsilon_8\gamma_{18} + \eta_8\delta_{18}) \\ + A\gamma_{18}\gamma_{28} + H(\gamma_{18}\delta_{28} + \gamma_{28}\delta_{18}) + B\delta_{18}\delta_{28}],\end{aligned}$$

arising out of the first line, together with the third and fourth lines in the expression for Ω.

Now

$$\begin{aligned}r\gamma_{18} &= \alpha_1 rp_8' + \beta_1 rq_8' \\ &= \alpha_1(up_1' + vp_2') + \beta_1(uq_1' + vq_2') \\ &= -up_1'' + v\gamma_{12} = -P_J,\end{aligned}$$

and so for the other quantities of this type: the tale is

$$r\gamma_{18} = -P_J, \quad r\delta_{18} = -Q_J; \quad r\gamma_{28} = -P_I, \quad r\delta_{28} = -Q_I.$$

The fourth line in Ω therefore

$$\begin{aligned}&= AP_IP_J + H(P_IQ_J + Q_IP_J) + BQ_IQ_J \\ &= r^2\{A\gamma_{18}\gamma_{28} + H(\gamma_{18}\delta_{28} + \gamma_{28}\delta_{18}) + B\delta_{18}\delta_{28}\}.\end{aligned}$$

The third line in Ω

$$\begin{aligned}= rP_I\left(p_1'\frac{dA}{ds_8} + q_1'\frac{dH}{ds_8}\right) + rQ_I\left(p_1'\frac{dH}{ds_8} + q_1'\frac{dB}{ds_8}\right) \\ + rP_J\left(p_2'\frac{dA}{ds_8} + q_2'\frac{dH}{ds_8}\right) + rQ_J\left(p_2'\frac{dH}{ds_8} + q_2'\frac{dH}{ds_8}\right),\end{aligned}$$

which, by means of the formulæ in § 98, is equal to

$$\begin{aligned}&rP_I\{(Ap_1' + Hq_1')\alpha_8 + (Hp_1' + Bq_1')\epsilon_8 + A\gamma_{18} + H\delta_{18}\} \\ &+ rQ_I\{(Ap_1' + Hq_1')\beta_8 + (Hp_1' + Bq_1')\eta_8 + H\gamma_{18} + B\delta_{18}\} \\ &+ rP_J\{(Ap_2' + Hq_2')\alpha_8 + (Hp_2' + Bq_2')\epsilon_8 + A\gamma_{28} + H\delta_{28}\} \\ &+ rQ_J\{(Ap_2' + Hq_2')\beta_8 + (Hp_2' + Bq_2')\eta_8 + H\gamma_{28} + B\delta_{28}\}.\end{aligned}$$

When we substitute for P_I, Q_I, P_J, Q_J, the values $-r\gamma_{28}$, $-r\delta_{28}$, $-r\gamma_{18}$, $-r\delta_{18}$, in this expression, we see that it cancels the foregoing aggregate of terms arising out of the first line as well as the modified aggregate of the fourth line.

Consequently the aggregate of terms in the set (iii) in Ω, being those which are independent of K and the quantities Γ_{ijk}, Δ_{ijk}, is zero. Hence the aggregate of the second-order terms, in the expansion of the expression for cos U in powers of small quantities, is given by

$$\Omega = -Kuv\sin^2\epsilon.$$

Thus, up to the second order of small quantities inclusive, we have

$$\cos U = \cos\epsilon - Kuv\sin^2\epsilon.$$

The magnitude $U-\epsilon$ is small; and, up to the required second order of small quantities, we thus have

$$U-\epsilon = Kuv\sin\epsilon.$$

This is the theorem (§ 120) due to Pérès, $uv\sin\epsilon$ being approximately (that is, to the retained order of small quantities) the area of the geodesic parallelogram. It has already been inferred (§ 113) from the corresponding proposition for a triangle.

Other developments are made when a surface, instead of being regarded as a configuration existing in its plenary space, is geodesic to some containing amplitude. The properties are affected by the geometry of that amplitude, and their establishment will be made in association with that geometry.

Geodesic parallelogram, with opposite sides equal.

127. The geodesic parallelograms, due to Levi-Civita and to Pérès and considered in §§ 122, 124, are defined: the former, by equality of angles at R and O (see Fig. 10, p. 320) with equal sides RU and OT; the latter, by a double equality of angles, at R and O, and at T and O. In a Euclidean parallelogram, a definition by equality of opposite sides leads to the same figure *. We therefore shall consider a convex geodesic quadrilateral, having the side TU equal to OR and the side UR equal to OT.

The direction-variables of OR are p_1', q_1'; those of OT are p_2', q_2'; also $OR=u=TU$, $OT=v=RU$, where u and v are small. After preceding investigations, the convex quadrilateral can be expected to have some properties connected with parallelism. We shall assume initially (and later shall justify this initial assumption) that, if p_3', q_3', are the direction-variables of TU, and p_4', q_4', are those of RU, we have

$$p_3' = p_1' - v\gamma_{12} + P_3, \quad p_4' = p_2' - u\gamma_{12} + P_4,$$
$$q_3' = q_1' - v\delta_{12} + Q_3, \quad q_4' = q_2' - u\delta_{12} + Q_4,$$

where

$$\gamma_{12} = \sum \Gamma_{11} p_1' p_2', \qquad \delta_{12} = \sum \Delta_{11} p_1' p_2',$$

* Two quadrilaterals can be constructed with these sides, RUT and $RU'T$ being equal triangles on opposite sides of TR; then $RU'TOR$ is such a quadrilateral but it is not convex, while $RUTOR$ is a parallelogram.

while P_3, Q_3, P_4, Q_4, are small quantities of the second order. These forms for the direction-variables of TU and RU, so far as concerns finite terms and terms of the first order, are in accord with the corresponding terms, in the direction-variables of RU in the Levi-Civita parallelogram, and in the direction-variables of RU and TU in the Pérès parallelogram.

As relations serving to justify the assumptions specified and to determine P_3, Q_3, P_4, Q_4, we have (i), the arc-relations

$$\sum A^{(T)} p_3'^2 = 1, \quad \sum A^{(R)} p_4'^2 = 1,$$

at T and R respectively: and (ii), the conditions arising from the necessity of having the same values for p and for q at U, by the path ORU as by the path OTU. In all instances, the equations must be satisfied up to the order of approximations retained, that order being the second for the direction-variables and being the third for the parameters.

(i). We begin with the arc-relations. At T, we have

$$A^{(T)} = A + v\frac{dA}{ds_2} + \tfrac{1}{2}v^2\frac{d^2A}{ds_2^2},$$

with similar values for $H^{(T)}$ and $B^{(T)}$; up to the retained order,

$$p_3'^2 = p_1'^2 - 2vp_1'\gamma_{12} + v^2\gamma_{12}^2 + 2p_1'P_3,$$

with similar values for $p_3'q_3'$ and $q_3'^2$; hence the relation becomes

$$\sum\left(A + v\frac{dA}{ds_2} + \tfrac{1}{2}v^2\frac{d^2A}{ds_2^2}\right)(p_1'^2 - 2vp_1'\gamma_{12} + v^2\gamma_{12}^2 + 2p_1'P_3) = 1.$$

On the left-hand side, the aggregate of finite terms

$$= \sum A p_1'^2 = 1,$$

thus cancelling the right-hand side. The aggregate of terms of the first order of small quantities on the left-hand side

$$= v\left(\sum\frac{dA}{ds_2}p_1'^2\right) - 2v\{(Ap_1' + Hq_1')\gamma_{12} + (Hp_1' + Bq_1')\delta_{12}\}:$$

but, by the results of § 98,

$$\begin{aligned}\sum\frac{dA}{ds_2}p_1'^2 &= 2[(A\alpha_2 + H\epsilon_2)p_1'^2 + (A\beta_2 + H\eta_2 + H\alpha_2 + B\epsilon_2)p_1'q_1' + (H\beta_2 + B\eta_2)q_1'^2]\\ &= 2[(Ap_1' + Hq_1')(\alpha_2 p_1' + \beta_2 q_1') + (Hp_1' + Bq_1')(\epsilon_2 p_1' + \eta_2 q_1')]\\ &= 2\{(Ap_1' + Hq_1')\gamma_{12} + (Hp_1' + Bq_1')\delta_{12}\},\end{aligned}$$

and therefore the aggregate of terms of the first order of small quantities is zero. The aggregate of terms of the second order is to vanish; and thus there is a residuary second-order condition

$$2\sum Ap_1'P_3 + v^2\sum A\gamma_{12}^2 - 2v^2\sum\frac{dA}{ds_2}p_1'\gamma_{12} + \tfrac{1}{2}v^2\sum\frac{d^2A}{ds_2^2}p_1'^2 = 0.$$

When substitution is made from § 98 for the arc-derivatives of A, H, B, and reduction is effected as in §§ 123, 124, this second-order condition can be expressed in the form

$$(Ap_1'+Hq_1')\{P_3+\tfrac{1}{2}v^2(\textstyle\sum\Gamma_{111}p_1'p_2'^2)\}$$
$$+(Hp_1'+Bq_1')\{Q_3+\tfrac{1}{2}v^2(\textstyle\sum\Delta_{111}p_1'p_2'^2)\}=\tfrac{1}{6}Kv^2\sin^2\epsilon,$$

where ϵ is the angle UOV.

It thus appears that the arc-relation at T is satisfied by the assumed forms of p_3' and q_3', provided this second-order condition is satisfied; and the condition is, in fact, an equation involving the unknown quantities P_3 and Q_3.

(ii). We proceed in the same manner with the arc-relation at R; and it similarly appears that the arc-relation is satisfied by the assumed forms of p_4' and q_4', provided a second-order condition

$$(Ap_2'+Hq_2')\{P_4+\tfrac{1}{2}u^2(\textstyle\sum\Gamma_{111}p_1'^2p_2')\}$$
$$+(Hp_2'+Bq_2')\{Q_4+\tfrac{1}{2}u^2(\textstyle\sum\Delta_{111}p_1'^2p_2')\}=\tfrac{1}{6}Ku^2\sin^2\epsilon$$

is satisfied. This residuary condition is, in fact, an equation involving the unknown quantities P_4 and Q_4.

(iii). In order that the p-parameter may acquire at U the same value by the path ORU as by the path OTU, the condition

$$p+up_1'+\tfrac{1}{2}u^2p_1''+\tfrac{1}{6}u^3p_1'''+vp_4'+\tfrac{1}{2}v^2p_4''+\tfrac{1}{6}v^3p_4'''$$
$$=p+vp_2'+\tfrac{1}{2}v^2p_2''+\tfrac{1}{6}v^3p_2'''+up_3'+\tfrac{1}{2}u^2p_3''+\tfrac{1}{6}u^3p_3'''$$

must be satisfied, the values of p_3'' and p_3''' being taken at T, and the values of p_4'' and p_4''' being taken at R.

Now

$$u^3p_3'''=-u^3\textstyle\sum\Gamma_{111}^{(T)}p_3'^3,$$

and we are retaining approximations towards the values of p and q only up to the third order of small quantities inclusive; in this term, therefore, we may take $\Gamma_{111}^{(T)}=\Gamma_{111}$, $p_3'=p_1'$; hence

$$u^3p_3'''=u^3p_1''',$$

and, similarly,

$$v^3p_4'''=v^3p_2'''.$$

Again, we have

$$p_3''=-\textstyle\sum\Gamma_{11}^{(T)}p_3'^2,$$

and p_3'' has a factor x^2, so that its value up to the first order must be retained; consequently

$$p_3''=-\sum\left(\Gamma_{11}+v\frac{d\Gamma_{11}}{ds_2}\right)(p_1'^2-2vp_1'\gamma_{12})$$
$$=p_1''+2v\sum\Gamma_{11}p_1'\gamma_{12}-v\sum\frac{d\Gamma_{11}}{ds_2}p_1'^2,$$

up to the required order. By the results in § 98, when $i=1, j=1, k=2$, we have

$$\sum p_1'^2 \frac{d\Gamma_{11}}{ds_2} = -\tfrac{2}{3}K(Hp_1'+Bq_1')(p_1'q_2'-p_2'q_1') + \sum \Gamma_{111}p_1'^2 p_2' + p_2'(2\alpha_1^2+2\beta_1\epsilon_1) + q_2'(2\alpha_1\beta_1+2\beta_1\eta_1)$$
$$= -\tfrac{2}{3}K(Hp_1'+Bq_1')(p_1'q_2'-p_2'q_1') + \sum \Gamma_{111}p_1'^2 p_2' + 2\sum \Gamma_{11}p_1'\gamma_{12} ;$$

and therefore, up to the first order of small quantities inclusive,

$$p_3'' = p_1'' + \tfrac{2}{3}vK(Hp_1'+Bq_1')(p_1'q_2'-p_2'q_1') - v\sum \Gamma_{111}p_1'^2 p_2'.$$

Similarly, up to the same order inclusive,

$$p_4'' = p_2'' - \tfrac{2}{3}uK(Hp_2'+Bq_2')(p_1'q_2'-p_2'q_1') - u\sum \Gamma_{111}p_1' p_2'^2.$$

Let these values, and the postulated values of p_3' and p_4', be substituted in the foregoing condition. It then appears that the finite terms cancel; likewise the terms of the first order of small quantities; and likewise the terms of the second order: in each instance, without any residuary condition. In order that the terms of the third order shall cancel, there remains a condition

$$vP_4 - \tfrac{1}{3}uv^2K(Hp_2'+Bq_2')(p_1'q_2'-p_2'q_1') - \tfrac{1}{2}uv^2\sum \Gamma_{111}p_1'p_2'^2$$
$$= uP_3 + \tfrac{1}{3}u^2vK(Hp_1'+Bq_1')(p_1'q_2'-p_2'q_1') - \tfrac{1}{2}u^2v\sum \Gamma_{111}p_1'^2p_2',$$

and therefore

$$v\{P_4 + \tfrac{1}{2}u^2\sum\Gamma_{111}p_1'^2p_2'\} - u\{P_3+\tfrac{1}{2}v^2\sum \Gamma_{111}p_1'p_2'^2\}$$
$$= \tfrac{1}{3}K(p_1'q_2'-p_2'q_1')uv\{u(Hp_1'+Bq_1')+v(Hp_2'+Bq_2')\},$$

this residuary condition being, in fact, an equation involving the unknown quantities P_3 and P_4.

(iv). In order that the q-parameter may attain, at U, the same value by the path ORU as by the path OTU, there is a similar relation. When the relation is developed, the finite terms cancel, as do the terms of the first order of small quantities, and likewise the terms of the second order; and, for the cancellation of the terms of the third order, there remains a condition of the form

$$v\{Q_4+\tfrac{1}{2}u^2\sum\Delta_{111}p_1'^2p_2'\} - u\{Q_3+\tfrac{1}{2}v^2\sum\Delta_{111}p_1'p_2'^2\}$$
$$= -\tfrac{1}{3}K(p_1'q_2'-p_2'q_1')uv\{u(Ap_1'+Hq_1')+v(Ap_2'+Hq_2')\},$$

this residuary condition being also, in fact, an equation involving the unknown quantities Q_3 and Q_4.

Let

$$P_3+\tfrac{1}{2}v^2\sum\Gamma_{111}p_1'p_2'^2 = \bar{P}_3, \quad P_4+\tfrac{1}{2}u^2\sum\Gamma_{111}p_1'^2p_2' = \bar{P}_4,$$
$$Q_3+\tfrac{1}{2}v^2\sum\Delta_{111}p_1'p_2'^2 = \bar{Q}_3, \quad Q_4+\tfrac{1}{2}u^2\sum\Delta_{111}p_1'^2p_2' = \bar{Q}_4;$$

then there are four equations

$$\left.\begin{aligned}(Ap_1'+Hq_1')\bar{P}_3 + (Hp_1'+Bq_1')\bar{Q}_3 &= \tfrac{1}{6}Kv^2\sin^2\epsilon,\\ (Ap_2'+Hq_2')\bar{P}_4 + (Hp_2'+Bq_2')\bar{Q}_4 &= \tfrac{1}{6}Ku^2\sin^2\epsilon\end{aligned}\right\},$$

and
$$\left.\begin{aligned} v\bar{P}_4 - u\bar{P}_3 &= \tfrac{1}{3}Kuv(p_1'q_2' - p_2'q_1')\{u(Hp_1' + Bq_1') + v(Hp_2' + Bq_2')\} \\ v\bar{Q}_4 - u\bar{Q}_3 &= -\tfrac{1}{3}Kuv(p_1'q_2' - p_2'q_1')\{u(Ap_1' + Hq_1') + v(Ap_2' + Hq_2')\} \end{aligned}\right\},$$

for the determination of the quantities P_3, Q_3, P_4, Q_4, of the second order. When these equations are resolved, they give the values

$$\left.\begin{aligned} \bar{P}_3 &= Kv(p_1'q_2' - p_2'q_1')\{-\tfrac{1}{2}u(Hp_1' + Bq_1') + \tfrac{1}{6}v(Hp_2' + Bq_2')\} \\ \bar{Q}_3 &= Kv(p_1'q_2' - p_2'q_1')\{\ \tfrac{1}{2}u(Ap_1' + Hq_1') - \tfrac{1}{6}v(Ap_2' + Hq_2')\} \end{aligned}\right\},$$
$$\left.\begin{aligned} \bar{P}_4 &= Ku(p_1'q_2' - p_2'q_1')\{-\tfrac{1}{6}u(Hp_1' + Bq_1') + \tfrac{1}{2}v(Hp_2' + Bq_2')\} \\ \bar{Q}_4 &= Ku(p_1'q_2' - p_2'q_1')\{\ \tfrac{1}{6}u(Ap_1' + Hq_1') - \tfrac{1}{2}v(Ap_2' + Hq_2')\} \end{aligned}\right\};$$

and now, up to the second order of small quantities inclusive, the direction-variables p_3', q_3', of TU, and the direction-variables p_4', q_4', of RU, are

$$\left.\begin{aligned} p_3' &= p_1' - v\gamma_{12} - \tfrac{1}{2}v^2\textstyle\sum \Gamma_{111}p_1'p_2'^2 + \bar{P}_3 \\ q_3' &= q_1' - v\delta_{12} - \tfrac{1}{2}v^2\textstyle\sum \Delta_{111}p_1'p_2'^2 + \bar{Q}_3 \end{aligned}\right\},$$
$$\left.\begin{aligned} p_4' &= p_2' - u\gamma_{12} - \tfrac{1}{2}u^2\textstyle\sum \Gamma_{111}p_1'^2p_2' + \bar{P}_4 \\ q_4' &= q_2' - u\delta_{12} - \tfrac{1}{2}u^2\textstyle\sum \Delta_{111}p_1'^2p_2' + \bar{Q}_4 \end{aligned}\right\}.$$

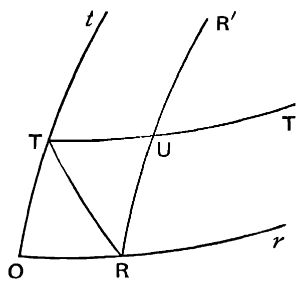

FIG. 13.

128. As the parallelogram $OTURO$ has been defined by the equality of opposite sides, and not (as before) by parallelism at R and at T respectively, it it important to know the magnitudes of the angles at R and at T, and also the magnitude of the angle at U. We have

$$\begin{aligned} \cos rRU &= \textstyle\sum A^{(R)}p_4'(p_1' + up_1'' + \tfrac{1}{2}u^2p_1''') \\ &= \textstyle\sum A^{(R)}\{p_2' - u\gamma_{12} - \tfrac{1}{2}u^2\sum\Gamma_{111}p_1'^2p_2' + \bar{P}_4\}(p_1' + up_1'' + \tfrac{1}{2}u^2p_1'''), \end{aligned}$$

where

$$A^{(R)} = A + u\frac{dA}{ds_1} + \tfrac{1}{2}u^2\frac{d^2A}{ds_1^2}.$$

The finite terms

$$= \textstyle\sum Ap_1'p_2' = \cos\epsilon.$$

The terms of the first order have the coefficient of u

$$= \sum\frac{dA}{ds_1}p_1'p_2' + \sum Ap_2'p_1'' - \sum Ap_1'\gamma_{12} = 0,$$

when the values of the first arc-derivative of the primary magnitudes, as given in § 98, are inserted.

The aggregate of terms of the second order

$$\begin{aligned} &= \tfrac{1}{2}u^2\sum\frac{d^2A}{ds_1^2}p_1'p_2' + \sum Ap_1'\bar{P}_4 - \tfrac{1}{2}u^2\{\textstyle\sum Ap_1'(\sum\Gamma_{111}p_1'^2p_2')\} \\ &\quad + \tfrac{1}{2}u^2\sum Ap_2'p_1''' - u^2\sum\frac{dA}{ds_1}p_1'\gamma_{12} + u^2\sum\frac{dA}{ds_1}p_2'p_1'' - u^2\sum Ap_1'\gamma_{12}. \end{aligned}$$

Now

$$\begin{aligned}\sum Ap_1'\bar{P}_4 &= (Ap_1'+Hq_1')\bar{P}_4+(Hp_1'+Bq_1')\bar{Q}_4\\ &= \tfrac{1}{2}Kuv(p_1'q_2'-p_2'q_1')V^2(p_1'q_2'-p_2'q_1')\\ &= \tfrac{1}{2}Kuv\sin^2\epsilon\,;\end{aligned}$$

and the values of the combinations of the first and the second arc-derivatives of the primary magnitudes have been given in § 98. When these are inserted, and reduction is effected, it is found that all the terms, except the cited value of $\sum Ap_1'\bar{P}_4$, cancel one another, so that the aggregate of terms of the second order

$$=\tfrac{1}{2}Kuv\sin^2\epsilon.$$

Hence, accurately up to the second order of small quantities,

$$\cos rRU=\cos\epsilon+\tfrac{1}{2}Kuv\sin^2\epsilon,$$

and therefore, also up to that order inclusive,

$$rRU=\epsilon-\tfrac{1}{2}Kuv\sin\epsilon.$$

In the same way, we find

$$tTU=\epsilon-\tfrac{1}{2}Kuv\sin\epsilon.$$

Now in the geodesic triangle ORT, the geodesic RT being drawn, we have seen (§ 112) that, in angles,

$$ROT+OTR+ORT-\pi=\tfrac{1}{2}Kuv\sin\epsilon,$$

ROT being ϵ; also

$$UTR+RTO=\pi-UTt=\pi-\epsilon+\tfrac{1}{2}Kuv\sin\epsilon\,;$$

and therefore, up to the second order,

$$UTR=ORT.$$

Similarly we find, also up to the second order,

$$URT=OTR.$$

Consequently, in the geodesic parallelogram $ORUTO$ on the surface, the sides RU and TO are geodesically parallel with reference to RT as a basic geodesic, and the sides TU and RO are geodesically parallel also with reference to RT as a basic geodesic.

One further inference can be drawn. The area of the small geodesic triangle RTU, to the second order of small quantities, is

$$\begin{aligned}&=\tfrac{1}{2}TU\,.\,TR\,.\,\sin UTR\\ &=\tfrac{1}{2}OR\,.\,RT\,.\,\sin TRO\\ &=\text{area of small geodesic triangle } ORT.\end{aligned}$$

As the triangles have a common vertex at T, so that the sphericity of the surface at T is the same for both, it follows that the sum of the three angles of RTU is equal to the sum of the three angles of RTO, to the second order of small

quantities. But $UTR = TRO$, and $URT = TRO$; and therefore, to the second order of small quantities,

$$TUR = TOR = \epsilon:$$

or the third and fourth sides of the small geodesic parallelogram cut at the same angle as the first and the second sides. Thus in the small parallelogram as drawn, the opposite angles are equal; and the opposite sides have been made equal, practically as the definition of the parallelogram. Moreover, the opposite sides are parallel geodesically, with reference to the diagonal of the parallelogram not passing through the initial point of reference.

The equality of the angles RUT and ROT, up to the second order of small quantities, can be verified analytically. The direction-variables of UT' at U, up to the second order, are

$$P_3' = p_3' + up_3'' + \tfrac{1}{2}u^2 p_3''', \quad Q_3' = q_3' + uq_3'' + \tfrac{1}{2}u^2 q_3''',$$

that is,

$$\begin{aligned} \bar{P}_3 + p_1' - \sum\{\Gamma_{11} p_1'(up_1' + vp_2')\} - \tfrac{1}{2}\sum\{\Gamma_{111} p_1'(up_1' + vp_2')^2\} \\ + \tfrac{2}{3}uvK(Hp_1' + Bq_1')(p_1'q_2' - p_2'q_1'), \\ \bar{Q}_3 + q_1' - \sum\{\varDelta_{11} p_1'(up_1' + vp_2')\} - \tfrac{1}{2}\sum\{\varDelta_{111} p_1'(up_1' + vp_2')^2\} \\ - \tfrac{2}{3}uvK(Ap_1' + Hq_1')(p_1'q_2' - p_2'q_1')\}; \end{aligned}$$

and the direction-variables of UR' at U, also up to the second order, are

$$P_4' = p_4' + vp_4'' + \tfrac{1}{2}v^2 p_4''', \quad Q_4' = q_4' + vq_4'' + \tfrac{1}{2}v^2 q_4''',$$

that is,

$$\begin{aligned} \bar{P}_4 + p_2' - \sum\{\Gamma_{11} p_2'(up_1' + vp_2')\} - \tfrac{1}{2}\sum\{\Gamma_{111} p_2'(up_1' + vp_2')^2\} \\ - \tfrac{2}{3}uvK(Hp_2' + Bq_2')(p_1'q_2' - p_2'q_1'), \\ \bar{Q}_4 + q_2' - \sum\{\varDelta_{11} p_2'(up_1' + vp_2')\} - \tfrac{1}{2}\sum\{\varDelta_{111} p_2'(up_1' + vp_2')^2\} \\ + \tfrac{2}{3}uvK(Ap_2' + Hq_2')(p_1'q_2' - p_2'q_1'). \end{aligned}$$

Further, the values of the parameters at U, up to the second order, are

$$\begin{aligned} p + up_1' + \tfrac{1}{2}u^2 p_1'' + vp_2' - uv\gamma_{12} + \tfrac{1}{2}v^2 p_2'', \quad &= p + P_1 + P_2, \\ q + uq_1' + \tfrac{1}{2}u^2 q_1'' + vq_2' - uv\delta_{12} + \tfrac{1}{2}v^2 q_2'', \quad &= q + Q_1 + Q_2; \end{aligned}$$

and therefore

$$\begin{aligned} A^{(U)} &= A(p + P_1 + P_2, q + Q_1 + Q_2) \\ &= A + u\frac{dA}{ds_1} + v\frac{dA}{ds_2} + \tfrac{1}{2}u^2\frac{d^2A}{ds_1^2} + uv\frac{d^2A}{ds_1 ds_2} + \tfrac{1}{2}v^2\frac{d^2A}{ds_2^2}, \end{aligned}$$

with the convention of § 99 as regards the simultaneous differentiation with respect to s_1 and s_2, a convention that is in accord with the present circumstances. There are similar values for $H^{(U)}$ and $B^{(U)}$.

Now we have $\cos T'UR' = \sum A^{(U)} P_3' P_4'$. When the values of $A^{(U)}$, $H^{(U)}$, $B^{(U)}$; P_3', Q_3'; P_4', Q_4'; are substituted, the resulting expression is valid up to the complete second-order aggregate of terms. The finite terms $= \sum A p_1' p_2' = \cos \epsilon$.

The terms of the first order of small quantities vanish, on substitution for the first arc-derivatives of the primary magnitudes. In the aggregate of terms of the second order, the various terms free from the sphericity cancel without modification; and the aggregate of terms involving the sphericity

$$
\begin{aligned}
=\text{such terms in } &\sum\left[p_1'p_2'\left(\tfrac{1}{2}u^2\frac{d^2A}{ds_1^2}+uv\frac{d^2A}{ds_1 ds_2}+\tfrac{1}{2}v^2\frac{d^2A}{ds_2^2}\right)\right] \\
&+[\ \ \tfrac{2}{3}vuK(p_1'q_2'-q_1'p_2')(Hp_1'+Bq_1')+\bar{P}_3](Ap_2'+Hq_2') \\
&+[-\tfrac{2}{3}vuK(p_1'q_2'-q_1'p_2')(Ap_1'+Hq_1')+\bar{Q}_3](Hp_2'+Bq_2') \\
&+[-\tfrac{2}{3}vuK(p_1'q_2'-q_1'p_2')(Hp_2'+Bq_2')+\bar{P}_4](Ap_1'+Hq_1') \\
&+[\ \ \tfrac{2}{3}vuK(p_1'q_2'-q_1'p_2')(Ap_2'+Hq_2')+\bar{Q}_4](Hp_1'+Bq_1').
\end{aligned}
$$

In the quantities $\sum\frac{d^2A}{ds_1^2}p_1'p_2'$ and $\sum\frac{d^2A}{ds_2^2}p_1'p_2'$, the terms involving K have vanishing coefficients (§ 98); and (§ 99) in $uv\sum\frac{d_2A}{ds_1 ds_2}p_1'p_2$, the term

$$=\tfrac{1}{3}uvKV^2(p_1'q_2'-q_1'p_2')^2.$$

When the values of $\bar{P}_3$, $\bar{Q}_3$, $\bar{P}_4$, $\bar{Q}_4$, are substituted in the last four lines, and the sum of the expressions is reduced, it is found to be

$$-\tfrac{1}{3}uvKV^2(p_1'q_2'-q_1'p_2')^2.$$

Consequently, we have $\cos T'UR'=\cos\epsilon$, up to the second order of small quantities inclusive; and the inference, as to the equality of the opposite angles of the geodesic parallelogram at O and U, is verified.

The result can also be inferred * by measuring RT in the triangle RUT. As $t^2=u^2+v^2-2uv\cos\epsilon$, a second-order difference between the angle TUR and ϵ would cause a third-order change in t, and therefore also in

$$t-\tfrac{1}{6}K\frac{u^2v^2}{t}\sin^2\epsilon,$$

contrary to the fact that, up to the third order, this expression gives the accurate value of RT.

NOTE. When such a parallelogram is drawn on the surface of a sphere in triple space, the opposite angles are equal whatever be the magnitude of the sides—a result due to the uniform sphericity of the surface. But for a surface, the sphericity of which varies both in place and in orientation at each place, the opposite angles of a small geodesic parallelogram will deviate from equality by magnitudes of the third order of small quantities—a deviation that evades estimate when only second-order quantities are retained in the measurement of angles.

Ex. Investigate the relations of the sides of the foregoing geodesic parallelogram to the geodesic diagonal OU, with special reference to the division of the angles at O and at U.

* This remark is due to Prof. E. H. Neville.

CHAPTER XI

Curvatures of Geodesics on Surfaces

Circular curvature and torsion of a geodesic.

129. Further developments of properties of the Riemann sphericity belong to amplitudes of more than two dimensions, and they arise mainly through geodesic surfaces in such amplitudes; they will therefore be deferred until the amplitudes themselves are considered. We now return to the consideration of geodesics on a free surface, with reference to their curvatures of rank higher than the circular curvature and the torsion.

The circular curvature and the direction of the prime normal of a geodesic are implied in the typical equation (§§ 93, 94)

$$\frac{Y}{\rho}=\eta_{11}p'^2+2\eta_{12}p'q'+\eta_{22}q'^2.$$

The actual value of ρ is given by the equation

$$\frac{1}{\rho^2}=\mathbf{a}p'^4+4\mathbf{h}p'^3q'+6\mathbf{k}p'^2q'^2+4\mathbf{f}p'q'^3+\mathbf{c}q'^4,$$

where $3\mathbf{k}=\mathbf{g}+2\mathbf{b}$, while the Riemann sphericity K is given by

$$V^2K=\mathbf{g}-\mathbf{b},$$

the symbols $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, being the quantities of the second order as defined in § 105. We shall assume that the plenary homaloidal space is of more than four dimensions, so that the determinant Υ, where

$$\Upsilon=\begin{vmatrix} \mathbf{a}, & \mathbf{h}, & \mathbf{g} \\ \mathbf{h}, & \mathbf{b}, & \mathbf{f} \\ \mathbf{g}, & \mathbf{f}, & \mathbf{c} \end{vmatrix},$$

does not vanish. We also have the relations (§ 104)

$$\frac{1}{\rho}=\bar{A}p'^2+2\bar{H}p'q'+\bar{B}q'^2,$$

where
$$\bar{A}=\sum Y\eta_{11},\quad \bar{H}=\sum Y\eta_{12},\quad \bar{B}=\sum Y\eta_{22};$$

also
$$\left.\begin{aligned}
\eta_{11}&=\frac{\partial^2 y}{\partial p^2}-\Gamma_{11}\frac{\partial y}{\partial p}-\Delta_{12}\frac{\partial y}{\partial q}, & \sum\frac{\partial y}{\partial p}\eta_{11}&=0=\sum\frac{\partial y}{\partial q}\eta_{11}\\
\eta_{12}&=\frac{\partial^2 y}{\partial p\,\partial q}-\Gamma_{12}\frac{\partial y}{\partial p}-\Delta_{12}\frac{\partial y}{\partial q}, & \sum\frac{\partial y}{\partial p}\eta_{12}&=0=\sum\frac{\partial y}{\partial q}\eta_{12}\\
\eta_{22}&=\frac{\partial^2 y}{\partial q^2}-\Gamma_{22}\frac{\partial y}{\partial p}-\Delta_{22}\frac{\partial y}{\partial q}, & \sum\frac{\partial y}{\partial p}\eta_{22}&=0=\sum\frac{\partial y}{\partial q}\eta_{22}
\end{aligned}\right\}.$$

The torsion of a geodesic and the direction of its binormal are implied in the typical equations (§ 106)

$$\frac{V}{\sigma}=\begin{vmatrix} \bar{A}p'+\bar{H}q', & \bar{H}p'+\bar{B}q' \\ Ap'+Hq', & Hp'+Bq' \end{vmatrix},$$

$$Vl_3=(Ap'+Hq')\frac{\partial y}{\partial q}-(Hp'+Bq')\frac{\partial y}{\partial p};$$

the binormal lies in the tangent plane of the surface and is at right angles to the tangent to the geodesic with the typical direction-cosine y', where

$$y'=\frac{\partial y}{\partial p}p'+\frac{\partial y}{\partial q}q'.$$

Also there is the relation

$$\frac{V}{\rho\sigma}=\begin{vmatrix} \mathbf{a}p'^3+3\mathbf{h}p'^2q'+3\mathbf{k}p'q'^2+\mathbf{f}q'^3, & Ap'+Hq' \\ \mathbf{h}p'^3+3\mathbf{k}p'^2q'+3\mathbf{f}p'q'^2+\mathbf{c}q'^3, & Hp'+Bq' \end{vmatrix}.$$

Obviously

$$\frac{\partial y}{\partial p}=y'(Ap'+Hq')-l_3Vq',\quad \frac{\partial y}{\partial q}=y'(Hp'+Bq')+l_3Vp',$$

results in accord with the property that, on the one hand, the magnitudes typified by $\frac{\partial y}{\partial p}$ and $\frac{\partial y}{\partial q}$ determine two leading lines in the tangent plane while, on the other, the tangent and the binormal are lines lying in that plane, so that the sets of spatial direction-cosines are linearly equivalent to one another.

In the orthogonal frame of any curve in multiple space, the prime normal, the trinormal, and all the principal lines of more advanced rank (these being lines with spatial direction-cosines typically represented by l_4, l_5, ...), when associated with the tangent and the binormal, constitute a complete orthogonal system of reference. For a geodesic on a surface, the tangent and the binormal lie in the tangent plane of the surface; and therefore all the remaining principal lines of a superficial geodesic lie in the orthogonal homaloid (§ 92) of the surface. The expression of this orthogonality can be taken in the form

$$\sum l_\mu y'=0,\quad \sum l_\mu l_3=0,$$

for $\mu=2$, when $l_2=Y$, and for $\mu=4, 5, \ldots, N$, where N is the dimension-number of the plenary space. Hence

$$\sum l_\mu\frac{\partial y}{\partial p}=(Ap'+Hq')(\sum l_\mu y')-Vq'(\sum l_\mu l_3)=0,$$

$$\sum l_\mu\frac{\partial y}{\partial q}=(Hp'+Bq')(\sum l_\mu y')+Vp'(\sum l_\mu l_3)=0,$$

for all the specified values of μ.

We shall find it convenient to write

$$\left.\begin{aligned} A_\mu &= \sum l_\mu \frac{\partial^2 y}{\partial p^2} = \sum l_\mu \eta_{11} \\ H_\mu &= \sum l_\mu \frac{\partial^2 y}{\partial p\,\partial q} = \sum l_\mu \eta_{12} \\ B_\mu &= \sum l_\mu \frac{\partial^2 y}{\partial q^2} = \sum l_\mu \eta_{22} \end{aligned}\right\},$$

for the values $\mu = 4, 5, \ldots, N$; the symbols $\bar{A}$, $\bar{H}$, $\bar{B}$, have already been used in connection with the value $\mu = 2$.

In connection with the Frenet system of equations, now to be applied to a geodesic, we write, as usual,

$$\rho_1 = \rho, \quad \rho_2 = \sigma, \quad \rho_3 = \tau, \quad \rho_4 = \kappa,$$

in connection with the circular curvature, the torsion, the tilt, and the coil, respectively, retaining the symbol ρ_n for values of n greater than four.

In the first place, when we differentiate the relations

$$\sum Y \frac{\partial y}{\partial p} = 0, \quad \sum Y \frac{\partial y}{\partial q} = 0,$$

along a geodesic, we have

$$\sum Y \left(\frac{\partial^2 y}{\partial p^2} p' + \frac{\partial^2 y}{\partial p\,\partial q} q'\right) = -\sum \frac{\partial y}{\partial p} Y' = -\sum \frac{\partial y}{\partial p} \left(\frac{l_3}{\sigma} - \frac{y'}{\rho}\right),$$

$$\sum Y \left(\frac{\partial^2 y}{\partial p\,\partial q} p' + \frac{\partial^2 y}{\partial q^2} q'\right) = -\sum \frac{\partial y}{\partial q} Y' = -\sum \frac{\partial y}{\partial q} \left(\frac{l_3}{\sigma} - \frac{y'}{\rho}\right),$$

that is,

$$\bar{A}p' + \bar{H}q' = \frac{1}{\rho}(Ap' + Hq') + \frac{1}{\sigma} Vq',$$

$$\bar{H}p' + \bar{B}q' = \frac{1}{\rho}(Hp' + Bq') - \frac{1}{\sigma} Vp',$$

which are in accord with the relations already cited.

In the second place, when we differentiate the relations

$$\sum l_4 \frac{\partial y}{\partial p} = 0, \quad \sum l_4 \frac{\partial y}{\partial q} = 0,$$

along a geodesic, we have

$$\sum l_4 \left(\frac{\partial^2 y}{\partial p^2} p' + \frac{\partial^2 y}{\partial p\,\partial q} q'\right) = -\sum \frac{\partial y}{\partial p} \left(\frac{l_5}{\kappa} - \frac{l_3}{\tau}\right) = \frac{1}{\tau} \sum l_3 \frac{\partial y}{\partial p},$$

$$\sum l_4 \left(\frac{\partial^2 y}{\partial p\,\partial q} p' + \frac{\partial^2 y}{\partial q^2} q'\right) = -\sum \frac{\partial y}{\partial q} \left(\frac{l_5}{\kappa} - \frac{l_3}{\tau}\right) = \frac{1}{\tau} \sum l_3 \frac{\partial y}{\partial q},$$

that is,

$$A_4 p' + H_4 q' = -Vq'\frac{1}{\tau}, \quad H_4 p' + B_4 q' = Vp'\frac{1}{\tau}.$$

We at once have

$$A_4 p'^2 + 2H_4 p'q' + B_4 q'^2 = 0\,;$$

and

$$\begin{vmatrix} A_4, & H_4 + \dfrac{V}{\tau} \\ H_4 - \dfrac{V}{\tau}, & B_4 \end{vmatrix} = 0,$$

leading to the result

$$A_4 B_4 - H_4^2 = -\frac{V^2}{\tau^2}.$$

In the next place, we take the relations

$$\sum l_\mu \frac{\partial y}{\partial p} = 0, \quad \sum l_\mu \frac{\partial y}{\partial q} = 0,$$

for the values of μ greater than 4, and we differentiate along the superficial geodesic; then

$$\sum l_\mu \left(\frac{\partial^2 y}{\partial p^2} p' + \frac{\partial^2 y}{\partial p\,\partial q} q'\right) = -\sum \frac{\partial y}{\partial p}\left(\frac{l_{\mu+1}}{\rho_\mu} - \frac{l_{\mu-1}}{\rho_{\mu-1}}\right) = 0,$$

$$\sum l_\mu \left(\frac{\partial^2 y}{\partial p\,\partial q} p' + \frac{\partial^2 y}{\partial q^2} q'\right) = -\sum \frac{\partial y}{\partial q}\left(\frac{l_{\mu+1}}{\rho_\mu} - \frac{l_{\mu-1}}{\rho_{\mu-1}}\right) = 0,$$

because of the values

$$\sum l_n \frac{\partial y}{\partial p} = 0, \quad \sum l_n \frac{\partial y}{\partial q} = 0,$$

for $n = 4, \ldots, N$, while there is no term in l_{n+1} when $n = N$. Hence

$$A_\mu p' + H_\mu q' = 0, \quad H_\mu p' + B_\mu q' = 0,$$

for all such values of μ; and therefore there exists, for each value of μ, a quantity T_μ such that

$$A_\mu = q'^2 T_\mu, \quad H_\mu = -q'p'T_\mu, \quad B_\mu = p'^2 T_\mu,$$

that is, for all the values $\mu = 5, 6, \ldots, N$. But these relations do not hold for $\mu = 4$.

Orthogonal flat of a surface.

130. Although the second parametric derivatives of the point-variables $y_1, y_2, \ldots$ are functions of position only, in space, and do not involve directions at their position, there is an analytic convenience in referring the total configuration of the surface to the organic frame of a geodesic of the surface in special association with the tangent plane. The association is due to the fact that the tangent plane

of the surface is also the organic plane determined by the tangent line and the binormal of any geodesic; so that we can substitute the direction-cosines of the lines typified by the magnitudes $\frac{\partial y}{\partial p}$ and $\frac{\partial y}{\partial q}$ for the typical direction-cosines y' and l_3.

Now the three quantities typified by $\eta_{11}\mathbf{a}^{-\frac{1}{2}}$, $\eta_{12}\mathbf{b}^{-\frac{1}{2}}$, $\eta_{22}\mathbf{c}^{-\frac{1}{2}}$, can be regarded as typical direction-cosines; for, because of the relations

$$\sum \eta_{11}^2 = \mathbf{a}, \quad \sum \eta_{12}^2 = \mathbf{b}, \quad \sum \eta_{22}^2 = \mathbf{c},$$
$$\sum \eta_{12}\eta_{22} = \mathbf{f}, \quad \sum \eta_{22}\eta_{11} = \mathbf{g}, \quad \sum \eta_{11}\eta_{12} = \mathbf{h},$$

they satisfy the requirements of the direction-cosines of three lines, such that, if the inclinations of the three directions, themselves indicated by 1, 2, 3, be denoted by 23, 31, 12, we have

$$(\mathbf{bc})^{\frac{1}{2}} \cos 23 = \mathbf{f}, \quad (\mathbf{ca})^{\frac{1}{2}} \cos 31 = \mathbf{g}, \quad (\mathbf{ab})^{\frac{1}{2}} \cos 12 = \mathbf{h}.$$

Accordingly, the quantities η_{11}, η_{12}, η_{22}, are expressible by means of the direction-cosines of the lines of the orthogonal frame of any geodesic referred to any set of axes in space: and thus, for a typical quantity η_{11}, there will be an equation

$$\eta_{11} = y'\bar{A}_1 + l_3\bar{A}_3 + Y\bar{A}_2 + l_4\bar{A}_4 + l_5\bar{A}_5 + \ldots + l_N\bar{A}_n,$$

with appropriately determined quantities $\bar{A}$ as coefficients. Owing to the specified relation between the geodesic and the tangent plane of the surface, we can substitute a combination

$$\frac{\partial y}{\partial p} C + \frac{\partial y}{\partial q} E$$

for the combination $y'\bar{A}_1 + l_3\bar{A}_3$.

The coefficients can be determined as follows. We have

$$\sum \eta_{11} \frac{\partial y}{\partial p} = 0, \quad \sum \eta_{11} \frac{\partial y}{\partial q} = 0, \quad \sum l \frac{\partial y}{\partial p} = 0, \quad \sum l \frac{\partial y}{\partial q} = 0,$$

for $l = Y, l_4, l_5, \ldots, l_N$. Hence multiplying the equation by $\frac{\partial y}{\partial p}$, and adding for the space-dimensions; then multiplying it by $\frac{\partial y}{\partial q}$, and again adding for the space-dimensions; we have, in turn,

$$AC + HE = 0, \quad HC + BE = 0,$$

that is,

$$C = 0, \quad E = 0.$$

Multiplying by Y, and adding as before, we have

$$\bar{A}_2 = \sum Y\eta_{11} = \bar{A},$$

where $\bar{A}$ is the quantity (§ 104) already known. Multiplying by l_μ, for $\mu=4, 5, \ldots, N$, and adding as before, we have

$$\bar{A}_\mu = \sum l_\mu \eta_{11} = A_\mu,$$

where A_μ is the magnitude defined in § 129. Thus we have

$$\eta_{11} = Y\bar{A} + l_4 A_4 + l_5 A_5 + \ldots + l_N A_N.$$

But (§ 129) we have had the results

$$A_\mu = q'^2 T_\mu,$$

for all the values $\mu=5, 6, \ldots, N$; and therefore, if we write

$$lT = l_5 T_5 + l_6 T_6 + \ldots + l_N T_N,$$

we have

$$\eta_{11} = Y\bar{A} + l_4 A_4 + lq'^2 T.$$

Proceeding similarly with η_{12}, η_{22}, we find

$$\eta_{12} = Y\bar{H} + l_4 H_4 - lq'p'T,$$
$$\eta_{22} = Y\bar{B} + l_4 B_4 + lp'^2 T.$$

As regards the quantity T and the magnitudes l, we have

$$T\sum ly' = 0, \quad T\sum ll_3 = 0, \quad T\sum lY = 0, \quad T\sum ll_4 = 0,$$

so that l can be regarded as a typical direction-cosine of a line which is at right angles to the tangent, the binormal, the prime normal, and the trinormal, of the geodesic. Hence, when we take $\sum l^2 = 1$, we have

$$T^2 = T_5^2 + T_6^2 + \ldots + T_N^2.$$

We have already (§ 92) noted the orthogonal $(N-2)$-fold homaloid of the surface; but it is possible to have an orthogonal homaloid of less extensive range (for $N>5$). Consider the flat, represented by the typical equations

$$\| \bar{y} - y, \quad \eta_{11}, \quad \eta_{12}, \quad \eta_{22} \| = 0.$$

Because the relations

$$\sum \eta_{ij} \frac{\partial y}{\partial p} = 0, \quad \sum \eta_{ij} \frac{\partial y}{\partial q} = 0,$$

are satisfied for the combinations $ij = 11, 12, 22$, the flat is orthogonal to the tangent plane of the surface; also, manifestly it is independent of any particular direction on the surface; it therefore will be entitled the ***orthogonal flat*** of the surface. The directions typified by η_{11}, η_{12}, η_{22}, are those of leading lines in the flat *; and the foregoing expressions shew that we can substitute, for these three

* It is assumed, at this stage, that the plenary homaloidal space is of more than four dimensions. Were that space quadruple, a linear relation would connect the quantities η_{11}, η_{12}, η_{22}, associated with the four space-coordinates: see *G.F.D.*, vol. i, § 214.

lines, three other contained directions, the typical direction-cosines of which are Y, l_4, l. But Y is the typical direction-cosine of the prime normal, and l_4 has the like relation to the trinormal: so that the flat certainly contains the prime normal and the trinormal of the geodesic. The magnitude l is the typical direction of a direction, certainly at right angles to the tangent and the binormal because it lies in the orthogonal flat of the surface: the direction is at right angles also to the prime normal and to the trinormal. Unless the plenary space is quintuple, a condition which would restrict the geodesic curvatures to circular curvature, torsion, tilt, and coil, we cannot infer that the direction typified by the quantity l in the expressions for η_{11}, η_{12}, η_{22}, is that of the quartinormal; it is a direction lying in the homaloid, which is orthogonal to the surface and which has the quartinormal and all the normals of more advanced rank for its leading lines.

Accordingly, there are relations

$$\left.\begin{aligned}\eta_{11}&=Y\bar{A}+l_4A_4+lq'^2\,T\\ \eta_{12}&=Y\bar{H}+l_4H_4-lq'p'T\\ \eta_{22}&=Y\bar{B}+l_4B_4+lp'^2\,T\end{aligned}\right\},$$

which, owing to the forms such as

$$\eta_{11}=\frac{\partial^2 y}{\partial p^2}-\Gamma_{11}\frac{\partial y}{\partial p}-\Gamma_{12}\frac{\partial y}{\partial q},$$

are of the nature of partial differential equations of the second order satisfied by the point-coordinates of the surface; but the quantities such as Y, $\bar{A}$, on the right-hand sides involve implicitly the direction-variables p', q', of the superficial geodesic, to the organic frame of which the whole configuration is referred. Further, we have obtained a flat

$$\|\ \bar{y}-y,\quad \eta_{11},\quad \eta_{12},\quad \eta_{22}\ \|=0,$$

which is the same for every geodesic and is, in fact, orthogonal to the surface.

131. In the construction of expressions for direction-cosines of the trinormal and for magnitude of the tilt, various subsidiary results are required.

We write

$$u_1=Ap'+Hq',\quad u_2=Hp'+Bq'.$$

Hence

$$\begin{aligned}\frac{du_1}{ds}&=\frac{dA}{ds}p'+\frac{dH}{ds}q'+Ap''+Hq''\\ &=p'\{2A(\Gamma_{11}p'+\Gamma_{12}q')+2H(\Delta_{11}p'+\Delta_{12}q')\}\\ &\qquad+q'\{A(\Gamma_{12}p'+\Gamma_{22}q')+H(\Delta_{12}p'+\Delta_{22}q')+H(\Gamma_{11}p'+\Gamma_{12}q')+B(\Delta_{11}p'+\Delta_{12}q')\}\\ &\qquad-A(\Gamma_{11}p'^2+2\Gamma_{12}p'q'+\Gamma_{22}q'^2)-H(\Delta_{11}p'^2+2\Delta_{12}p'q'+\Delta_{22}q'^2)\\ &=(Ap'+Hq')(\Gamma_{11}p'+\Gamma_{12}q')+(Hp'+Bq')(\Delta_{11}p'+\Delta_{12}q')\\ &=\alpha u_1+\epsilon u_2,\end{aligned}$$

with the former significance (§ 98) for α and ϵ. Similarly for the arc-derivative of u_2; the two results are

$$\left.\begin{aligned}\frac{du_1}{ds}&=\alpha u_1+\epsilon u_2\\ \frac{du_2}{ds}&=\beta u_1+\eta u_2\end{aligned}\right\}.$$

Also

$$\frac{1}{V}\frac{dV}{ds}=(\Gamma_{11}+\Delta_{12})p'+(\Gamma_{12}+\Delta_{22})q'=\alpha+\eta\,;$$

hence

$$\left.\begin{aligned}\frac{d}{ds}\left(\frac{u_1}{V}\right)&=-\eta\frac{u_1}{V}+\epsilon\frac{u_2}{V}\\ \frac{d}{ds}\left(\frac{u_2}{V}\right)&=\quad\beta\frac{u_1}{V}-\alpha\frac{u_2}{V}\end{aligned}\right\}.$$

In connection with the quantity which is equal to Y/ρ, we take

$$\xi_1=\eta_{11}p'+\eta_{12}q',\quad \xi_2=\eta_{12}p'+\eta_{22}q'.$$

Then

$$\begin{aligned}\frac{d}{ds}\left(\frac{\partial y}{\partial p}\right)&=\frac{\partial^2 y}{\partial p^2}p'+\frac{\partial^2 y}{\partial p\,\partial q}q'\\ &=\eta_{11}p'+\eta_{12}q'+\alpha\frac{\partial y}{\partial p}+\epsilon\frac{\partial y}{\partial q}=\xi_1+\alpha\frac{\partial y}{\partial p}+\epsilon\frac{\partial y}{\partial q}.\end{aligned}$$

Similarly for the arc-derivative of $\dfrac{\partial y}{\partial q}$; the two results are

$$\left.\begin{aligned}\frac{d}{ds}\left(\frac{\partial y}{\partial p}\right)&=\xi_1+\alpha\frac{\partial y}{\partial p}+\epsilon\frac{\partial y}{\partial q}\\ \frac{d}{ds}\left(\frac{\partial y}{\partial q}\right)&=\xi_2+\beta\frac{\partial y}{\partial p}+\eta\frac{\partial y}{\partial q}\end{aligned}\right\}.$$

We take quantities v_1, v_2, connected with the expression of the circular curvature, in the same way as u_1 and u_2 are connected with the arc-relation, and defined by the equations

$$v_1=\bar{A}p'+\bar{H}q',\quad v_2=\bar{H}p'+\bar{B}q'.$$

By differentiating the relation $\sum Y\dfrac{\partial y}{\partial p}=0$ along the geodesic, we have

$$\sum Y'\frac{\partial y}{\partial p}=-\sum Y\left(\frac{\partial^2 y}{\partial p^2}p'+\frac{\partial^2 y}{\partial p\,\partial q}q'\right)=-(\bar{A}p'+\bar{H}q')=-v_1\,;$$

and similarly

$$\sum Y'\frac{\partial y}{\partial q}=-v_2.$$

Now

$$Y'=\frac{l_3}{\sigma}-\frac{y'}{\rho}=\frac{1}{V\sigma}\left\{(Ap'+Hq')\frac{\partial y}{\partial q}-(Hp'+Bq')\frac{\partial y}{\partial p}\right\}-\frac{1}{\rho}\left(p'\frac{\partial y}{\partial p}+q'\frac{\partial y}{\partial q}\right),$$

and we have

$$\sum\eta_{ij}\frac{\partial y}{\partial p}=0, \quad \sum\eta_{ij}\frac{\partial y}{\partial q}=0,$$

for all combinations of $i, j=11$, 12, 22 ; consequently

$$\sum Y'\eta_{ij}=0,$$

for all these combinations. Thus

$$\left.\begin{aligned}\sum Y'\frac{\partial^2 y}{\partial p^2}&=\sum\left(\eta_{11}+\Gamma_{11}\frac{\partial y}{\partial p}+\Delta_{11}\frac{\partial y}{\partial q}\right)Y'=-(\Gamma_{11}v_1+\Delta_{11}v_2)\\ \sum Y'\frac{\partial^2 y}{\partial p\,\partial q}&=\sum\left(\eta_{12}+\Gamma_{12}\frac{\partial y}{\partial p}+\Delta_{12}\frac{\partial y}{\partial q}\right)Y'=-(\Gamma_{12}v_1+\Delta_{12}v_2)\\ \sum Y'\frac{\partial^2 y}{\partial q^2}&=\sum\left(\eta_{22}+\Gamma_{22}\frac{\partial y}{\partial p}+\Delta_{22}\frac{\partial y}{\partial q}\right)Y'=-(\Gamma_{22}v_1+\Delta_{22}v_2)\end{aligned}\right\}.$$

We have

$$\bar{A}=\sum Y\frac{\partial^2 y}{\partial p^2},$$

and therefore, on differentiating along the geodesic

$$\begin{aligned}\frac{d\bar{A}}{ds}&=\sum Y\left(\frac{\partial^3 y}{\partial p^3}p'+\frac{\partial^3 y}{\partial p^2\partial q}q'\right)+\sum Y'\frac{\partial^2 y}{\partial p^2}\\ &=p'\{e_{111}+3(\Gamma_{11}\bar{A}+\Delta_{11}\bar{H})\}\\ &\qquad+q'\{e_{112}+2(\Gamma_{12}\bar{A}+\Delta_{12}\bar{H})+(\Gamma_{11}\bar{H}+\Delta_{11}\bar{B})\}-(\Gamma_{11}v_1+\Delta_{11}v_2)\end{aligned}$$

by the results in § 101 : hence

$$\frac{d\bar{A}}{ds}=e_{111}p'+e_{112}q'+2\alpha\bar{A}+2\epsilon\bar{H}.$$

Similarly, we find

$$\frac{d\bar{H}}{ds}=e_{112}p'+e_{122}q'+\beta\bar{A}+\eta\bar{H}+\alpha\bar{H}+\epsilon\bar{B},$$

$$\frac{d\bar{B}}{ds}=e_{122}p'+e_{222}q'+2\beta\bar{H}+2\eta\bar{B}.$$

For the arc-derivatives of v_1 and v_2, we have

$$\begin{aligned}\frac{dv_1}{ds}&=\frac{d\bar{A}}{ds}p'+\frac{d\bar{H}}{ds}q'+\bar{A}p''+\bar{H}q''\\ &=e_{111}p'^2+2e_{112}p'q'+e_{122}q'^2\\ &\qquad+2(\alpha\bar{A}+\epsilon\bar{H})p'+(\beta\bar{A}+\eta\bar{H}+\alpha\bar{H}+\epsilon\bar{B})q'\\ &\qquad-\bar{A}(\alpha p'+\beta q')-\bar{H}(\epsilon p'+\eta q')\\ &=w_1+\alpha v_1+\epsilon v_2,\end{aligned}$$

with the significance for w_1 as defined in § 101. Similarly for the arc-derivative of v_2; the two results are

$$\left.\begin{aligned}\frac{dv_1}{ds}&=w_1+\alpha v_1+\epsilon v_2\\ \frac{dv_2}{ds}&=w_2+\beta v_1+\eta v_2\end{aligned}\right\}.$$

These results can be verified in association with the expression for the arc-derivative of the circular curvature. We have

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{\rho}\right)&=\frac{d}{ds}(v_1p'+v_2q')\\ &=p'(w_1+\alpha v_1+\epsilon v_2)+q'(w_2+\beta v_1+\eta v_2)+v_1p''+v_2q'',\end{aligned}$$

or, as

$$p''=-\alpha p'-\beta q',\quad q''=-\epsilon p'-\eta q',$$

the relation becomes

$$\frac{d}{ds}\left(\frac{1}{\rho}\right)=p'w_1+q'w_2,$$

in accord with the former value (§ 101) for the arc-derivative of the curvature of the geodesic.

Also, when we differentiate

$$\sum Y'\frac{\partial y}{\partial p}=-v_1$$

along the geodesic, we have

$$\sum Y''\frac{\partial y}{\partial p}=-\sum Y'\left(\frac{\partial^2 y}{\partial p^2}p'+\frac{\partial^2 y}{\partial p\,\partial q}q'\right)-\frac{dv_1}{ds}=-w_1,$$

on substitution; and similarly for $\sum Y''\frac{\partial y}{\partial q}$. The two results are

$$\sum Y''\frac{\partial y}{\partial p}=-w_1,\quad \sum Y''\frac{\partial y}{\partial q}=-w_2.$$

Trinormal of a geodesic: the tilt.

132. The typical direction-cosine l_3 of the binormal of a geodesic on a surface is given (§ 95) by

$$Vl_3=(Ap'+Hq')\frac{\partial y}{\partial q}-(Hp'+Bq')\frac{\partial y}{\partial p},$$

that is,

$$l_3=\frac{u_1}{V}\frac{\partial y}{\partial q}-\frac{u_2}{V}\frac{\partial y}{\partial p}.$$

Differentiating along the arc of the geodesic, we have

$$\frac{l_4}{\tau}-\frac{Y}{\sigma}=\frac{\partial y}{\partial q}\left(-\eta\frac{u_1}{V}+\epsilon\frac{u_2}{V}\right)-\frac{\partial y}{\partial p}\left(\beta\frac{u_1}{V}-\alpha\frac{u_2}{V}\right)$$
$$+\frac{u_1}{V}\left(\xi_2+\beta\frac{\partial y}{\partial p}+\eta\frac{\partial y}{\partial q}\right)-\frac{u_2}{V}\left(\xi_1+\alpha\frac{\partial y}{\partial p}+\epsilon\frac{\partial y}{\partial q}\right)$$
$$=\frac{1}{V}(u_1\xi_2-u_2\xi_1).$$

To this result many forms can be given. In the first place, we have

$$\frac{V}{\sigma}=v_1u_2-v_2u_1,\quad \frac{Y}{\rho}=\xi_1p'+\xi_2q',\quad \frac{1}{\rho}=v_1p'+v_2q'\ ;$$

hence

$$V\frac{l_4}{\rho\tau}=(u_1\xi_2-u_2\xi_1)\frac{1}{\rho}+(\xi_1p'+\xi_2q')(v_1u_2-v_2u_1).$$

On the right-hand side, the coefficient of ξ_1

$$=-\frac{u_2}{\rho}+p'(v_1u_2-v_2u_1)$$
$$=u_2(-v_2q')-p'v_2u_1=-v_2(u_1p'+u_2q')=-v_2\ ;$$

similarly, the coefficient of ξ_2 is equal to v_1; and therefore

$$\frac{l_4}{\rho\tau}=\frac{1}{V}(v_1\xi_2-v_2\xi_1).$$

We have

$$\xi_1=\eta_{11}p'+\eta_{12}q',\quad \xi_2=\eta_{12}p'+\eta_{22}q',$$

and therefore

$$\sum l_4\xi_1=p'\sum l_4\eta_{11}+q'\sum l_4\eta_{12}$$
$$=A_4p'+H_4q'=-Vq'\frac{1}{\tau},$$

$$\sum l_4\xi_2=H_4p'+B_4q'=Vp'\frac{1}{\tau}.$$

When we multiply the first expression for l_4 by l_4, and add for all the space-dimensions, using the relation $\sum l_4 Y=0$, we find

$$\frac{1}{\tau}=\frac{1}{V}\frac{V}{\tau}(u_1p'+u_2q'),$$

which is an identity because

$$u_1p'+u_2q'=1.$$

When we multiply the second expression for l_4 by l_4, and add for all the space-dimensions, we find

$$\frac{1}{\rho\tau}=\frac{1}{V}\frac{V}{\tau}(v_1p'+v_2q'),$$

which is an identity because

$$v_1 p' + v_2 q' = \frac{1}{\rho}.$$

Further, when we multiply the second expression for l_4 by l_4, and add for all the space-dimensions, we also have

$$\begin{aligned}\frac{V}{\rho\tau} &= v_1(\textstyle\sum l_4\xi_2) - v_2(\textstyle\sum l_4\xi_1)\\ &= v_1(H_4p' + B_4q') - v_2(A_4p' + H_4q')\\ &= \begin{vmatrix} \bar{A}p' + \bar{H}q', & \bar{H}p' + \bar{B}q' \\ A_4p' + H_4q', & H_4p' + B_4q' \end{vmatrix}.\end{aligned}$$

When we substitute the values of $A_4p' + H_4q'$, $H_4p' + B_4q'$, again an identity ensues. But we retain the form, as being a covariant (the Jacobian, in fact) of the two quadratics

$$\bar{A}p'^2 + 2\bar{H}p'q' + \bar{B}^2q'^2, \quad A_4p'^2 + 2H_4p'q' + B_4q'^2;$$

it is analogous to the expression (§ 106)

$$\frac{V}{\sigma} = \begin{vmatrix} \bar{A}p' + \bar{H}q', & \bar{H}p' + \bar{B}q' \\ Ap' + Hq', & Hp' + Bq' \end{vmatrix}$$

for the torsion of a geodesic, which likewise is a covariant (again the Jacobian) of the two quadratics

$$\bar{A}p'^2 + 2\bar{H}p'q' + \bar{B}q'^2, \quad Ap'^2 + 2Hp'q' + Bq'^2.$$

Similarly when we multiply the first expression for l_4 by l_4, and add for all the space-dimensions, using the equation $\sum l_4 Y = 0$, we also have

$$\begin{aligned}\frac{V}{\tau} &= u_1(H_4p' + B_4q') - u_2(A_4p' + H_4q')\\ &= \begin{vmatrix} Ap' + Hq', & Hp' + Bq' \\ A_4p' + H_4q', & H_4p' + B_4q' \end{vmatrix},\end{aligned}$$

once more a covariant (also the Jacobian) of the two quadratics

$$Ap'^2 + 2Hp'q' + Bq'^2, \quad A_4p'^2 + 2H_4p'q' + B_4q'^2.$$

This equation, like the others, becomes an identity when the values of

$$A_4p' + H_4q', \quad H_4p' + B_4q',$$

are substituted.

Again, from the relations established in § 130 in the form

$$\begin{aligned}\eta_{11} &= Y\bar{A} + l_4A_4 + lq'^2T,\\ \eta_{12} &= Y\bar{H} + l_4H_4 - lq'p'T,\\ \eta_{22} &= Y\bar{B} + l_4B_4 + lp'^2T,\end{aligned}$$

we have

$$\begin{aligned} \xi_1 &= \eta_{11}p' + \eta_{12}q' \\ &= Yv_1 + l_4(A_4p' + H_4q') = Yv_1 - l_4\frac{Vq'}{\tau}, \\ \xi_2 &= \eta_{12}p' + \eta_{22}q' \\ &= Yv_2 + l_4(H_4p' + B_4q') = Yv_2 + l_4\frac{Vp'}{\tau}. \end{aligned}$$

When these values are substituted in the two expressions for l_4, the result is an identity in each instance.

It is therefore convenient to retain the two forms of value for l_4, which are

$$\begin{aligned} \frac{l_4}{\tau} &= \frac{1}{\sigma}Y + \frac{1}{V}(u_1\xi_2 - u_2\xi_1), \\ \frac{l_4}{\rho\tau} &= \qquad\quad \frac{1}{V}(v_1\xi_2 - v_2\xi_1), \end{aligned}$$

together with the foregoing values of ξ_1 and ξ_2, as well as the relations

$$A_4p' + H_4q' = -Vq'\frac{1}{\tau}, \quad H_4p' + B_4q' = Vp'\frac{1}{\tau}.$$

Moreover, we have

$$\frac{Y}{\rho} = \eta_{11}p'^2 + 2\eta_{12}p'q' + \eta_{22}q'^2,$$

and therefore

$$\frac{1}{\rho}Y\eta_{11} = \xi_1^2 + (\eta_{11}\eta_{22} - \eta_{12}^2)q'^2,$$

so that, on adding for all the space-dimensions, there results the equation

$$\frac{1}{\rho}\bar{A} = \sum\xi_1^2 + q'^2\left(\sum\eta_{11}\eta_{22} - \sum\eta_{12}^2\right) = \sum\xi_1^2 + V^2Kq'^2.$$

But

$$\begin{aligned} \sum\xi_1^2 &= \sum\left(Yv_1 - l_4\frac{Vq'}{\tau}\right)^2 \\ &= v_1^2 + V^2\frac{q'^2}{\tau^2}; \end{aligned}$$

and therefore

$$\frac{1}{\rho}\bar{A} = v_1^2 \;+ \left(\frac{1}{\tau^2} + K\right)V^2q'^2.$$

Similarly

$$\begin{aligned} \frac{1}{\rho}\bar{H} &= v_1v_2 - \left(\frac{1}{\tau^2} + K\right)V^2q'p', \\ \frac{1}{\rho}\bar{B} &= v_2^2 \;+ \left(\frac{1}{\tau^2} + K\right)V^2p'^2. \end{aligned}$$

Consequently

$$\frac{1}{\rho^2}(\bar{A}\bar{B}-\bar{H}^2)=V^2\left(\frac{1}{\tau^2}+K\right)(v_1p'+v_2q')^2,$$

that is,

$$\bar{A}\bar{B}-\bar{H}^2=V^2\left(\frac{1}{\tau^2}+K\right),$$

a relation of frequent occurrence.

Finally, for this set of relations, we have

$$\frac{1}{\rho}(A\bar{B}-2H\bar{H}+B\bar{A})=Av_2^2-2Hv_1v_2+Bv_2^2+V^2\left(\frac{1}{\tau^2}+K\right).$$

Now

$$\begin{aligned} u_1^2&=(Ap'+Hq')^2\\ &=A-V^2q'^2,\\ u_1u_2&=H+V^2q'p',\\ u_2^2&=B-V^2p'^2; \end{aligned}$$

and therefore

$$\begin{aligned} \frac{V^2}{\sigma^2}&=(u_1v_2-u_2v_1)^2\\ &=Av_2^2-2Hv_1v_2+Bv_1^2-V^2(v_1p'+v_2q')^2, \end{aligned}$$

that is,

$$Av_2^2-2Hv_1v_2+Bv_1^2=V^2\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right).$$

Hence

$$\frac{1}{\rho}(A\bar{B}-2H\bar{H}+B\bar{A})=V^2\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}+\frac{1}{\tau^2}+K\right),$$

another relation that frequently recurs.

The former relation

$$\bar{A}\bar{B}-\bar{H}^2=V^2\left(\frac{1}{\tau^2}+K\right),$$

when, in it, we substitute the values (§ 104)

$$\begin{aligned} \frac{\bar{A}}{\rho}&=\mathbf{a}p'^2+2\mathbf{h}p'q'+\mathbf{g}q'^2,\\ \frac{\bar{H}}{\rho}&=\mathbf{h}p'^2+2\mathbf{b}p'q'+\mathbf{f}q'^2,\\ \frac{\bar{B}}{\rho}&=\mathbf{g}p'^2+2\mathbf{f}p'q'+\mathbf{c}q'^2, \end{aligned}$$

and use the equation

$$V^2K=\mathbf{g}-\mathbf{b},$$

leads to the equation

$$\frac{V^2}{\rho^2\tau^2} = \bar{\mathbf{a}}q'^4 - 2\bar{\mathbf{h}}q'^3p' + (\bar{\mathbf{b}} + 2\bar{\mathbf{g}})q'^2p'^2 - 2\bar{\mathbf{f}}q'p'^3 + \bar{\mathbf{c}}p'^4,$$

where $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$, $\bar{\mathbf{f}}$, $\bar{\mathbf{g}}$, $\bar{\mathbf{h}}$, denote the minors of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, in the determinant Y of § 105. Hence, inserting the value of ρ^2, we obtain an expression for the tilt in the form

$$\frac{V^2}{\tau^2} = \frac{\bar{\mathbf{a}}q'^4 - 2\bar{\mathbf{h}}q'^2p' + (\bar{\mathbf{b}} + 2\bar{\mathbf{g}})q'^2p'^2 - 2\bar{\mathbf{f}}q'p'^3 + \bar{\mathbf{c}}p'^4}{\mathbf{a}p'^4 + 4\mathbf{h}p'^3q' + (2\mathbf{g} + 4\mathbf{b})p'^2q'^2 + 4\mathbf{f}p'q'^3 + \mathbf{c}q'^4}.$$

The value will be obtained, later (§ 133), in a different manner.

Again, we have the relation (p. 364)

$$\frac{l_4}{\rho\tau} = \frac{1}{V}(v_1\xi_2 - v_2\xi_1).$$

Also we have

$$\begin{aligned} \sum \eta_{11}\xi_1 &= \mathbf{a}p' + \mathbf{h}q', & \sum \eta_{11}\xi_2 &= \mathbf{h}p' + \mathbf{g}q', \\ \sum \eta_{12}\xi_1 &= \mathbf{h}p' + \mathbf{b}q', & \sum \eta_{12}\xi_2 &= \mathbf{b}p' + \mathbf{f}q', \\ \sum \eta_{22}\xi_1 &= \mathbf{g}p' + \mathbf{f}q', & \sum \eta_{22}\xi_2 &= \mathbf{f}p' + \mathbf{c}q', \end{aligned}$$

while

$$A_4 = \sum \eta_{11}l_4, \quad H_4 = \sum \eta_{12}l_4, \quad B_4 = \sum \eta_{22}l_4;$$

consequently

$$\begin{aligned} \frac{A_4}{\rho\tau} &= \frac{1}{V}\{(\mathbf{h}p' + \mathbf{g}q')v_1 - (\mathbf{a}p' + \mathbf{h}q')v_2\}, \\ \frac{H_4}{\rho\tau} &= \frac{1}{V}\{(\mathbf{b}p' + \mathbf{f}q')v_1 - (\mathbf{h}p' + \mathbf{b}q')v_2\}, \\ \frac{B_4}{\rho\tau} &= \frac{1}{V}\{(\mathbf{f}p' + \mathbf{c}q')v_1 - (\mathbf{g}p' + \mathbf{f}q')v_2\}. \end{aligned}$$

Further,

$$\begin{aligned} \frac{v_1}{\rho} &= \mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + 3\mathbf{k}p'q'^2 + \mathbf{f}q'^3, \\ \frac{v_2}{\rho} &= \mathbf{h}p'^3 + 3\mathbf{k}p'^2q' + 3\mathbf{f}p'q'^2 + \mathbf{c}q'^3; \end{aligned}$$

and therefore

$$\left.\begin{aligned} -\frac{V}{\rho^2\tau}A_4 &= \quad 2\bar{\mathbf{c}}p'^3q' - 3\bar{\mathbf{f}}p'^2q'^2 + (\bar{\mathbf{b}} + 2\bar{\mathbf{g}})p'q'^3 - \bar{\mathbf{h}}q'^4 \\ -\frac{V}{\rho^2\tau}H_4 &= -\bar{\mathbf{c}}p'^4 + \bar{\mathbf{f}}p'^3q' - \bar{\mathbf{h}}p'q'^3 + \bar{\mathbf{a}}q'^4 \\ -\frac{V}{\rho^2\tau}B_4 &= \quad \bar{\mathbf{f}}p'^4 - (\bar{\mathbf{b}} + 2\bar{\mathbf{g}})p'^3q' + 3\bar{\mathbf{h}}p'^2q'^2 - 2\bar{\mathbf{a}}p'q'^3 \end{aligned}\right\}.$$

NOTE. When the plenary homaloidal space of the surface is quadruple, the determinant Υ vanishes and its minors satisfy the relations

$$\bar{\mathfrak{b}}\bar{\mathfrak{c}}-\bar{\mathfrak{f}}^2=0, \quad \bar{\mathfrak{c}}\bar{\mathfrak{a}}-\bar{\mathfrak{g}}^2=0, \quad \bar{\mathfrak{a}}\bar{\mathfrak{b}}-\bar{\mathfrak{h}}^2=0,$$
$$\bar{\mathfrak{g}}\bar{\mathfrak{h}}-\bar{\mathfrak{a}}\bar{\mathfrak{f}}=0, \quad \bar{\mathfrak{h}}\bar{\mathfrak{f}}-\bar{\mathfrak{b}}\bar{\mathfrak{g}}=0, \quad \bar{\mathfrak{f}}\bar{\mathfrak{g}}-\bar{\mathfrak{c}}\bar{\mathfrak{h}}=0.$$

The expression for $V^2/\rho^2\tau^2$ is easily seen to be a perfect square, *quâ* quartic function of p' and q'; and this result is in accordance with the known formula * for the tilt of a geodesic on a surface in quadruple space

$$-\frac{V}{\rho\tau}=\bar{\mathfrak{a}}^{\frac{1}{2}}q'^2+\bar{\mathfrak{b}}^{\frac{1}{2}}q'p'+\bar{\mathfrak{c}}^{\frac{1}{2}}p'^2.$$

It may be remarked, in passing, that the quantities $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$, and therefore Υ, are invariantive for all orthogonal transformations. If therefore it is found that, for a surface in multiple space, the determinant Υ vanishes, we should infer that the surface exists in a plenary quadruple homaloidal space. The quantity Υ is not expressible, as is the magnitude $\mathfrak{g}-\mathfrak{b}$, solely in terms of A, H, B, and their parametric derivatives; and therefore this criterion, as to the dimensionality of the plenary space of a surface, cannot be framed solely from the constituents of the arc-element of the surface.

Ex. Through any point on a surface, two geodesics with direction-variables p_1', q_1', and p_2', q_2', are drawn. Their inclination is θ, the inclination of their prime normals is $\bar{\theta}$, and their radii of circular curvature are denoted by ρ_1, ρ_2; prove that

$$\frac{V^2}{\rho_1^2\rho_2^2}\frac{\sin^2\bar{\theta}}{\sin^2\theta}=\bar{\mathfrak{a}}q_1'^2q_2'^2+\bar{\mathfrak{b}}(p_1'q_2'+p_2'q_1')^2+4\bar{\mathfrak{c}}p_1'^2p_2'^2$$
$$-4\bar{\mathfrak{f}}p_1'p_2'(p_1'q_2'+p_2'q_1')+8\bar{\mathfrak{g}}p_1'p_2'q_1'q_2'-4\bar{\mathfrak{h}}q_1'q_2'(p_1'q_2'+p_2'q_1').$$

Shew also that, when the plenary space of the surface is quadruple, the right-hand side can be expressed in the form

$$\frac{4V^2}{\rho_1\rho_2\tau_1\tau_2}+\frac{1}{V^2}(\bar{\mathfrak{b}}-4\bar{\mathfrak{g}})\sin^2\theta,$$

the radii of tilt of the geodesics being denoted by τ_1, τ_2.

Magnitudes connected with the trinormal.

133. When the quantities $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$, are constructed from the values of η_{11}, η_{12}, η_{22}, which have been obtained in § 130, the properties

$$\sum l^2=1, \quad \sum lY=0, \quad \sum ll_4=0,$$

being used, we find

$$\mathfrak{a}=\bar{A}^2+A_4^2+\quad q'^4T^2, \quad \mathfrak{f}=\bar{H}\bar{B}+H_4B_4-q'p'^3T^2,$$
$$\mathfrak{b}=\bar{H}^2+H_4^2+q'^2p'^2T^2, \quad \mathfrak{g}=\bar{B}\bar{A}+B_4A_4+q'^2p'^2T^2,$$
$$\mathfrak{c}=\bar{B}^2+B_4^2+\quad p'^4T^2, \quad \mathfrak{h}=\bar{A}\bar{H}+A_4H_4-q'^3p'T^2.$$

* *G.F.D.*, vol. i, §§ 214, 235.

In the first place, we have

$$\mathbf{g}-\mathbf{b}=\bar{A}\bar{B}-\bar{H}^2+A_4B_4-H_4{}^2\,;$$

it has been proved (§ 130) that

$$A_4B_4-H_4{}^2=-\frac{V^2}{\tau^2},$$

and we have $\mathbf{g}-\mathbf{b}=V^2K$; hence

$$\bar{A}\bar{B}-\bar{H}^2=V^2\left(\frac{1}{\tau^2}+K\right),$$

a relation already obtained (§ 132).

Next, the determinant

$$\begin{vmatrix} \bar{A}^2+\ A_4{}^2, & \bar{A}\bar{H}+A_4H_4, & \bar{A}\bar{B}+A_4B_4 \\ \bar{A}\bar{H}+A_4H_4, & \bar{H}^2+\ H_4{}^2, & \bar{H}\bar{B}+H_4B_4 \\ \bar{A}\bar{B}+A_4B_4, & \bar{H}\bar{B}+H_4B_4, & \bar{B}^2+\ B_4{}^2 \end{vmatrix}$$

vanishes identically, whatever be the values of $\bar{A}$, $\bar{H}$, $\bar{B}$, A_4, H_4, B_4; and we therefore have

$$\begin{vmatrix} \mathbf{a}-\ \ q'^4T^2, & \mathbf{h}+q'^3p'T^2, & \mathbf{g}-q'^2p'^2T^2 \\ \mathbf{h}+q'^3p'T^2, & \mathbf{b}-q'^2p'^2T^2, & \mathbf{f}+\ q'p'^3T^2 \\ \mathbf{g}-q'^2p'^2T^2, & \mathbf{f}+\ q'p'^3T^2, & \mathbf{c}-\ \ p'^4T^2 \end{vmatrix}=0.$$

With the earlier significance of Υ in § 105, and using the symbols $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$, $\bar{\mathbf{f}}$, $\bar{\mathbf{g}}$, $\bar{\mathbf{h}}$, to denote the minors of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, in Υ, this equation becomes

$$T^2\{\bar{\mathbf{a}}q'^4-2\bar{\mathbf{h}}q'^3p'+(\bar{\mathbf{b}}+2\bar{\mathbf{g}})q'^2p'^2-2\bar{\mathbf{f}}q'p'^3+\bar{\mathbf{c}}p'^4\}=\Upsilon\,;$$

and therefore

$$V^2T^2=\Upsilon\rho^2\tau^2,$$

thus giving a value * for the quantity T.

Again, with the values of $\bar{A}$, $\bar{H}$, $\bar{B}$, cited on p. 367, and using the value of ρ given by

$$\frac{1}{\rho^2}=\mathbf{a}p'^4+4\mathbf{h}p'^3q'+(2\mathbf{g}+4\mathbf{b})p'^2q'^2+4\mathbf{f}p'q'^3+\mathbf{c}q'^4,$$

* When the plenary homaloidal space of the surface is quadruple, the determinant Υ vanishes (*G.F.D.*, vol. i, § 214). The quantity T^2 vanishes—a result to be expected, because the orthogonal frame of any superficial geodesic is then complete in the tangent, the prime normal, the binormal, and the trinormal: there are no principal lines of higher rank in the frame, so that the quantities l_5, l_6, ... do not exist, and therefore the composite typical direction-cosine does not exist. The vanishing of T secures that the terms in lT disappear from the expressions for η_{11}, η_{12}, η_{22}.

we find

$$\left.\begin{aligned}
\frac{1}{\rho^2}(\mathfrak{a}-\bar{A}^2) &= \qquad 4\bar{\mathfrak{c}}p'^2q'^2 - 4\bar{\mathfrak{f}}p'q'^3 + \bar{\mathfrak{b}}q'^4 = A_0\\
\frac{1}{\rho^2}(\mathfrak{b}-\bar{H}^2) &= \bar{\mathfrak{c}}p'^4 - 2\bar{\mathfrak{g}}p'^2q'^2 + \bar{\mathfrak{a}}q'^4 = B_0\\
\frac{1}{\rho^2}(\mathfrak{c}-\bar{B}^2) &= \bar{\mathfrak{b}}p'^4 - 4\bar{\mathfrak{h}}p'^3q' + 4\bar{\mathfrak{a}}p'^2q'^2 = C_0\\
\frac{1}{\rho^2}(\mathfrak{f}-\bar{H}\bar{B}) &= -\bar{\mathfrak{f}}p'^4 + 2\bar{\mathfrak{g}}p'^3q' + \bar{\mathfrak{h}}p'^2q'^2 - 2\bar{\mathfrak{a}}p'q'^3 = F_0\\
\frac{1}{\rho^2}(\mathfrak{g}-\bar{B}\bar{A}) &= \qquad 2\bar{\mathfrak{f}}p'^3q' - (\bar{\mathfrak{b}}+4\bar{\mathfrak{g}})p'^2q'^2 + 2\bar{\mathfrak{h}}p'q'^3 = G_0\\
\frac{1}{\rho^2}(\mathfrak{h}-\bar{A}\bar{H}) &= \qquad -2\bar{\mathfrak{c}}p'^3q' + \bar{\mathfrak{f}}p'^2q'^2 + 2\bar{\mathfrak{g}}p'q'^3 - \bar{\mathfrak{h}}q'^4 = H_0
\end{aligned}\right\},$$

the symbols A_0, B_0, C_0, F_0, G_0, H_0, being used solely for brevity. Thus we have

$$\begin{aligned}
\frac{1}{\rho^2}(A_4^2+q'^4T^2) &= A_0, & \frac{1}{\rho^2}(H_4B_4-q'p'^3T^2) &= F_0,\\
\frac{1}{\rho^2}(H_4^2+q'^2p'^2T^2) &= B_0, & \frac{1}{\rho^2}(B_4A_4+q'^2p'^2T^2) &= G_0,\\
\frac{1}{\rho^2}(B_4^2+p'^4T^2) &= C_0, & \frac{1}{\rho^2}(A_4H_4-q'^3p'T^2) &= H_0.
\end{aligned}$$

Hence

$$\frac{1}{\rho_2}(A_4^2p'+A_4H_4q') = A_0p'+H_0q',$$

and, as before,

$$A_4p'+H_4q' = -\frac{Vq'}{\tau};$$

therefore

$$-\frac{Vq'}{\rho^2\tau}A_4 = A_0p'+H_0q'.$$

Similarly we obtain expressions for H_4 and for B_4; and when the values of the quantities, such as A_0 and H_0, are inserted, we find the set of values for A_4, H_4, B_4, as given by

$$\left.\begin{aligned}
-\frac{V}{\rho^2\tau}A_4 &= \qquad 2\bar{\mathfrak{c}}p'^3q' - 3\bar{\mathfrak{f}}p'^2q'^2 + (\bar{\mathfrak{b}}+2\bar{\mathfrak{g}})p'q'^3 - \bar{\mathfrak{h}}q'^4\\
-\frac{V}{\rho^2\tau}H_4 &= -\bar{\mathfrak{c}}p'^4 + \bar{\mathfrak{f}}p'^3q' - \bar{\mathfrak{h}}p'q'^3 + \bar{\mathfrak{a}}q'^4\\
-\frac{V}{\rho^2\tau}B_4 &= \bar{\mathfrak{f}}p'^4 - (\bar{\mathfrak{b}}+2\bar{\mathfrak{g}})p'^3q' + 3\bar{\mathfrak{h}}p'^2q'^2 - 2\bar{\mathfrak{a}}p'q'^3
\end{aligned}\right\},$$

agreeing with the previous results (§ 132).

When these values of A_4 and H_4 are substituted in either of the relations

$$A_4p'+H_4q'=-\frac{Vq'}{\tau},\quad H_4p'+B_4q'=\frac{Vp'}{\tau},$$

they lead to the equation

$$\frac{V^2}{\rho^2\tau^2}=\bar{\mathfrak{a}}q'^4-2\bar{\mathfrak{h}}q'^3p'+(\bar{\mathfrak{b}}+2\bar{\mathfrak{g}})q'^2p'^2-2\bar{\mathfrak{f}}q'p'^3+\bar{\mathfrak{c}}p'^4,$$

already (§ 132) obtained.

134. The values of A_4, H_4, B_4, in terms of magnitudes connected with the surface are known; and a value of T has been obtained. Accordingly, the typical partial differential equations of the second order satisfied (§ 130) by the point-coordinates on the surface become

$$\left.\begin{aligned}\eta_{11}&=Y\bar{A}+l_4A_4+l\frac{\rho\tau}{V}Y^{\frac{1}{2}}q'^2\\ \eta_{12}&=Y\bar{H}+l_4H_4-l\frac{\rho\tau}{V}Y^{\frac{1}{2}}q'p'\\ \eta_{22}&=Y\bar{B}+l_4B_4+l\frac{\rho\tau}{V}Y^{\frac{1}{2}}p'^2\end{aligned}\right\},$$

where the coefficients of the typical direction-cosines are now expressed in terms of surface-magnitudes.

It has been pointed out (§ 130) that, in the orthogonal flat of the surface, three leading lines can be selected with typical direction-cosines determined by η_{11}, η_{12}, η_{22}. Another set of three leading lines can be selected, viz. the prime normal, the trinormal, and a line perpendicular to these (which, however, is not necessarily the quartinormal unless the plenary space is quintuple), their typical direction-cosines being Y, l_4, l. The preceding relations can be regarded as the equations of transformation from the typical direction-cosines of the one set of leading lines to the typical direction-cosines of the other set. The determinant of the coefficients of Y, l_4, l, on the right-hand sides, is equal to $Y^{\frac{1}{2}}$; and, by resolving the equations so as to provide explicit expressions for Y, l_4, l, in terms of η_{11}, η_{12}, η_{22}, the values for Y and l_4 are obtained in the same form as before, the values of Y and l_4 being given by

$$\frac{Y}{\rho}=\xi_1p'+\xi_2q',\quad -\frac{V}{\rho\tau}l_4=\xi_1v_2-\xi_2v_1.$$

Now

$$\begin{aligned}\sum\eta_{11}\xi_1&=\mathfrak{a}p'+\mathfrak{h}q',&\sum\eta_{11}\xi_2&=\mathfrak{h}p'+\mathfrak{g}q',\\ \sum\eta_{12}\xi_1&=\mathfrak{h}p'+\mathfrak{b}q',&\sum\eta_{12}\xi_2&=\mathfrak{b}p'+\mathfrak{f}q',\\ \sum\eta_{22}\xi_1&=\mathfrak{g}p'+\mathfrak{f}q',&\sum\eta_{22}\xi_2&=\mathfrak{f}p'+\mathfrak{c}q'.\end{aligned}$$

Multiplying the foregoing typical equation by η_{11}, η_{12}, η_{22}, in turn, and adding each of the several products for the space-dimensions, we obtain the results

$$\left.\begin{aligned} -\frac{V}{\rho\tau}A_4 &= (\mathbf{a}p' + \mathbf{h}q')v_2 - (\mathbf{h}p' + \mathbf{b}q')v_1 \\ -\frac{V}{\rho\tau}H_4 &= (\mathbf{h}p' + \mathbf{b}q')v_2 - (\mathbf{b}p' + \mathbf{f}q')v_1 \\ -\frac{V}{\rho\tau}B_4 &= (\mathbf{g}p' + \mathbf{f}q')v_2 - (\mathbf{f}p' + \mathbf{c}q')v_1 \end{aligned}\right\}.$$

When the values of v_1 and v_2, as given by

$$\frac{v_1}{\rho} = \mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + 3\mathbf{k}p'q'^2 + \mathbf{f}q'^3,$$

$$\frac{v_2}{\rho} = \mathbf{h}p'^3 + 3\mathbf{k}p'^2q' + 3\mathbf{f}p'q'^2 + \mathbf{c}q'^3,$$

are substituted, we return to the expressions already (§ 133) obtained for A_4, H_4, B_4.

Again, when the equations expressing η_{11}, η_{12}, η_{22}, in terms of X, l_4, l, are resolved for the latter magnitudes, we find an expression of the form

$$l = P\eta_{11} + Q\eta_{12} + R\eta_{22}.$$

Multiplying those also by η_{11}, η_{12}, η_{22}, in turn, and adding for the space-dimensions, we have

$$P\mathbf{a} + Q\mathbf{h} + R\mathbf{g} = \sum l\eta_{11} = \frac{\rho\tau}{V}\,\Upsilon^{\frac{1}{2}}q'^2,$$

$$P\mathbf{h} + Q\mathbf{b} + R\mathbf{f} = \sum l\eta_{12} = -\frac{\rho\tau}{V}\,\Upsilon^{\frac{1}{2}}q'p',$$

$$P\mathbf{g} + Q\mathbf{f} + R\mathbf{c} = \sum l\eta_{22} = \frac{\rho\tau}{V}\,\Upsilon^{\frac{1}{2}}p'^2.$$

The determinant of the coefficients on the left-hand side is equal to Υ; hence

$$\left.\begin{aligned} \frac{V}{\rho\tau}\Upsilon^{\frac{1}{2}}P &= \bar{\mathbf{a}}q'^2 - \bar{\mathbf{h}}q'p' + \bar{\mathbf{g}}p'^2 \\ \frac{V}{\rho\tau}\Upsilon^{\frac{1}{2}}Q &= \bar{\mathbf{h}}q'^2 - \bar{\mathbf{b}}q'p' + \bar{\mathbf{f}}p'^2 \\ \frac{V}{\rho\tau}\Upsilon^{\frac{1}{2}}R &= \bar{\mathbf{g}}q'^2 - \bar{\mathbf{f}}q'p' + \bar{\mathbf{c}}p'^2 \end{aligned}\right\}.$$

We therefore have an expression for the value of l.

Further, resolving the original equations directly, we have

$$l\Upsilon^{\frac{1}{2}} = \begin{vmatrix} \eta_{11}, & \eta_{12}, & \eta_{22} \\ \bar{A}, & \bar{H}, & \bar{B} \\ A_4, & H_4, & B_4 \end{vmatrix};$$

and therefore, comparing the two expressions for l, we find

$$\frac{V}{\rho\tau}(\bar{H}B_4 - \bar{B}H_4) = \mathfrak{a}q'^2 - \mathfrak{h}q'p' + \mathfrak{g}p'^2,$$

$$\frac{V}{\rho\tau}(\bar{B}A_4 - \bar{A}B_4) = \mathfrak{h}q'^2 - \mathfrak{b}q'p' + \mathfrak{f}p'^2,$$

$$\frac{V}{\rho\tau}(\bar{A}H_4 - \bar{H}A_4) = \mathfrak{g}q'^2 - \mathfrak{f}q'p' + \mathfrak{c}p'^2,$$

formulæ which can be verified by substituting the values of the quantities $\bar{A}$, $\bar{H}$, $\bar{B}$, A_4, H_4, B_4, in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$.

Ex. 1. Some of the results can be associated with the theory of binary forms. In particular, if

$$u = ax^2 + 2hxy + by^2, \quad u' = a'x^2 + 2h'xy + b'y^2,$$
$$J = (a'x + h'y)(hx + by) - (b'x + a'y)(ax + hy),$$
$$D = ab - h^2, \quad I = ab' - 2hh' + ba', \quad D' = a'b' - h'^2,$$

there exists the purely algebraical relation

$$J^2 = uu'I - u^2D' - u'^2D.$$

(i) When this is applied to the two forms

$$u = 1 = Ap'^2 + 2Hp'q' + Bq'^2, \quad u' = \frac{1}{\rho} = \bar{A}p'^2 + 2\bar{H}pq + \bar{B}q'^2,$$

taken without reference to the implicit values of the coefficients $\bar{A}$, $\bar{H}$, $\bar{B}$, then

$$J = \frac{V}{\sigma}, \quad D = V^2, \quad D' = \bar{A}\bar{B} - \bar{H}^2 = V^2\left(\frac{1}{\tau^2} + K\right),$$
$$I = A\bar{B} - 2H\bar{H} + B\bar{A};$$

and the relation gives

$$\frac{1}{\rho}(A\bar{B} - 2H\bar{H} + B\bar{A}) = V^2\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} + \frac{1}{\tau^2} + K\right),$$

in accordance with the result on p. 367.

(ii) When it is applied to the two forms

$$u = 1 = Ap'^2 + 2Hp'q' + Bq'^2, \quad u' = 0 = A_4p'^2 + 2H_4p'q' + B_4q'^2,$$

then

$$J = -\frac{V}{\tau}, \quad D = V^2, \quad D' = A_4B_4 - H_4^2 = -\frac{V^2}{\tau^2};$$

and the relation becomes an identity.

(iii) When it is applied to the two forms

$$u = 0 = A_4p'^2 + 2H_4p'q' + B_4q'^2, \quad u' = \frac{1}{\rho} = \bar{A}p'^2 + 2\bar{H}p'q' + \bar{B}q'^2,$$

then

$$J = \frac{V}{\rho\tau}, \quad D = A_4 B_4 - H_4^2 = -\frac{V^2}{\tau^2};$$

and again the relation becomes an identity.

In cases (ii) and (iii), the term involving the intermediate invariant I disappears because the value of the quantic $A_4 p'^2 + 2H_4 p'q' + B_4 q'^2$ is zero. The relation can also be regarded as determining (except for sign) the values of the respective Jacobians.

Ex. 2. By means of the formulæ

$$V\left(\frac{l_4}{\tau} - \frac{Y}{\sigma}\right) = u_1 \xi_2 - u_2 \xi_1,$$

or otherwise, obtain the formulæ

$$\left.\begin{aligned} V\left(\frac{A_4}{\tau} - \frac{\bar{A}}{\sigma}\right) &= u_1(\mathbf{h}p' + \mathbf{g}q') - u_2(\mathbf{a}p' + \mathbf{h}q') \\ V\left(\frac{H_4}{\tau} - \frac{\bar{H}}{\sigma}\right) &= u_1(\mathbf{b}p' + \mathbf{f}q') - u_2(\mathbf{h}p' + \mathbf{b}q') \\ V\left(\frac{B_4}{\tau} - \frac{\bar{B}}{\sigma}\right) &= u_1(\mathbf{f}p' + \mathbf{c}q') - u_2(\mathbf{g}p' + \mathbf{f}q') \end{aligned}\right\}.$$

Quartinormal of a geodesic : the coil.

135. To obtain the magnitude of the coil and the direction of the quartinormal for a superficial geodesic, we proceed from the equivalent equations for the direction of the trinormal

$$\frac{l_4}{\tau} - \frac{Y}{\sigma} = \frac{1}{V}(u_1 \xi_2 - u_2 \xi_1),$$

$$\frac{l_4}{\rho\tau} = \frac{1}{V}(v_1 \xi_2 - v_2 \xi_1).$$

Differentiating the former along the geodesic, and using the relations (§ 101)

$$\frac{d\xi_1}{ds} = \theta_1 + \alpha\xi_1 + \epsilon\xi_2 - \tfrac{1}{3}q' l_3 VK, \quad \frac{d\xi_2}{ds} = \theta_2 + \beta\xi_1 + \eta\xi_2 + \tfrac{1}{3}p' l_3 VK,$$

we have

$$\begin{aligned} &\frac{1}{\tau}\left(\frac{l_5}{\kappa} - \frac{l_3}{\tau}\right) + l_4 \frac{d}{ds}\left(\frac{1}{\tau}\right) - \frac{1}{\sigma}\left(\frac{l_3}{\sigma} - \frac{y'}{\rho}\right) - Y\frac{d}{ds}\left(\frac{1}{\sigma}\right) \\ &\qquad = -\frac{1}{V}(\alpha + \eta)(u_1\xi_2 - u_2\xi_1) + \frac{1}{V}\{(\alpha u_1 + \epsilon u_2)\xi_2 - (\beta u_1 + \eta u_2)\xi_1\} \\ &\qquad\qquad + \frac{1}{V}[u_1(\theta_2 + \beta\xi_1 + \eta\xi_2) - u_2(\theta_1 + \alpha\xi_1 + \epsilon\xi_2) + \tfrac{1}{3}l_3 VK] \\ &\qquad = \frac{1}{V}(u_1\theta_2 - u_2\theta_1) + \tfrac{1}{3}l_3 K ; \end{aligned}$$

and therefore

$$\frac{l_5}{\tau\kappa}+l_4\frac{d}{ds}\left(\frac{1}{\tau}\right)-l_3\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}+\tfrac{1}{3}K\right)-Y\frac{d}{ds}\left(\frac{1}{\sigma}\right)+\frac{y'}{\sigma\rho}=\frac{1}{V}(u_1\theta_2-u_2\theta_1).$$

Differentiating the second relation also along the geodesic, and using the further relations (§ 131)

$$\frac{dv_1}{ds}=w_1+\alpha v_1+\epsilon v_2,\quad \frac{dv_2}{ds}=w_2+\beta v_1+\eta v_2,$$

we have

$$\begin{aligned}\frac{1}{\rho\tau}&\left(\frac{l_5}{\kappa}-\frac{l_3}{\tau}\right)+l_4\frac{d}{ds}\left(\frac{1}{\rho\tau}\right)\\
&=-\frac{1}{V}(\alpha+\eta)(v_1\xi_2-v_2\xi_1)+\frac{1}{V}\{(w_1+\alpha v_1+\epsilon v_2)\xi_2-(w_2+\beta v_1+\eta v_2)\xi_1\}\\
&\qquad+\frac{1}{V}\left[v_1(\theta_2+\beta\xi_1+\eta\xi_2)-v_2(\theta_1+\alpha\xi_1+\epsilon\xi_2)+\tfrac{1}{3}\frac{l_3}{\rho}VK\right]\\
&=\frac{1}{V}(\xi_2w_1-\xi_1w_2)+\frac{1}{V}(v_1\theta_2-v_2\theta_1)+\tfrac{1}{3}\frac{l_3}{\rho}K.\end{aligned}$$

Now we have (§ 132)

$$\xi_1=Yv_1-l_4\frac{Vq'}{\tau},\quad \xi_2=Yv_2+l_4\frac{Vp'}{\tau},$$

and therefore

$$\begin{aligned}\xi_2w_1-\xi_1w_2&=l_4\frac{V}{\tau}\frac{d}{ds}\left(\frac{1}{\rho}\right)+Y(v_2w_1-v_1w_2)\\
&=l_4\frac{V}{\tau}\frac{d}{ds}\left(\frac{1}{\rho}\right)-Y\frac{V}{\sigma^2}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right),\end{aligned}$$

by the result in § 107. Hence the equation becomes

$$\frac{l_5}{\rho\tau\kappa}+\frac{l_4}{\rho}\frac{d}{ds}\left(\frac{1}{\tau}\right)-\frac{l_3}{\rho}\left(\frac{1}{\tau^2}+\tfrac{1}{3}K\right)+\frac{Y}{\sigma^2}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)=\frac{1}{V}(v_1\theta_2-v_2\theta_1).$$

We thus obtain two expressions, equivalent to one another, for a typical direction-cosine of the quartinormal; they involve the magnitudes θ_1 and θ_2, connected with the surface, as well as the magnitudes ρ, σ, τ, the values of which have already been obtained. But they also involve the magnitude of the coil; and, by squaring the first of the initial expressions for l_5 and adding for all spatial dimensions, an equation for the coil can be found in the form

$$\begin{aligned}\frac{1}{\tau^2\kappa^2}+\frac{\tau'^2}{\tau^4}+\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}+\tfrac{1}{3}K\right)^2+2\frac{\sigma'^2}{\sigma^4}+\frac{2}{\sigma^2\rho^2}+\frac{1}{\rho^4}&\\
=\frac{1}{V^2}\left(A\sum\theta_2^2-2H\sum\theta_1\theta_2+B\sum\theta_1^2\right),&\end{aligned}$$

where, again, the sums on the right-hand side are magnitudes belonging solely to the surface. Proceeding in the same way with the second of the initial expressions for l_5, we find

$$\frac{1}{\rho^2\tau^2\kappa^2}+\frac{\tau'^2}{\rho^2\tau^4}+\frac{1}{\rho^2}\left(\frac{1}{\tau^2}+\tfrac{1}{3}K\right)^2+\frac{1}{\sigma^4\rho^4}(\sigma\rho'-\rho\sigma')^2+\left(\frac{1}{\tau^2}+K\right)^2\left(\frac{1}{\rho^4}+\frac{\rho'^2}{\rho^4}+\frac{1}{\rho^2\sigma^2}\right)$$
$$=\frac{1}{V^2}(\bar{A}\sum\theta_2{}^2-2\bar{H}\sum\theta_1\theta_2+\bar{B}\sum\theta_1{}^2).$$

Both these relations involve the quantities $\sum\theta_1{}^2$, $\sum\theta_1\theta_2$, $\sum\theta_2{}^2$; that is, when expressed in full, they involve new magnitudes belonging to the surface of such types as

$$\sum\eta_{111}{}^2, \quad \sum\eta_{111}\eta_{112},$$

analogous to, but of higher order of derivation than, the magnitudes **a**, **b**, **c**, **f**, **g**, **h**.

When the two equations for l_5 are resolved for θ_1 and θ_2, they give

$$\left.\begin{aligned}\theta_1&=l_3\frac{v_1}{\sigma}+Yu_1\frac{d}{ds}\left(\frac{1}{\rho}\right)-y'\frac{v_1}{\rho}-Vq'\Theta\\ \theta_2&=l_3\frac{v_2}{\sigma}+Yu_2\frac{d}{ds}\left(\frac{1}{\rho}\right)-y'\frac{v_2}{\rho}+Vp'\Theta\end{aligned}\right\},$$

where

$$\Theta=\frac{l_5}{\tau\kappa}+l_4\frac{d}{ds}\left(\frac{1}{\tau}\right)-l_3\left(\frac{1}{\tau^2}+\tfrac{1}{3}K\right)-Y\frac{d}{ds}\left(\frac{1}{\sigma}\right).$$

A slight modification in the expressions for θ_1 and θ_2 can be made. In θ_1, the whole coefficient of Y

$$\begin{aligned}&=u_1\frac{d}{ds}\left(\frac{1}{\rho}\right)+Vq'\frac{d}{ds}\left(\frac{1}{\sigma}\right)\\ &=u_1(w_1p'+w_2q')+q'(w_1u_2-w_2u_1)=w_1,\end{aligned}$$

by the formula in § 131; and, in θ_2, the whole coefficient of Y similarly is equal to w_2. Hence

$$\left.\begin{aligned}\theta_1&=l_3\frac{v_1}{\sigma}+Yw_1-y'\frac{v_1}{\rho}-Vq'\left\{\frac{l_5}{\tau\kappa}+l_4\frac{d}{ds}\left(\frac{1}{\tau}\right)-l_3\left(\frac{1}{\tau^2}+\tfrac{1}{3}K\right)\right\}\\ \theta_2&=l_3\frac{v_2}{\sigma}+Yw_2-y'\frac{v_2}{\rho}+Vp'\left\{\frac{l_5}{\tau\kappa}+l_4\frac{d}{ds}\left(\frac{1}{\tau}\right)-l_3\left(\frac{1}{\tau^2}+\tfrac{1}{3}K\right)\right\}\end{aligned}\right|,$$

which may be taken as the definite interpretations of θ_1 and θ_2.

Various analytical results, in the form of summations affecting θ_1 and θ_2, may be placed on record for reference.

I. There is a set involving the direction-cosines of the principal lines of the geodesics. We have, at once,

$$\sum y'\theta_1 = -\frac{v_1}{\rho}, \quad \sum y'\theta_2 = -\frac{v_2}{\rho}$$
$$\sum Y\theta_1 = w_1, \qquad \sum Y\theta_2 = w_2,$$

admitting of partial verification through the relation

$$\theta_1 p' + \theta_2 q' = y'''.$$

Again,

$$\sum l_3\theta_1 = \frac{v_1}{\sigma} + \left(\frac{1}{\tau^2} + \tfrac{1}{3}K\right) Vq'$$
$$= -\tfrac{2}{3}q'VK + \frac{1}{V\rho}(u_2\bar{A} - u_1\bar{H}),$$

on using the relation

$$\bar{A}\bar{B} - \bar{H}^2 = V^2\left(\frac{1}{\tau^2} + K\right);$$

and similarly

$$\sum l_3\theta_2 = \frac{v_2}{\sigma} - \left(\frac{1}{\tau^2} + \tfrac{1}{3}K\right) Vp'$$
$$= -\tfrac{2}{3}p'VK + \frac{1}{V\rho}(u_2\bar{H} - u_1\bar{B}).$$

Also, there are relations

$$\sum l_4\theta_1 = -Vq'\frac{d}{ds}\left(\frac{1}{\tau}\right), \quad \sum l_4\theta_2 = Vp'\frac{d}{ds}\left(\frac{1}{\tau}\right),$$
$$V\frac{d}{ds}\left(\frac{1}{\tau}\right) = \sum l_4(u_1\theta_2 - u_2\theta_1);$$

and relations

$$\sum l_5\theta_1 = -\frac{Vq'}{\tau\kappa}, \quad \sum l_5\theta_2 = \frac{Vp'}{\tau\kappa}, \quad \frac{V}{\tau\kappa} = \sum l_5(u_1\theta_2 - u_2\theta_1).$$

II. There is a set of relations, which involve the direction-cosines of the parametric lines of reference in the tangent plane and the orthogonal flat of the surface. We have

$$\left.\begin{aligned} \sum \frac{\partial y}{\partial p}\theta_1 &= -\frac{\bar{A}}{\rho} + \tfrac{2}{3}V^2Kq'^2 \\ \sum \frac{\partial y}{\partial q}\theta_1 = \sum \frac{\partial y}{\partial p}\theta_2 &= -\frac{\bar{H}}{\rho} - \tfrac{2}{3}V^2Kq'p' \\ \sum \frac{\partial y}{\partial q}\theta_2 &= -\frac{\bar{B}}{\rho} + \tfrac{2}{3}V^2Kp'^2 \end{aligned}\right\},$$

by direct substitution. Again, as

$$\xi_1 = Yv_1 - l_4 \frac{Vq'}{\tau}, \quad \xi_2 = Yv_2 + l_4 \frac{Vp'}{\tau},$$

we have

$$\sum \xi_1\theta_1 = v_1w_1 - V^2q'^2 \frac{\tau'}{\tau^2},$$

$$\sum \xi_1\theta_2 = v_1w_2 + V^2q'p' \frac{\tau'}{\tau^2},$$

$$\sum \xi_2\theta_1 = v_2w_1 + V^2q'p' \frac{\tau'}{\tau^2},$$

$$\sum \xi_2\theta_2 = v_2w_2 - V^2p'^2 \frac{\tau'}{\tau^2};$$

and therefore

$$\sum (\xi_1\theta_2 - \xi_2\theta_1) = v_1w_2 - v_2w_1 = \frac{V}{\sigma^2} \frac{d}{ds}\left(\frac{\sigma}{\rho}\right),$$

by the formulæ of § 107.

We easily find, by using the formulæ of § 101,

$$\left.\begin{aligned}
\sum y'\eta_{111} &= -(\mathbf{a}p' + \mathbf{h}q'), & V\sum l_3\eta_{111} &= \mathbf{a}u_2 - \mathbf{h}u_1 \\
\sum y'\eta_{112} &= -(\mathbf{h}p' + \mathbf{k}q'), & V\sum l_3\eta_{112} &= \mathbf{h}u_2 - \mathbf{k}u_1 \\
\sum y'\eta_{122} &= -(\mathbf{k}p' + \mathbf{f}\,q'), & V\sum l_3\eta_{122} &= \mathbf{k}u_2 - \mathbf{f}u_1 \\
\sum y'\eta_{122} &= -(\mathbf{f}p' + \mathbf{c}q'), & V\sum l_3\eta_{222} &= \mathbf{f}u_2 - \mathbf{c}u_1
\end{aligned}\right\}.$$

Other relations, arising out of the magnitudes $\sum\eta_{ij}\theta_k$, for $i, j, k = 1, 2$, independently, lead to equations involving the coefficients A_{ijk}, H_{ijk}, B_{ijk}, of p. 269.

Because

$$\sum \xi_1\eta_{ijk} = A_{ijk}p' + H_{ijk}q', \quad \sum \xi_2\eta_{ijk} = H_{ijk}p' + B_{ijk}q',$$

and

$$\frac{V}{\rho\tau} l_4 = v_1\xi_2 - v_2\xi_1,$$

we have

$$\frac{V}{\rho\tau} \sum l_4\eta_{ijk} = \begin{vmatrix} v_1, & A_{ijk}p' + H_{ijk}q' \\ v_2, & H_{ijk}p' + B_{ijk}q' \end{vmatrix}.$$

For the expression of the quantity as above given, it is necessary to use the quantities $\sum\theta_1^2$, $\sum\theta_1\theta_2$, $\sum\theta_2^2$, each of which is a homogeneous quartic in p' and q', with new coefficients which arise from the twelve magnitudes

$$\begin{aligned}
\sum\eta_{111}^2 = A_3, \quad \sum\eta_{111}\eta_{112} = H_3, \quad \sum\eta_{111}\eta_{122} = G_3, \quad \sum\eta_{111}\eta_{222} = L_3, \\
\sum\eta_{112}^2 = B_3, \quad \sum\eta_{112}\eta_{122} = F_3, \quad \sum\eta_{112}\eta_{222} = M_3, \\
\sum\eta_{122}^2 = C_3, \quad \sum\eta_{122}\eta_{222} = N_3, \\
\sum\eta_{222}^2 = D_3.
\end{aligned}$$

For the present purpose, we require magnitudes of the type $\sum \eta_{ijk}\theta_l$, for $i, j, k, l, =1, 2$; thus

$$\sum \eta_{111}\theta_1 = A_3 p'^2 + 2H_3 p'q' + G_3 q'^2,$$
$$\sum \eta_{111}\theta_2 = H_3 p'^2 + 2G_3 p'q' + L_3 q'^2.$$

When the relation

$$\frac{l_5}{\tau\kappa} + l_4 \frac{d}{ds}\left(\frac{1}{\tau}\right) - l_3\left(\frac{1}{\sigma^2} + \frac{1}{\tau^2} + \tfrac{1}{3}K\right) - Y\frac{d}{ds}\left(\frac{1}{\sigma}\right) + \frac{y'}{\sigma\rho} = \frac{1}{V}(u_1\theta_2 - u_2\theta_1),$$

in the text, is multiplied by η_{111} and summation is taken over all the space-dimensions, we obtain an expression for the magnitude $\sum l_5\eta_{111}$; and similarly for the quantities $\sum l_5\eta_{112}$, $\sum l_5\eta_{122}$, $\sum l_5\eta_{122}$.

By means of these relations, we can obtain the coefficients in the equations

$$\eta_{ijk} = \frac{\partial y}{\partial p}\phi_{ijk} + \frac{\partial y}{\partial q}\psi_{ijk} + YE_{ijk} + l_4 R_{ijk} + l_5 S_{ijk} + l_6 T_{ijk} + \dots .$$

Thus we have

$$\frac{V}{\rho\tau} R_{ijk} = \begin{vmatrix} v_1, & A_{ijk}p' + H_{ijk}q' \\ v_2, & H_{ijk}p' + B_{ijk}q' \end{vmatrix};$$

and there are relations among the coefficients of l_6, l_7, ... of the type

$$T_{111}p'^2 + 2T_{112}p'q' + T_{122}q'^2 = 0,$$
$$T_{112}p'^2 + 2T_{122}p'q' + T_{222}q'^2 = 0.$$

CHAPTER XII

Curves of Curvature on Surfaces

Curves of circular curvature.

136. We have seen (§ 54) that, in any amplitude in multiple space, there are curves of circular curvature at any point, their directions being defined by each of the properties characteristic of such curves upon a surface in homaloidal triple space. According to one definition, their directions at a point provide a maximum or a minimum circular curvature for their geodesic tangent among all geodesics through the point; according to the other, their directions are such that the prime normals of the successive tangential geodesics intersect; the two definitions are equivalent as regards analytical representation. Moreover, this representation shews that they have the geometrical property that, at every point in their course, the torsion of a geodesic tangent is zero.

For most amplitudes, the analytical equations, which are to determine the directions of the curves of curvature at a point, cannot be resolved explicitly by present methods. It is possible, however, to construct the single equation which determines their directions on a surface in any plenary homaloidal space; and it is also possible to determine the superficial measures of geodesic circular curvature. To some of the results we now proceed; and it will appear that (contrary to the result in the theory of surfaces in triple space) there exists no simple relation between these measures and the Riemann sphericity for a surface in space of more than three dimensions. This divergence is already known * for a surface in quadruple space: we shall therefore assume that the plenary space is of more than four dimensions, so that (§ 105) the determinant Υ is not zero.

The magnitude of the circular curvature of a superficial geodesic with direction-variables p', q', subject to the permanent relation

$$Ap'^2 + 2Hp'q' + Bq'^2 = 1,$$

is given by the equation

$$\frac{1}{\rho^2} = \mathbf{a}p'^4 + 4\mathbf{h}p'^3q' + 6\mathbf{k}p'^2q'^2 + 4\mathbf{f}p'q'^3 + \mathbf{c}q'^4,$$

where $\mathbf{k}$ and the Riemann sphericity K are such that

$$3\mathbf{k} = \mathbf{g} + 2\mathbf{b}, \quad V^2K = \mathbf{g} - \mathbf{b},$$

the magnitudes $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, being defined as in § 105.

* *G.F.D.*, vol. i, ch. xiii.

For brevity, we shall call the directions of the curves of curvature at a point, the principal directions, and the circular curvatures of the geodesics, in the principal directions at a point, the principal curvatures. These principal directions and principal curvatures can be determined from the requirement that the curvature is a maximum or a minimum among all the possible directions on the surface at the point; and the critical conditions are that the magnitude, expressing the value of $1/\rho^2$, shall have a maximum or a minimum for direction-variables satisfying the permanent arc-relation. These critical conditions are

$$\begin{aligned} \mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + 3\mathbf{k}p'q'^2 + \mathbf{f}q'^3 &= u(Ap' + Hq'), \\ \mathbf{h}p'^3 + 3\mathbf{k}p'^2q' + 3\mathbf{f}p'q'^2 + \mathbf{c}q'^3 &= u(Hp' + Bq'), \end{aligned}$$

where u is a multiplier left undetermined in the actual formation of the equations. The value of u is at once obtained: we multiply these equations by p' and q' respectively, and add; and then we find

$$u = \frac{1}{\rho^2}.$$

The equation for the principal directions follows from the elimination of u between the two critical equations; it is

$$\begin{vmatrix} \mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + 3\mathbf{k}p'q'^2 + \mathbf{f}q'^3, & Ap' + Hq' \\ \mathbf{h}p'^3 + 3\mathbf{k}p'^2q' + 3\mathbf{f}p'q'^2 + \mathbf{c}q'^3, & Hp' + Bq' \end{vmatrix} = 0.$$

In the first place, this equation shews that the torsion of a geodesic in the direction p', q', thus determined, that is, of a geodesic tangent to a curve of curvature, vanishes (§ 106), verifying the general descriptive property of a curve of curvature.

Next, the equation manifestly is a quartic in the ratio p'/q', so that there are four principal directions at any point of a surface; in this respect, there is a divergence from the property of a surface in triple space, which has only two principal directions *. Moreover, it is easy to verify that the discriminant of the quartic equation for the principal directions does not vanish †: thus the four principal directions are distinct.

* There is a similar divergence as regards asymptotic directions on a surface, according as the plenary space is triple or of more than three dimensions. For triple space, there are two asymptotic directions on a surface: for space of higher dimensionality, there are four asymptotic directions. Such directions are given by the vanishing values of the circular curvature.

† It is a purely numerical multiple of the invariant

$$\left(A\frac{\partial^2}{\partial q'^2} - 2H\frac{\partial^2}{\partial q'\,\partial p'} + B\frac{\partial^2}{\partial p'^2}\right)^6 (I_0 Z^2 \mathbf{H} - J_0 Z^3),$$

where Z is the quartic for $\frac{1}{\rho^2}$, $\mathbf{H}$ is its Hessian (which occurs in the expression for $\frac{1}{\tau^2}$ in § 132), while I_0 and J_0 are the quadrinvariant and the cubinvariant of Z.

Equation for the principal radii of circular curvature.

137. The magnitudes of the principal curvatures of the surface are given by the equation, which results from the elimination of the direction-variables between the two critical equations. These can be written in the forms

$$\mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + 3\mathbf{k}p'q'^2 + \mathbf{f}q'^3 - u(Ap' + Hq')(Ap'^2 + 2Hp'q' + Bq'^2) = 0,$$
$$\mathbf{h}p'^3 + 3\mathbf{k}p'^2q' + 3\mathbf{f}p'q'^2 + \mathbf{c}q'^3 - u(Hp' + Bq')(Ap'^2 + 2Hp'q' + Bq'^2) = 0;$$

and, in their turn, these two equations imply that the partial derivatives, with regard to p' and q', of the quartic

$$\Theta = \mathbf{a}p'^4 + 4\mathbf{h}p'^3q' + 6\mathbf{k}p'^2q'^2 + 4\mathbf{f}p'q'^3 + \mathbf{c}q'^4 - u(Ap'^2 + 2Hp'q' + Bq'^2)^2,$$

are to vanish. The eliminant of these partial derivatives is, in fact, the discriminant of Θ; and therefore the equation, which determines the magnitudes of the principal curvatures of the surface, is provided by the condition that the discriminant of Θ shall vanish.

When we write

$$A^2 = \mathbf{a}_0,\quad AH = \mathbf{h}_0,\quad AB = \mathbf{g}_0,\quad H^2 = \mathbf{b}_0,\quad HB = \mathbf{f}_0,\quad B^2 = \mathbf{c}_0,\quad AB + 2H^2 = 3\mathbf{k}_0,$$

and also

$$a_0 = \mathbf{a} - u\mathbf{a}_0,\quad a_1 = \mathbf{h} - u\mathbf{h}_0,\quad a_2 = \mathbf{k} - u\mathbf{k}_0,\quad a_3 = \mathbf{f} - u\mathbf{f}_0,\quad a_4 = \mathbf{c} - u\mathbf{c}_0,$$

the quartic Θ becomes

$$\Theta = (a_0, a_1, a_2, a_3, a_4 \between p', q')^4.$$

The quadrinvariant I and the cubinvariant J of the quartic Θ are

$$I = a_0a_4 - 4a_1a_3 + 3a_2^2,$$
$$J = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3;$$

and the vanishing discriminant of the quartic, denoted by D, gives

$$D = I^3 - 27J^2 = 0,$$

as the required equation. To obtain the explicit form of this equation, which should be a quartic in u to correspond with the four principal directions of curvature, we must calculate the values of I and J.

When substitution is effected in I, we find

$$I = I_0 - uI_1 + u^2I_2,$$

where

$$I_0 = \mathbf{ac} - 4\mathbf{hf} + 3\mathbf{k}^2,$$
$$I_1 = \mathbf{ac}_0 - 4\mathbf{hf}_0 + 6\mathbf{kk}_0 - 4\mathbf{fh}_0 + \mathbf{ca}_0,$$
$$I_2 = \mathbf{a}_0\mathbf{c}_0 - 4\mathbf{h}_0\mathbf{f}_0 + 3\mathbf{k}_0^2 = \tfrac{4}{3}V^4.$$

Similarly, after substitution in J, we find

$$J = J_0 - uJ_1 + u^2J_2 - u^3J_3,$$

where

$$
\begin{aligned}
J_0 &= \mathbf{akc} + 2\mathbf{hkf} - \mathbf{af}^2 - \mathbf{ch}^2 - \mathbf{k}^3 ;\\
J_1 &= \mathbf{c}_0(\mathbf{ak} - \mathbf{h}^2) + 2\mathbf{f}_0(\mathbf{hk} - \mathbf{af})\\
&\qquad + \mathbf{a}_0(\mathbf{kc} - \mathbf{f}^2) + 2\mathbf{h}_0(\mathbf{kf} - \mathbf{ch}) + \mathbf{k}_0(\mathbf{ac} + 2\mathbf{hf} - 3\mathbf{k}^2) ;\\
J_2 &= \mathbf{c}(\mathbf{a}_0\mathbf{k}_0 - \mathbf{h}_0^2) + 2\mathbf{f}(\mathbf{h}_0\mathbf{k}_0 - \mathbf{a}_0\mathbf{f}_0)\\
&\qquad + \mathbf{a}(\mathbf{k}_0\mathbf{c}_0 - \mathbf{f}_0^2) + 2\mathbf{h}(\mathbf{k}_0\mathbf{f}_0 - \mathbf{c}_0\mathbf{h}_0) + \mathbf{k}(\mathbf{a}_0\mathbf{c}_0 + 2\mathbf{h}_0\mathbf{f}_0 - 3\mathbf{k}_0^2)\\
&= \tfrac{1}{3}V^2 I_1 ;\\
J_3 &= \mathbf{a}_0\mathbf{k}_0\mathbf{c}_0 + 2\mathbf{h}_0\mathbf{k}_0\mathbf{f}_0 - \mathbf{a}_0\mathbf{f}_0^2 - \mathbf{c}_0\mathbf{h}_0^2 - \mathbf{k}_0^3\\
&= \tfrac{8}{27}V^6.
\end{aligned}
$$

Of these quantities which occur in I and J, the magnitudes I_0 and J_0 are the respective quadrinvariant and cubinvariant of the quartic form which gives the value of $1/\rho^2$, while I_1 and J_1 are invariants intermediate to this quartic form and the quadratic form $Ap'^2 + 2Hp'q' + Bq'^2$.

The discriminant of a binary quartic is of the sixth degree in its coefficients; and therefore the discriminant of Θ normally would be of degree six in u. But when substitution is effected in the equation

$$I^3 - 27J^2 = 0,$$

the terms in u^6 and u^5 disappear owing to the values of J_3, J_2, I_2; and the resulting equation is found to be the quartic

$$C_0 - C_1 u + C_2 u^2 - C_3 u^3 + C_4 u^4 = 0,$$

where

$$
\begin{aligned}
C_0 &= I_0^3 - 27J_0^2,\\
C_1 &= 3(I_0^2 I_1 - 18 J_0 J_1),\\
C_2 &= 3I_0 I_1^2 - 18 J_0 I_1 V^2 - 27 J_1^2 + 4 I_0^2 V^4,\\
C_3 &= I_1^3 - 18 I_1 J_1 V^2 + 18 I_1 I_0 V^4 - 16 J_0 V^6,\\
C_4 &= I_1^2 V^4 - 16 J_1 V^6 + \tfrac{16}{3} I_0 V^8.
\end{aligned}
$$

Thus there are four measures of circular curvature of the surface, being the four symmetrical combinations of the principal curvatures of the surface represented by

$$\frac{C_3}{C_4},\quad \frac{C_2}{C_4},\quad \frac{C_1}{C_4},\quad \frac{C_0}{C_4};$$

but a more specific geometrical significance of the measures is not forthcoming.

138. Two remarks may be made. In the first place, we are dealing with two quantics

$$
\begin{aligned}
U &= Ap'^2 + 2Hp'q' + Bq'^2,\\
Z &= \mathbf{a}p'^4 + 4\mathbf{h}p'^3 q' + 6\mathbf{k}p'^2 q'^2 + 4\mathbf{f}p'q'^3 + \mathbf{c}q'^4.
\end{aligned}
$$

The full asyzygetic system of concomitants of a binary quartic and a binary quadratic is known *; and it appears that there are six invariants in that system.

* J. H. Grace and A. Young, *Algebra of Invariants*, § 143.

But the number * of algebraically independent invariants is only five, so that a single relation must subsist among the six invariants; and, in fact, the five independent invariants can be taken to be V^2, I_0, J_0, I_1, J_1. Of these, V^2 is the discriminant of U, while I_0 and J_0 are respectively the quadrinvariant and the cubinvariant of Z; the invariants I_1 and J_1 are intermediate between the two quantics, being merely numerical multiples of the quantities symbolically represented by

$$I_1=(ZU^2)^4, \quad J_1=(\mathbf{H}U^2)^4,$$

respectively, where $\mathbf{H}$ is the Hessian of Z. The remaining invariant, expressible in terms of V^2, I_0, J_0, I_1, J_1, is a numerical multiple of the quantity symbolically represented by

$$(\Phi U^3)^6,$$

where Φ is the cubicovariant of Z.

Again, the total number of concomitants (invariants and covariants together) in that asyzygetic system is eighteen. But the number † of algebraically independent concomitants is only seven, so that eleven relations must subsist among the members of the system. When regard is paid to invariants alone, five can be taken; and therefore two other magnitudes would suffice to constitute an algebraically complete system. It is, however, convenient to retain certain other magnitudes, although the aggregate of members in the full retained system may not be independent of one another: and we therefore retain, in addition to the original quantics U and Z, the Jacobian J_{24} of U and Z, the Hessian $\mathbf{H}$ of Z, and the cubicovariant Φ of Z.

A geometric significance can be given to the covariants. The quantic U is equal to unity; and the quantic Z is equal to $1/\rho^2$. We can take J_{24} in the form

$$\begin{aligned} J_{24} &= \tfrac{1}{8}\left(\frac{\partial Z}{\partial p'}\frac{\partial U}{\partial q'} - \frac{\partial Z}{\partial q'}\frac{\partial U}{\partial p'}\right) = \tfrac{1}{8}(ZU) \\ &= \begin{vmatrix} \mathbf{a}p'^3+3\mathbf{h}p'^2q'+3\mathbf{k}p'q'^2+\mathbf{f}q'^3, & Ap'+Hq' \\ \mathbf{h}p'^3+3\mathbf{k}p'^2q'+3\mathbf{f}p'q'^2+\mathbf{c}q'^3, & Hp'+Bq' \end{vmatrix} \\ &= \frac{V}{\rho\sigma}, \end{aligned}$$

* The invariants of any binary form, or any system of binary forms, satisfy a complete (Jacobian) system of three linearly independent homogeneous partial differential equations of the first order. For the invariants in question, eight parametric variables A, H, B, $\mathbf{a}$, $\mathbf{h}$, $\mathbf{k}$, $\mathbf{f}$, $\mathbf{c}$, occur in the equations; and therefore, by the usual theorem concerning the algebraically independent integrals of a Jacobian system, the number of algebraically independent invariants is five.

† For the system of concomitants, the total number of arguments occurring in the complete Jacobian system is ten, being made up of the direction-variables p' and q', in addition to the eight coefficients A, H, B, $\mathbf{a}$, $\mathbf{h}$, $\mathbf{k}$, $\mathbf{f}$, $\mathbf{c}$; by the same theorem as before, the number of algebraically independent concomitants is seven.

by § 106. Again, the Hessian $\mathbf{H}$ is

$$\mathbf{H}=\begin{vmatrix} \mathbf{a}p'^2+2\mathbf{h}p'q'+\mathbf{k}q'^2, & \mathbf{h}p'^2+2\mathbf{k}p'q'+\mathbf{f}q'^2 \\ \mathbf{h}p'^2+2\mathbf{k}p'q'+\mathbf{f}q'^2, & \mathbf{k}p'^2+2\mathbf{f}p'q'+\mathbf{c}q'^2 \end{vmatrix};$$

but

$$\mathbf{a}p'^2+2\mathbf{h}p'q'+\mathbf{k}p'^2=\frac{\bar{A}}{\rho}-\tfrac{2}{3}V^2Kq'^2,$$

$$\mathbf{h}p'^2+2\mathbf{k}p'q'+\ \mathbf{f}q'^2=\frac{\bar{H}}{\rho}+\tfrac{2}{3}V^2Kq'p',$$

$$\mathbf{k}p'^2+\ 2\mathbf{f}p'q'+\ \mathbf{c}q'^2=\frac{\bar{B}}{\rho}-\tfrac{2}{3}V^2Kp'^2,$$

so that

$$\mathbf{H}=\frac{1}{\rho^2}(\bar{A}\bar{B}-\bar{H}^2)-\tfrac{2}{3}V^2K\frac{1}{\rho^2}=\frac{V^2}{\rho^2}\left(\frac{1}{\tau^2}+\tfrac{1}{3}K\right).$$

Further, the cubicovariant Φ is

$$\Phi=\tfrac{1}{8}\left(\frac{\partial Z}{\partial p'}\frac{\partial \mathbf{H}}{\partial q'}-\frac{\partial Z}{\partial q'}\frac{\partial \mathbf{H}}{\partial p'}\right)$$
$$=(\mathbf{a}^2\mathbf{f}-3\mathbf{ahk}+2\mathbf{h}^3)p'^6+\dots,$$

retaining only the first term which always is sufficient to identify the whole binariant. Connected with the quadratic forms

$$\bar{A}p'^2+2\bar{H}p'q'+\bar{B}q'^2,\quad A_4p'^2+2H_4p'q'+B_4q'^2,$$

there is an invariantive form *

$$A_4\bar{B}-2H_4\bar{H}+B_4\bar{A};$$

and so, having regard to the values of A_4, H_4, B_4, obtained in § 133, we find

$$-\frac{V}{\rho^2\tau}\left(A_4\frac{\bar{B}}{\rho}-2H_4\frac{\bar{H}}{\rho}+B_4\frac{\bar{A}}{\rho}\right)=\{2\mathbf{h}(\mathbf{ab}-\mathbf{h}^2)+\mathbf{a}(\mathbf{gh}-\mathbf{af})\}p'^6+\dots$$
$$=\{-\mathbf{a}^2\mathbf{f}+\mathbf{ah}(\mathbf{g}+2\mathbf{b})-2\mathbf{h}^3\}p'^6+\dots$$
$$=-\Phi,$$

so that

$$\frac{V}{\rho^3\tau}(A_4\bar{B}-2H_4\bar{H}+B_4\bar{A})=\Phi.$$

Again, there exists the algebraical identity

$$\Phi^2+4\mathbf{H}^3=I_0\mathbf{H}Z^2-J_0Z^3,$$

among concomitants of a binary quantic, so that, if we write

$$z=-\frac{\mathbf{H}}{Z}=-V^2\left(\frac{1}{\tau^2}+\tfrac{1}{3}K\right),$$

* See also the later result on p. 399.

we have

$$\Phi^2 = \frac{1}{\rho^6}(4z^3 - I_0 z - J_0).$$

Two instances of the method of reduction of any concomitant, to algebraical dependence upon selected and retained forms, may suffice. We have

$$J^2_{24} = p'^8 (\mathbf{a}H - \mathbf{h}A)^2 + \dots;$$

but

$$\begin{aligned}(\mathbf{a}H - \mathbf{b}A)^2 &= \mathbf{a}^2(AB - V^2) - 2\mathbf{ah}AH + A^2\{\mathbf{ak} - (\mathbf{ak} - \mathbf{h}^2)\}\\ &= \mathbf{a}A(B\mathbf{a} - 2H\mathbf{h} + A\mathbf{k}) - V^2\mathbf{a}^2 - A^2(\mathbf{ak} - \mathbf{h}^2).\end{aligned}$$

Now $B\mathbf{a} - 2H\mathbf{h} + A\mathbf{k}$ is the leading coefficient of the concomitant symbolically represented by

$$\tfrac{1}{24}(UZ)^2,$$

being of the second order in p', q'; the quantity A is the leading coefficient of U which is unity, the quantity $\mathbf{a}$ is the leading coefficient of Z which is $1/\rho^2$, and the quantity $\mathbf{ak} - \mathbf{h}^2$ is the leading coefficient of $\mathbf{H}$ the value of which is given in the text. Hence, substituting the value of J_{24}, we have

$$\begin{aligned}\tfrac{1}{24}(UZ)^2 &= (B\mathbf{a} - 2H\mathbf{h} + A\mathbf{k})p'^2 + 2(B\mathbf{h} - 2H\mathbf{k} + A\mathbf{f})p'q' + (B\mathbf{k} - 2H\mathbf{f} + A\mathbf{c})q'^2\\ &= V^2\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} + \frac{1}{\tau^2} + \tfrac{1}{3}K\right).\end{aligned}$$

Using the same method of identification by leading coefficients, we can obtain an expression for the covariant

$$\begin{aligned}\tfrac{1}{4}(\mathbf{H}U) &= \tfrac{1}{4}\left(\frac{\partial \mathbf{H}}{\partial p'}\frac{\partial U}{\partial q'} - \frac{\partial \mathbf{H}}{\partial q'}\frac{\partial U}{\partial p'}\right)\\ &= \{2(\mathbf{ak} - \mathbf{h}^2)H - A(\mathbf{af} - \mathbf{hk})\}p'^4 + \dots .\end{aligned}$$

Hence

$$\begin{aligned}\tfrac{1}{4}Z(\mathbf{H}U) - 2\mathbf{H}J_{24} &= \{2(\mathbf{a}^2\mathbf{k} - \mathbf{ah}^2)H - A(\mathbf{a}^2\mathbf{f} - \mathbf{ahk})\}p'^8 + \dots\\ &\quad - \{2(\mathbf{ak} - \mathbf{h}^2)(\mathbf{a}H - \mathbf{h}A)p'^8 + \dots\}\\ &= -A(\mathbf{a}^2\mathbf{f} - 3\mathbf{ahk} + 2\mathbf{h}^3)p'^8 + \dots\\ &= -U\Phi = -\Phi;\end{aligned}$$

and therefore

$$\begin{aligned}\tfrac{1}{4}(\mathbf{H}U) &= \frac{2V^3}{\rho\sigma}\left(\frac{1}{\tau^2} + \tfrac{1}{3}K\right) - \rho^2\Phi\\ &= -\frac{2zV}{\rho\sigma} - \frac{1}{\rho}(4z^3 - I_0 z - J_0)^{\frac{1}{2}},\end{aligned}$$

with the foregoing value of z.

The Riemann sphericity in the system of concomitants.

139. In the second place, the Riemann sphericity K itself is an invariant of the surface, independent of directions through a point on the surface. It is not algebraically expressible in terms of the foregoing invariants; they are measures

of (curvilinear) circular curvature, while K is a measure of superficial curvature. Also, there has been the invariant denoted by Υ, which vanishes when the plenary space is quadruple, and is different from zero when the plenary space is quintuple or is more extensive than a quintuple space. It appears, as follows, that the invariants Υ, K, I_0, J_0, V^2, are connected by an algebraical relation. We have

$$\Upsilon = \begin{vmatrix} \mathbf{a}, & \mathbf{h}, & \mathbf{g} \\ \mathbf{h}, & \mathbf{b}, & \mathbf{f} \\ \mathbf{g}, & \mathbf{f}, & \mathbf{c} \end{vmatrix}, \qquad J_0 = \begin{vmatrix} \mathbf{a}, & \mathbf{h}, & \mathbf{k} \\ \mathbf{h}, & \mathbf{k}, & \mathbf{f} \\ \mathbf{k}, & \mathbf{f}, & \mathbf{c} \end{vmatrix};$$

and therefore

$$\Upsilon - J_0 = \mathbf{ac}(\mathbf{b}-\mathbf{k}) + 2\mathbf{fh}(\mathbf{g}-\mathbf{k}) + \mathbf{k}^3 - \mathbf{bg}^2.$$

Now $\mathbf{g}-\mathbf{b} = V^2K$, $\mathbf{g}+2\mathbf{b} = 3\mathbf{k}$, so that

$$\mathbf{b}-\mathbf{k} = -\tfrac{1}{3}V^2K, \quad \mathbf{g}-\mathbf{k} = \tfrac{2}{3}V^2K,$$
$$\mathbf{k}^3 - \mathbf{bg}^2 = -\mathbf{k}^2V^2K + \tfrac{4}{27}V^6K^3;$$

hence

$$\Upsilon - J_0 = -\tfrac{1}{3}V^2K(\mathbf{ac} - 4\mathbf{fh} + 3\mathbf{k}^2) + \tfrac{4}{27}V^6K^3.$$

Consequently

$$\Upsilon = \tfrac{4}{27}V^6K^3 - \tfrac{1}{3}I_0V^2K + J_0,$$

which is the algebraical relation expressing Υ in terms of the invariants already retained.

Ex. The determination of the geometrical significance of many of the concomitants manifestly depends upon the establishment of the respective algebraical relations expressing them in terms of concomitants already known in geometrical significance; and this establishment belongs to the domain of the invariant-theory which, so far as concerns binariants, allows the leading coefficients alone to be taken into consideration.

With the customary significance of u_1, u_2, v_1, v_2, w_1, w_2, given by

$$\left.\begin{aligned} u_1 &= Ap' + Hq' \\ u_2 &= Hp' + Bq' \end{aligned}\right\}, \quad \left.\begin{aligned} v_1 &= \bar{A}p' + \bar{H}q' \\ v_2 &= \bar{H}p' + \bar{B}q' \end{aligned}\right\}, \quad \left.\begin{aligned} w_1 &= e_{111}p'^2 + 2e_{112}p'q' + e_{122}q'^2 \\ w_2 &= e_{112}p'^2 + 2e_{122}p'q' + e_{222}q'^2 \end{aligned}\right\},$$

the following relations may be verified:

$$\left.\begin{aligned} Au_2^2 - 2Hu_2u_1 + Bu_1^2 &= V^2 \\ Au_2v_2 - H(u_1v_2 + u_2v_1) + Bu_1v_1 &= \frac{V^2}{\rho} \\ Av_2^2 - 2Hv_2v_1 + Bv_1^2 &= V^2\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) \\ Au_2w_2 - H(u_1w_2 + u_2w_1) + Bu_1w_1 &= V^2\frac{d}{ds}\left(\frac{1}{\rho}\right) \\ Av_2w_2 - H(v_1w_2 + v_2w_1) + Bv_1w_1 &= V^2\frac{d}{ds}\left(\frac{1}{\rho\sigma}\right) \\ Aw_2^2 - 2Hw_2w_1 + Bw_1^2 &= V^2\left(\frac{\rho'^2}{\rho^4} + \frac{\sigma'^2}{\sigma^4}\right) \end{aligned}\right\};$$

$$\left.\begin{aligned}
\bar{A}u_2{}^2 - 2\bar{H}u_2u_1 + \bar{B}u_1{}^2 &= V^2\rho\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}+K\right)\\
\bar{A}u_2v_2 - \bar{H}(u_1v_2+u_2v_1) + \bar{B}u_1v_1 &= V^2\left(\frac{1}{\tau^2}+K\right)\\
\bar{A}v_2{}^2 - 2\bar{H}v_2v_1 + \bar{B}v_1{}^2 &= \frac{V^2}{\rho}\left(\frac{1}{\tau^2}+K\right)\\
\bar{A}u_2w_2 - \bar{H}(u_1w_2+u_2w_1) + \bar{B}u_1w_1 &= V^2\rho\left\{\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}+K\right)\frac{d}{ds}\left(\frac{1}{\rho}\right) - \frac{\rho}{\sigma}\frac{d}{ds}\left(\frac{1}{\sigma}\right)\right\}\\
\bar{A}v_2w_2 - \bar{H}(v_1w_2+v_2w_1) + \bar{B}v_1w_1 &= V^2\left(\frac{1}{\tau^2}+K\right)\frac{d}{ds}\left(\frac{1}{\rho}\right)\\
\bar{A}w_2{}^2 - 2\bar{H}w_2w_1 + \bar{B}w_1{}^2 &= \frac{V^2}{\rho^3}\left\{\frac{1}{\sigma^4}(\sigma\rho'-\rho\sigma')^2 + \rho'^2\left(\frac{1}{\tau^2}+K\right)\right\}
\end{aligned}\right\};$$

$$\left.\begin{aligned}
\mathbf{a}u_2{}^4 - 4\mathbf{h}u_2{}^3u_1 + 6\mathbf{k}u_2{}^2u_1{}^2 - 4\mathbf{f}u_2u_1{}^3 + \mathbf{c}u_1{}^4 &= I_1 - V^2\left(\frac{1}{\rho^2}+\frac{2}{\sigma^2}+\frac{2}{\tau^2}+\tfrac{2}{3}K\right)\\
\mathbf{a}v_2{}^4 - 4\mathbf{h}v_2{}^3v_1 + 6\mathbf{k}v_2{}^2v_1{}^2 - 4\mathbf{f}v_2v_1{}^3 + \mathbf{c}v_1{}^4 &= \frac{1}{\rho^2}\left\{I_0 - 3V^4\left(\frac{1}{\tau^2}+\tfrac{1}{3}K\right)^2\right\}
\end{aligned}\right\}.$$

Stationary tilt.

140. The geometrical values of $\mathbf{H}$ and Z give the relation

$$\frac{\mathbf{H}}{Z} = \frac{V^2}{\tau^2} + \tfrac{1}{3}KV^2.$$

Hence, as K is a constant, so far as concerns directions through a point on the surface, the value of τ^2 will be a maximum or a minimum for maximum or minimum values of $\mathbf{H}/Z$. Moreover, the algebraical minimum of $1/\tau^2$ cannot be negative, though it might be zero: that is, along a curve of spherical curvature.

To determine the directions on the surface which, at a point, provide a maximum or a minimum tilt, we have to make $\mathbf{H}/Z$ a maximum or a minimum subject to the values p', q', satisfying the permanent relation. The critical conditions are

$$\frac{1}{Z}\frac{\partial \mathbf{H}}{\partial p'} - \frac{\mathbf{H}}{Z^2}\frac{\partial Z}{\partial p'} = \lambda(Ap'+Hq'),$$

$$\frac{1}{Z}\frac{\partial \mathbf{H}}{\partial q'} - \frac{\mathbf{H}}{Z^2}\frac{\partial Z}{\partial q'} = \lambda(Hp'+Bq'),$$

where λ is undetermined by these conditions themselves. To find λ, we multiply the equations by p', q', and add; then, as

$$p'\frac{\partial \mathbf{H}}{\partial p'} + q'\frac{\partial \mathbf{H}}{\partial q'} = 4\mathbf{H},\quad p'\frac{\partial Z}{\partial p'} + q'\frac{\partial Z}{\partial q'} = 4Z,$$

we have

$$\lambda = 0.$$

Consequently

$$\frac{\partial \mathrm{H}}{\partial p'}\frac{\partial Z}{\partial q'}-\frac{\partial \mathrm{H}}{\partial q'}\frac{\partial Z}{\partial p'}=0,$$

that is, having regard to the definition of the cubicovariant,

$$\Phi=0.$$

Hence, owing to the identical relation among the concomitants of Z, we have

$$4\mathrm{H}^3=I_0\mathrm{H}Z-J_0Z^3,$$

that is,

$$4\left(\frac{V^2}{\tau^2}+\tfrac{1}{3}KV^2\right)^3=I_0\left(\frac{V^2}{\tau^2}+\tfrac{1}{3}KV^2\right)-J_0.$$

Consequently the directions of stationary tilt are given by $\Phi=0$; and the magnitudes are given by the non-negative roots, if any, of this cubic in $1/\tau^2$.

When the cubic is developed, it becomes

$$\frac{4}{\tau^6}+\frac{4K}{\tau^4}+\left(\tfrac{4}{3}K^2-\frac{I_0}{V^4}\right)\frac{1}{\tau^2}+\frac{\Upsilon}{V^6}=0,$$

where Υ is the determinant

$$\begin{vmatrix} \mathfrak{a}, & \mathfrak{h}, & \mathfrak{g} \\ \mathfrak{h}, & \mathfrak{g}, & \mathfrak{f} \\ \mathfrak{g}, & \mathfrak{f}, & \mathfrak{c} \end{vmatrix}.$$

Now Υ is an essentially positive quantity when the plenary space is of five or more than five dimensions, for it is equal to the magnitude

$$\sum_i\sum_j\sum_k\begin{vmatrix} \eta_{11}{}^{(i)}, & \eta_{12}{}^{(i)}, & \eta_{22}{}^{(i)} \\ \eta_{11}{}^{(j)}, & \eta_{12}{}^{(j)}, & \eta_{22}{}^{(j)} \\ \eta_{11}{}^{(k)}, & \eta_{12}{}^{(k)}, & \eta_{22}{}^{(k)} \end{vmatrix}^2,$$

where the summation is taken over all the values $i, j, k, =1, \ldots, N$, the plenary space being of N dimensions. When the plenary space is of four dimensions, $\Upsilon=0$.

When the space is quadruple, we can have a value zero for $1/\tau^2$; and then * the corresponding directions on the surface are given by the equation

$$\bar{\mathfrak{c}}^{\frac{1}{2}}p'^2+\bar{\mathfrak{b}}^{\frac{1}{2}}p'q'+\bar{\mathfrak{a}}^{\frac{1}{2}}q'^2=0,$$

if such directions are real. For such a space, the other possible values of $1/\tau^2$ are the roots of

$$\frac{1}{\tau^4}+\frac{K}{\tau^2}+\tfrac{1}{3}K^2-\frac{I_0}{4V^4}=0.$$

* *G.F.D.*, vol. i, § 235.

When the space is of quintuple or higher extension, the quantity Y is definitely positive; hence the product of the three roots of the foregoing algebraical cubic is negative. Thus at least one algebraical root is negative: it has no significance for the immediate aim. As there can be no zero root, it follows that there must be two positive roots; and we therefore infer that all three roots are real.

Now return to the equation in the undeveloped form, and write

$$t = V^2 \left(\frac{1}{\tau^2} + \tfrac{1}{3}K\right),$$

so that the cubic for t is

$$4t^3 = I_0 t - J_0.$$

This cubic is to have three real roots; accordingly

$$I_0^3 - 27J_0^2 > 0,$$

and we must have I_0 a positive quantity. We write

$$I_0^3 - 27J_0^2 = 27D^2;$$

and the three roots of the cubic are

$$t_1 = -\frac{1}{\sqrt{3}} I_0^{\frac{1}{2}} \cos\theta,$$

$$t_2 = \frac{1}{2\sqrt{3}} I_0^{\frac{1}{2}} (\cos\theta + \sqrt{3}\sin\theta),$$

$$t_3 = \frac{1}{2\sqrt{3}} I_0^{\frac{1}{2}} (\cos\theta - \sqrt{3}\sin\theta),$$

where 3θ, taken to lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, is given by

$$J \tan 3\theta = D,$$

a positive value D being assigned to the square root of D^2, and similarly to $I_0^{\frac{1}{2}}$.

If K is a positive quantity, no negative value of t is admissible; thus t_1 would be excluded, while t_2 and t_3 must be positive. If K is a negative quantity, all three values of t would be admissible, from this form of the equation; but the earlier shews that there are only two admissible values, and therefore

$$\frac{1}{3}K + \frac{1}{\sqrt{3}} I_0^{\frac{1}{2}} \cos\theta$$

is positive for this negative value of K.

Ex. A geodesic is drawn on a surface in a direction at right angles to a given geodesic; the magnitude of its circular curvature is denoted by $1/\rho_0$, and the inclination

of the prime normals is denoted by θ. Prove that ρ_0 and θ are given by the equations

$$\frac{\cos\theta}{\rho\rho_0}=\frac{1}{\sigma^2}+\frac{1}{\tau^2}+K,$$

$$\frac{1}{\rho_0{}^2}+\frac{1}{\rho^2}=\frac{I_1}{V^4}-2\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}+\tfrac{1}{3}K\right),$$

where I_1 is the superficial invariant of p. 383, while $1/\sigma$ and $1/\tau$ denote the torsion and the tilt of the given invariant.

Obviously $\frac{1}{\sigma_0{}^2}+\frac{1}{\tau_0{}^2}=\frac{1}{\sigma^2}+\frac{1}{\tau^2}$, where σ_0 and τ_0 are the radii of torsion and of tilt of the perpendicular geodesic. In particular, the torsions of two perpendicular geodesics on a Gaussian surface are equal: and it is easy to see that these two torsions are in opposite senses.

Locus of centre of curvature of concurrent geodesics.

141. When the plenary homaloidal space of a surface is triple, the locus of the centre of circular curvature of all its geodesics through a point is merely the normal to the surface: that is, it is either the internal or the external portion of the normal bounded by the two centres of principal curvature at the point. When the plenary space is quadruple, the locus of the centre of circular curvature of all the superficial geodesics through the point is a lemniscate, which lies in the orthogonal plane of the surface. We therefore shall assume, for the present purpose, that the plenary space has five dimensions at least: and thus, with the surface, there is associated an orthogonal flat (§ 130), given by the equations

$$\|\ \bar{y}-y,\quad \eta_{11},\quad \eta_{12},\quad \eta_{22}\ \|=0,$$

this flat being orthogonal to the tangent plane of the surface.

The typical direction-cosine of the prime normal of a superficial geodesic in a direction p', q', and the magnitude of the radius of curvature, are given by the equation

$$\frac{Y}{\rho}=\eta_{11}p'^2+2\eta_{12}p'q'+\eta_{22}q'^2.$$

If then y_0 denotes the typical space-coordinate of the centre of circular curvature of the geodesic, so that

$$y_0-y=Y\rho,$$

we have

$$\frac{y_0-y}{\rho^2}=\eta_{11}p'^2+2\eta_{12}p'q'+\eta_{22}q'^2.$$

Hence the coordinates of this centre satisfy the equations of the orthogonal flat; and therefore the locus of the centre lies in the flat. As the coordinates of the centre are expressible in terms of p' and q', which are subject to the permanent arc-

relation $Ap'^2+2Hp'q'+Bq'^2=1$, so that they are expressible in terms of a single parameter, the locus of the centre is a curve in the orthogonal flat.

To obtain the curve, we refer the configuration to three non-complanar axes $O\bar{X}$, $O\bar{Y}$, $O\bar{Z}$, in the flat, with typical direction-cosines

$$\frac{\eta_{11}}{\mathrm{a}^{\frac{1}{2}}},\quad \frac{\eta_{12}}{\mathrm{b}^{\frac{1}{2}}},\quad \frac{\eta_{22}}{\mathrm{c}^{\frac{1}{2}}},$$

respectively; and we write

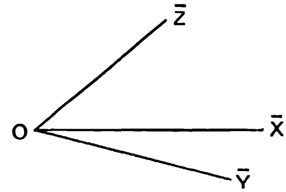

FIG. 14.

$$i=\cos\bar{Y}O\bar{Z}=\frac{1}{(\mathrm{bc})^{\frac{1}{2}}}\sum\eta_{12}\eta_{22}=\frac{\mathrm{f}}{(\mathrm{bc})^{\frac{1}{2}}},$$

$$j=\cos\bar{Z}O\bar{X}=\frac{1}{(\mathrm{ca})^{\frac{1}{2}}}\sum\eta_{22}\eta_{11}=\frac{\mathrm{g}}{(\mathrm{ca})^{\frac{1}{2}}},$$

$$k=\cos\bar{X}O\bar{Y}=\frac{1}{(\mathrm{ab})^{\frac{1}{2}}}\sum\eta_{11}\eta_{12}=\frac{\mathrm{h}}{(\mathrm{ab})^{\frac{1}{2}}},$$

the axes usually being oblique. Let x_c, y_c, z_c, denote the coordinates of the centre of curvature relative to these axes; then by projecting upon the three axes in succession, we have

$$\left.\begin{aligned} x_c+ky_c+jz_c&=\frac{1}{\mathrm{a}^{\frac{1}{2}}}\sum\eta_{11}(y_0-y)=\bar{X}\\ kx_c+\ y_c+iz_c&=\frac{1}{\mathrm{b}^{\frac{1}{2}}}\sum\eta_{12}(y_0-y)=\bar{Y}\\ jx_c+\ iy_c+\ z_c&=\frac{1}{\mathrm{c}^{\frac{1}{2}}}\sum\eta_{22}(y_0-y)=\bar{Z}\end{aligned}\right\},$$

while

$$\rho^2=x_c^{\,2}+y_c^{\,2}+z_c^{\,2}+2iy_cz_c+2jz_cx_c+2kx_cy_c,$$

the last relation being the expression of the distance of x_c, y_c, z_c, from the origin without specific reference to the significance of the point.

Now we have

$$\frac{\bar{X}\mathrm{a}^{\frac{1}{2}}}{\rho^2}=\frac{1}{\rho^2}\sum\eta_{11}(y_0-y)=\frac{1}{\rho}\sum Y\eta_{11}=\frac{\bar{A}}{\rho},$$

and similarly for $\bar{Y}$, $\bar{Z}$: that is,

$$\frac{1}{\rho^2}\bar{X}\mathrm{a}^{\frac{1}{2}}=\mathrm{a}p'^2+2\mathrm{h}p'q'+\mathrm{g}q'^2.$$

Hence, as

$$\frac{1}{\rho^2}\left(\mathrm{a}\frac{x_c}{\mathrm{a}^{\frac{1}{2}}}+\mathrm{h}\frac{y_c}{\mathrm{b}^{\frac{1}{2}}}+\mathrm{g}\frac{z_c}{\mathrm{c}^{\frac{1}{2}}}\right)=\frac{1}{\rho^2}\bar{X}\mathrm{a}^{\frac{1}{2}},$$

we have

$$\mathrm{a}\left(\frac{x_c}{\mathrm{a}^{\frac{1}{2}}\rho^2}-p'^2\right)+\mathrm{h}\left(\frac{y_c}{\mathrm{b}^{\frac{1}{2}}\rho^2}-2p'q'\right)+\mathrm{g}\left(\frac{z_c}{\mathrm{c}^{\frac{1}{2}}\rho^2}-q'^2\right)=0.$$

Similarly from the values of $\overline{Y}$ and $\overline{Z}$, we have the respective equations

$$\mathrm{h}\left(\frac{x_c}{\mathrm{a}^{\frac{1}{2}}\rho^2}-p'^2\right)+\mathrm{b}\left(\frac{y_c}{\mathrm{b}^{\frac{1}{2}}\rho^2}-2p'q'\right)+\mathrm{f}\left(\frac{z_c}{\mathrm{c}^{\frac{1}{2}}\rho^2}-q'^2\right)=0,$$

$$\mathrm{g}\left(\frac{x_c}{\mathrm{a}^{\frac{1}{2}}\rho^2}-p'^2\right)+\mathrm{f}\left(\frac{y_c}{\mathrm{b}^{\frac{1}{2}}\rho^2}-2p'q'\right)+\mathrm{c}\left(\frac{z_c}{\mathrm{c}^{\frac{1}{2}}\rho^2}-q'^2\right)=0.$$

The determinant of the coefficients of the three terms on the left-hand side, being

$$\begin{vmatrix} \mathrm{a}, & \mathrm{h}, & \mathrm{g} \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f} \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c} \end{vmatrix},$$

is not zero ; and therefore

$$\mathrm{a}^{\frac{1}{2}}\rho^2 p'^2 = x_c, \quad 2\mathrm{b}^{\frac{1}{2}}\rho^2 p'q' = y_c, \quad \mathrm{c}^{\frac{1}{2}}\rho^2 q'^2 = z_c.$$

The equations of the locus result from the elimination of p', q', among these equations and $Ap'^2+2Hp'q'+Bq'^2=1$; and they can be at once obtained in the form

$$\frac{y_c^2}{\mathrm{b}}=4\frac{x_c z_c}{(\mathrm{ac})^{\frac{1}{2}}},$$

$$x_c^2+y_c^2+z_c^2+2iy_cz_c+2jz_cx_c+2kx_cy_c=\frac{A}{\mathrm{a}^{\frac{1}{2}}}x_c+\frac{H}{\mathrm{b}^{\frac{1}{2}}}y_c+\frac{B}{\mathrm{c}^{\frac{1}{2}}}z_c.$$

The former equation represents a quadric cone with its vertex at the origin ; the latter equation represents a sphere, passing through the origin. The locus is therefore a skew quartic curve in the orthogonal flat, being the intersection of the sphere and the cone.

Let the coordinates of the centre of the sphere be denoted by ξ_c, η_c, ζ_c; then, as usual, they are given by the equations

$$\left.\begin{aligned} \xi_c+k\eta_c+j\zeta_c&=\tfrac{1}{2}\frac{A}{\mathrm{a}^{\frac{1}{2}}} \\ k\xi_c+\eta_c+i\zeta_c&=\tfrac{1}{2}\frac{H}{\mathrm{b}^{\frac{1}{2}}} \\ j\xi_c+i\eta_c+\zeta_c&=\tfrac{1}{2}\frac{B}{\mathrm{c}^{\frac{1}{2}}} \end{aligned}\right\}.$$

If the quantities on the left-hand sides of these equations be denoted by $\bar{X}_c$, $\bar{Y}_c$, $\bar{Z}_c$, respectively, we have

$$\begin{aligned}(1-i^2)\bar{X}_c^2+(1-j^2)\bar{Y}_c^2+(1-k^2)\bar{Z}_c^2 \\ -2(i-jk)\bar{Y}_c\bar{Z}_c-2(j-ki)\bar{Z}_c\bar{X}_c-2(k-ij)\bar{X}_c\bar{Y}_c \\ =(1-i^2-j^2-k^2+2ijk)(\xi_c^2+\eta_c^2+\zeta_c^2+2i\eta_c\zeta_c+2j\zeta_c\xi_c+2k\xi_c\eta_c) \\ =\frac{\Upsilon}{4\mathbf{abc}}\mathbf{D}^2,\end{aligned}$$

if $\mathbf{D}$ denote the diameter of the sphere ; and therefore that diameter is given by

$$\Upsilon\mathbf{D}^2=A^2\bar{\mathfrak{a}}+2AH\bar{\mathfrak{h}}+H^2\bar{\mathfrak{b}}+2AB\bar{\mathfrak{g}}+2HB\bar{\mathfrak{f}}+B^2\bar{\mathfrak{c}}.$$

The magnitude $\mathbf{D}$, an intrinsic quantity belonging to the surface and independent of directions on the surface, will naturally be an invariant appertaining to the surface. By direct algebra, its invariantive expression is found to be

$$\Upsilon\mathbf{D}^2=J_1-\tfrac{1}{3}V^2I_0-\tfrac{1}{3}V^2KI_1+\tfrac{4}{9}V^6K^2,$$

where K is the Riemann sphericity, and I_0, I_1, J_1, are the invariants connected (§ 137) with the principal circular curvatures of the surface.

142. Thus the locus of the centre of circular curvature of geodesics through any point of a surface is a quartic curve in a triple space, when the number of dimensions of the plenary space of the surface is five or more than five. (As already noted, the locus is part of a straight line when the plenary space is triple, and is a plane lemniscate when the plenary space is quadruple.)

When the plenary space of the surface actually is quintuple, there exists a property of the surface which is peculiar to surfaces in a quintuple space. The equations of the orthogonal flat, which now is the complete orthogonal homaloid of the surface, can be taken in the form

$$\sum(\bar{y}-y)\frac{\partial y}{\partial p}=0, \quad \sum(\bar{y}-y)\frac{\partial y}{\partial q}=0,$$

for they obviously provide a homaloid orthogonal to the tangent plane

$$\left\| \bar{y}-y, \ \frac{\partial y}{\partial p}, \ \frac{\partial y}{\partial q} \right\|=0.$$

To obtain the equations of the envelope of this flat, we associate its pair of equations in the stated form with the additional equations of the type,

$$\sum(\bar{y}-y)\frac{\partial^2 y}{\partial p^2}-A=0,$$

that is, when account is taken of the earlier equations,

and
$$\left.\begin{aligned}\textstyle\sum(\bar{y}-y)\eta_{11}&=A\\ \textstyle\sum(\bar{y}-y)\eta_{12}&=H\\ \textstyle\sum(\bar{y}-y)\eta_{22}&=B\end{aligned}\right\}.$$

The five equations also give the limiting position of the intersection of the orthogonal flat with the orthogonal flats at all points of the surface immediately contiguous to O; and they involve the five magnitudes $\bar{y}-y$ linearly. Now the determinant of the coefficients of the five magnitudes is equal to

$$V\begin{vmatrix} \mathfrak{a}, & \mathfrak{h}, & \mathfrak{g} \\ \mathfrak{h}, & \mathfrak{b}, & \mathfrak{f} \\ \mathfrak{g}, & \mathfrak{f}, & \mathfrak{c} \end{vmatrix}^{\frac{1}{2}},$$

and therefore it is distinct from zero. Thus the five equations determine a unique point, which will be called * the *normal centre* of the surface in quintuple space.

As the normal centre lies in the orthogonal flat, we denote temporarily its coordinates referred to the former axes in that flat by x_n, y_n, z_n; then as $\bar{\mathfrak{a}}^{\frac{1}{2}}\{\sum(\bar{y}-y)\eta_{11}\}$ is the projection, upon the x-axis, of the line joining it to the origin, we have

$$x_n+ky_n+jz_n=\bar{\mathfrak{a}}^{\frac{1}{2}}\{\textstyle\sum(\bar{y}-y)\eta_{11}\}=\bar{\mathfrak{a}}^{\frac{1}{2}}A,$$

in the present instance; and similarly

$$kx_n+y_n+iz_n=\bar{\mathfrak{b}}^{\frac{1}{2}}H,$$
$$jx_n+iy_n+z_n=\bar{\mathfrak{c}}^{\frac{1}{2}}B.$$

Hence

$$x_n=2\xi_c,\quad y_n=2\eta_c,\quad z_n=2\zeta_c;$$

or the normal centre is the extremity, other than O, of the diameter through O of the sphere, which occurs as containing the locus of the centre of circular curvature of the geodesics on the surface through that point.

Moreover, the orthogonal flat of the surface contains both the prime normal and the trinormal of any geodesic; and it contains a line at right angles both to the prime normal and to the trinormal. This line also is at right angles to the tangent and to the binormal of the geodesic, because it lies in the orthogonal flat; and therefore, when the plenary space is quintuple, the line is the quartinormal of the geodesic. We consider the projections of the diameter of the sphere upon the directions of these three organic lines of the geodesic.

* Distinct from the orthogonal centre for a direction on a surface in quadruple space, which is the limiting position of the point of intersection of the orthogonal plane by the orthogonal plane at a contiguous point along the selected direction. There is an orthogonal centre for each direction on the surface in quadruple space.

(i) The projection of the spherical diameter on the prime normal

$$\begin{aligned}&= \sum Y(\bar{y}-y)\\ &= \rho \sum \{(\eta_{11}p'^2 + 2\eta_{12}p'q' + \eta_{22}q'^2)(\bar{y}-y)\}\\ &= \rho(Ap'^2 + 2Hp'q' + Bq'^2) = \rho.\end{aligned}$$

(ii) The projection of the spherical diameter on the trinormal

$$\begin{aligned}&= \sum l_4(\bar{y}-y)\\ &= \frac{\rho\tau}{V} \sum \{(v_1\xi_2 - v_2\xi_1)(\bar{y}-y)\},\end{aligned}$$

;ause of the value of l_4 on p. 364. Now

$$\begin{aligned}\sum \xi_1(\bar{y}-y) &= p' \sum \{\eta_{11}(\bar{y}-y)\} + q' \sum \{\eta_{12}(\bar{y}-y)\}\\ &= Ap' + Hq' = u_1,\end{aligned}$$

l, similarly,

$$\sum \xi_2(\bar{y}-y) = Hp' + Bq' = u_2\,;$$

ıce the required projection

$$= \frac{\rho\tau}{V}(v_1u_2 - v_2u_1) = \frac{\rho\tau}{\sigma},$$

;ause of the value of the torsion (§ 129).

iii) In the present instance, when the plenary space of the surface is quintuple, typical direction-cosine of the quartinormal is

$$P\eta_{11} + Q\eta_{12} + R\eta_{22},$$

ere P, Q, R, have the values given in § 134. Hence the projection of the ıerical diameter on the quartinormal

$$\begin{aligned}&= \sum l_5(\bar{y}-y)\\ &= \sum \{(P\eta_{11} + Q\eta_{12} + R\eta_{22})(\bar{y}-y)\}\\ &= AP + HQ + BR\\ &= \frac{\rho\tau}{VY^{\frac{1}{2}}}\{A(\bar{\mathfrak{a}}q'^2 - \bar{\mathfrak{h}}q'p' + \bar{\mathfrak{g}}p'^2) + H(\bar{\mathfrak{h}}q'^2 - \bar{\mathfrak{b}}q'p' + \bar{\mathfrak{f}}p'^2) + B(\bar{\mathfrak{g}}q'^2 - \bar{\mathfrak{f}}q'p' + \bar{\mathfrak{c}}p'^2)\}.\end{aligned}$$

; this length be denoted by L.

Thus, in the orthogonal flat, the diameter of the sphere, denoted on p. 395 by when projected upon the prime normal, the trinormal, and the quartinormal, lds the respective components

$$\rho, \quad \frac{\rho\tau}{\sigma}, \quad L\,;$$

l therefore, as these three lines of reference are at right angles, we must have

$$\mathbf{D}^2 = \rho^2 + \frac{\rho^2\tau^2}{\sigma^2} + L^2.$$

When the values of the various magnitudes are substituted, the relation becomes an identity. Further, the relations which give the normal centre of the surface can be represented by the typical equation

$$\bar{y}-y=X\rho+l_4\frac{\rho\tau}{\sigma}+l_5L\,;$$

but it must be remembered that the results apply only to a surface which exists in a quintuple plenary homaloidal space.

We may note, in passing, that this magnitude L can be used for the expression of the two quantities

$$AB_4-2HH_4+BA_4,\quad \bar{A}B_4-2\bar{H}H_4+\bar{B}H_4,$$

which are covariants of the system of forms belonging to the surface, the value of $A\bar{B}-2H\bar{H}+B\bar{A}$ being known (p. 367) to be given by the equation

$$\frac{1}{\rho}(A\bar{B}-2H\bar{H}+B\bar{A})=V^2\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}+\frac{1}{\tau^2}+K\right).$$

The coordinates of the normal centre of a surface in quintuple space can be represented by the typical equation

$$\bar{y}-y=\frac{\partial y}{\partial p}\theta+\frac{\partial y}{\partial q}\phi+EX+Fl_4+Gl_5,$$

with appropriate values of θ, ϕ, E, F, G. Owing to the relations

$$\sum(\bar{y}-y)\frac{\partial y}{\partial p}=0,\quad \sum(\bar{y}-y)\frac{\partial y}{\partial q}=0,$$

satisfied by those coordinates, we at once have

$$\theta=0,\quad \phi=0.$$

Also, as we have just established the properties

$$\sum X(\bar{y}-y)=\rho,\quad \sum l_4(\bar{y}-y)=\frac{\rho\tau}{\sigma},\quad \sum l_5(\bar{y}-y)=L,$$

we have

$$E=\rho,\quad F=\frac{\rho\tau}{\sigma},\quad G=L,$$

so that our typical equation becomes

$$\bar{y}-y=X\rho+l_4\frac{\rho\tau}{\sigma}+l_5L.$$

When these values are substituted in the earlier equations (p. 396)

$$\sum(\bar{y}-y)\eta_{11}=A,\quad \sum(\bar{y}-y)\eta_{12}=H,\quad \sum(\bar{y}-y)\eta_{22}=B,$$

we have

$$A=\rho\bar{A}+\frac{\rho\tau}{\sigma}A_4+LTq'^2,$$

$$H=\rho\bar{H}+\frac{\rho\tau}{\sigma}H_4-LTq'p',$$

$$B=\rho\bar{B}+\frac{\rho\tau}{\sigma}B_4+LTp'^2,$$

where

$$T = \frac{Y^{\frac{1}{2}}}{V} \rho\tau.$$

Let these values of A, H, B, be substituted in $A\bar{B} - 2H\bar{H} + B\bar{A}$, so that

$$A\bar{B} - 2H\bar{H} + B\bar{A} = 2\rho(\bar{A}\bar{B} - \bar{H}^2) + \frac{\rho\tau}{\sigma}(A_4\bar{B} - 2H_4\bar{H} + B_4\bar{A}) + L\frac{Y^{\frac{1}{2}}}{V}\tau;$$

or, when we substitute

$$\bar{A}\bar{B} - \bar{H}^2 = V^2\left(\frac{1}{\tau^2} + K\right),$$

and adopt the known value of the left-hand side, we have

$$A_4\bar{B} - 2H_4\bar{H} + B_4\bar{A} = V^2\frac{\sigma}{\tau}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} - \frac{1}{\tau^2} - K\right) - L\frac{Y^{\frac{1}{2}}}{V}\frac{\sigma}{\rho}.$$

Similarly, let these values of A, H, B, be substituted in $AB_4 - 2HH_4 + BA_4$, so that

$$AB_4 - 2HH_4 + BA_4 = \rho(\bar{A}B_4 - 2\bar{H}H_4 + \bar{B}A_4) + 2\frac{\rho\tau}{\sigma}(A_4B_4 - H_4^2);$$

or, when we substitute

$$A_4B_4 - H_4^2 = -\frac{V^2}{\tau^2},$$

and insert the value of $\bar{A}B_4 - 2\bar{H}H_4 + \bar{B}A_4$, which has just been obtained, we find

$$AB_4 - 2HH_4 + BA_4 = V^2\frac{\rho\sigma}{\tau}\left(\frac{1}{\rho^2} - \frac{1}{\sigma^2} - \frac{1}{\tau^2} - K\right) - L\frac{Y^{\frac{1}{2}}}{V}\sigma.$$

Property of the locus of centres of circular curvature of concurrent geodesics.

143. Returning to the more general case, when the plenary space of the surface is not restricted to be quintuple, we can associate one property, of curves of circular curvature through a point O on the surface, with the skew curve in the orthogonal flat which is the locus of the centre of circular curvatures of superficial geodesics at the point. This skew curve is the intersection of the sphere

$$x^2 + y^2 + z^2 + 2iyz + 2jzx + 2kxy = 2lx + 2my + 2nz,$$

with the cone

$$\mu y^2 = 2xz,$$

where

$$2l = A\bar{\mathrm{a}}^{\frac{1}{2}}, \quad 2m = H\bar{\mathrm{b}}^{\frac{1}{2}}, \quad 2n = B\bar{\mathrm{c}}^{\frac{1}{2}}, \quad 2\mu\mathrm{b} = (\mathrm{ac})^{\frac{1}{2}},$$

the two surfaces being referred to selected axes in the flat. The radius vector r from the origin O to a point on the curve, in direction and magnitude, is a radius of circular curvature ρ of a superficial geodesic in the direction p', q', through O, such that

$$\frac{x}{\mathrm{a}^{\frac{1}{2}}p'^2} = \frac{y}{2\mathrm{b}^{\frac{1}{2}}p'q'} = \frac{z}{\mathrm{c}^{\frac{1}{2}}q'^2} = r^2 = \rho^2.$$

Now one definition (§ 136) of a curve of circular curvature on a surface requires the circular curvature of a geodesic, touching the curve, to be a maximum or a minimum among the circular curvatures of all the geodesics through the point. Accordingly, among all the radii vectores from O to the foregoing curve, those belonging to a curve of curvature on the surface must be either a maximum or a minimum; and therefore each such radius vector must be a normal to the curve-locus drawn from O. Let ξ_0, η_0, ζ_0, denote the coordinates of the foot of such a normal, referred to the axes in the flat. The direction-variables x', y', z', of the tangent to the curve-locus at ξ_0, η_0, ζ_0, are given by the two equations

$$(\xi_0+k\eta_0+j\zeta_0-l)x'+(k\xi_0+\eta_0+i\zeta_0-m)y'+(j\xi_0+i\eta_0+\zeta_0-n)z'=0,$$
$$\zeta_0 x'-\mu\eta_0 y'+\xi_0 z'=0.$$

The equations of the radius vector, referred to the flat, are

$$\frac{x}{\xi_0}=\frac{y}{\eta_0}=\frac{z}{\zeta_0}.$$

The condition, that the tangent to the curve and this radius vector shall be at right angles, is

$$\xi_0 x'+\eta_0 y'+\zeta_0 z'+i(\eta_0 z'+\zeta_0 y')+j(\zeta_0 x'+\xi_0 z')+k(\xi_0 y'+\eta_0 x')=0.$$

Thus there are three equations, homogeneous and linear in x', y', z'. For the first of them, when combined with the third, an equation

$$lx'+my'+nz'=0,$$

can be substituted; and then the result of eliminating x', y', z', gives an equation

$$\begin{vmatrix} \xi_0+k\eta_0+j\zeta_0, & \zeta_0, & l \\ k\xi_0+\eta_0+i\zeta_0, & -\mu\eta_0, & m \\ j\xi_0+i\eta_0+\zeta_0, & \xi_0, & n \end{vmatrix}=0,$$

to be satisfied by ξ_0, η_0, ζ_0. This equation represents a quadric cone in the flat, having its vertex at O.

But the coordinates ξ_0, η_0, ζ_0, also satisfy the equation

$$\mu\eta_0^2=2\xi_0\zeta_0.$$

Hence every such normal to the curve-locus, that is, every principal radius of circular curvature of the surface at the point, is the intersection of two quadric cones in the flat, the cones having a common vertex at O. Accordingly, there are four such principal radii of circular curvature. Their directions are those of the four generators common to the two cones. Their magnitudes are the intercepts made on the sphere

$$x^2+y^2+z^2+2iyz+2jzx+2kxy=2lx+2my+2nz$$

by the four generators.

To obtain more specific expression for the generators, we represent any generator of the cone $\mu y^2 = 2xz$ parametrically by the equations

$$x = 2\mu\theta^2 z, \quad y = 2\theta z.$$

Then the values of the principal parameters, corresponding to the four principal radii of circular curvature, are the roots of the quartic equation

$$\begin{vmatrix} 2\mu\theta^2 + 2k\theta + j, & 1, & l \\ 2k\mu\theta^2 + 2\theta + i, & -2\mu\theta, & m \\ 2j\mu\theta^2 + 2i\theta + 1, & 2\mu\theta^2, & n \end{vmatrix} = 0.$$

The generator, as to direction-variables p', q', is given by

$$\frac{x}{\mathrm{a}^{\frac{1}{2}} p'^2} = \frac{y}{2\mathrm{b}^{\frac{1}{2}} p'q'} = \frac{z}{\mathrm{c}^{\frac{1}{2}} q'^2},$$

so that, in this connection, we have

$$\frac{2\mu\theta^2}{\mathrm{a}^{\frac{1}{2}} p'^2} = \frac{2\theta}{2\mathrm{b}^{\frac{1}{2}} p'q'} = \frac{1}{\mathrm{c}^{\frac{1}{2}} q'^2}, \quad \theta = \frac{\mathrm{b}^{\frac{1}{2}} p'}{\mathrm{c}^{\frac{1}{2}} q'}.$$

When these values are substituted, together with the values of μ, l, m, n, and the values of the axial constants i, j, k, the equation is easily transformed so as to become

$$\begin{vmatrix} \mathrm{a}p'^2 + 2\mathrm{h}p'q' + \mathrm{g}q'^2, & A, & q'^2 \\ \mathrm{h}p'^2 + 2\mathrm{b}p'q' + \mathrm{f}q'^2, & H, & -q'p' \\ \mathrm{g}p'^2 + 2\mathrm{f}p'q' + \mathrm{c}q'^2, & B, & p'^2 \end{vmatrix} = 0,$$

that is, in effect,

$$\begin{vmatrix} \bar{A}, & A, & q'^2 \\ \bar{H}, & H, & -q'p' \\ \bar{B}, & B, & p'^2 \end{vmatrix} = 0,$$

being the former equation (§§ 106, 136) for the direction-variables of the curves of circular curvature on the surface.

Locus of centre of spherical curvature of concurrent geodesics.

144. The locus of the centre of spherical curvature of all the geodesics passing through any point O on a surface (this centre will be called the *spherical centre* of the surface) is obtainable in the same way as the locus of the centre of circular curvature of those geodesics. We measure a length ρ, along the prime normal of a geodesic up to the centre of circular curvature; and from that centre of circular curvature, along a direction parallel to the geodesic binormal at O, we measure a

distance $\sigma\rho'$. The point S, thus obtained, is the centre of spherical curvature at O. If y_S denote its typical space-coordinates, we have

$$\begin{aligned} y_S - y &= Y\rho + l_3\sigma\rho' \\ &= \rho^2(\eta_{11}p'^2 + 2\eta_{12}p'q' + \eta_{22}q'^2) + \frac{1}{V}\sigma\rho'\left(\frac{\partial y}{\partial q}u_1 - \frac{\partial y}{\partial p}u_2\right). \end{aligned}$$

Hence the locus of S lies in the homaloid

$$\left\| \, y_S - y, \quad \eta_{11}, \quad \eta_{12}, \quad \eta_{22}, \quad \frac{\partial y}{\partial p}, \quad \frac{\partial y}{\partial q} \, \right\| = 0,$$

that is, in the quintuple homaloid generated by the two leading lines of the tangent plane and the three leading lines of the orthogonal flat of the surface.

The last statement implicitly assumes that the plenary space is at least quintuple. When the space is only quadruple, there is no orthogonal flat of the surface: the orthogonal homaloid is a plane. When the plenary space is only triple, there is no orthogonal flat: the orthogonal homaloid is a line. These two exceptions require separate discussions, respectively, which will come later; and, for the immediate purpose, we shall assume that the plenary space is quintuple at the lowest.

The locus is referred to five axes in the quintuple homaloid. Two of these are constituted by the parametric directions at O, taken along tangents in the tangent plane, inclined at the angle ϵ; when the coordinates relative to these axes are denoted by t_s and v_s, we have

$$\left.\begin{aligned} t_s + v_s\cos\epsilon &= \frac{1}{A^{\frac{1}{2}}}\sum \frac{\partial y}{\partial p}(y_S - y) \\ v_s + t_s\cos\epsilon &= \frac{1}{B^{\frac{1}{2}}}\sum \frac{\partial y}{\partial q}(y_S - y) \end{aligned}\right\}.$$

The other three axes are constituted by the same three lines in the orthogonal flat as are used (§ 141) for the locus of the centre of circular curvature; when the coordinates relative to these axes are denoted by x_s, y_s, z_s, we have

$$\left.\begin{aligned} x_s + ky_s + jz_s &= \frac{1}{\mathrm{a}^{\frac{1}{2}}}\sum \eta_{11}(y_S - y) \\ kx_s + y_s + iz_s &= \frac{1}{\mathrm{b}^{\frac{1}{2}}}\sum \eta_{12}(y_S - y) \\ jx_s + iy_s + z_s &= \frac{1}{\mathrm{c}^{\frac{1}{2}}}\sum \eta_{22}(y_S - y) \end{aligned}\right\},$$

where, as before,

$$i(\mathrm{bc})^{\frac{1}{2}} = \mathrm{f}, \quad j(\mathrm{ca})^{\frac{1}{2}} = \mathrm{g}, \quad k(\mathrm{ab})^{\frac{1}{2}} = \mathrm{h}.$$

Also, regard being paid to the distances measured along the prime normal and

along the direction parallel to the binormal (these directions being perpendicular to one another), it follows that

$$w^2 = t_s^2 + 2t_s v_s \cos\epsilon + v_s^2 = (\sigma\rho')^2,$$
$$r^2 = x_s^2 + y_s^2 + z_s^2 + 2iy_s z_s + 2jz_s x_s + 2kx_s y_s = \rho^2,$$

the plane of t_s, v_s, and the distance w, being orthogonal to the flat of x_s, y_s, z_s, and the distance r.

Moreover, we require the parametric expressions of ρ and of $\sigma\rho'$. For ρ, we have

$$\frac{1}{\rho^2} = \mathrm{a}p'^4 + 4\mathrm{h}p'^3q' + 6\mathrm{k}p'^2q'^2 + 4\mathrm{f}p'q'^3 + \mathrm{c}q'^4.$$

The parametric expression of $\sigma\rho'$ has not yet been obtained. We have had (§ 106)

$$\frac{V}{\sigma} = \begin{vmatrix} \bar{A}, & A, & q'^2 \\ \bar{H}, & H, & -q'p' \\ \bar{B}, & B, & p'^2 \end{vmatrix};$$

and therefore

$$\frac{V\rho}{\sigma} = \begin{vmatrix} \dfrac{\bar{A}}{\rho}, & \dfrac{\bar{H}}{\rho}, & \dfrac{\bar{B}}{\rho} \\ A, & H, & B \\ \rho^2 q'^2, & -\rho^2 q'p', & \rho^2 p'^2 \end{vmatrix} = T_2,$$

a form that will prove useful for substitution. Also (§ 101)

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = e_{111}p'^3 + 3e_{112}p'^2q' + 3e_{122}p'q'^2 + e_{222}q'^3,$$

and therefore (p. 270)

$$\begin{aligned} -\frac{\rho'}{\rho^3} = p'^3(A_{111}p'^2 + 2H_{111}p'q' + B_{111}q'^2) \\ + 3p'^2q'(A_{112}p'^2 + 2H_{112}p'q' + B_{112}q'^2) \\ + 3p'q'^2(A_{122}p'^2 + 2H_{122}p'q' + B_{122}q'^2) \\ + q'^3(A_{222}p'^2 + 2H_{222}p'q' + B_{222}q'^2) \\ = \mathrm{e}_0p'^5 + 5\mathrm{e}_1p'^4q' + 10\mathrm{e}_2p'^3q'^2 + 10\mathrm{e}_3p'^2q'^3 + 5\mathrm{e}_4p'q'^4 + \mathrm{e}_5q'^5 = Q_5, \end{aligned}$$

for brevity; hence

$$\sigma\rho' = -\frac{V}{T_2}\rho^4 Q_5,$$

the form which will be used.

To obtain the locus of the centre of spherical curvature, the direction-variables p', q', have to be eliminated. As we are to obtain a curve in the quintuple homaloid of reference, it will be sufficient if four equations among the five coordinates of reference to that homaloid are obtained: and they can be derived as follows.

When the general value of $y_S - y$ is substituted in the five equations for t_s, v_s, x_s, y_s, z_s, we at once have

$$t_s + v_s \cos\epsilon = -\frac{1}{A^{\frac{1}{2}}} V\sigma\rho' q' = -\frac{Vw}{A^{\frac{1}{2}}} q',$$

$$v_s + t_s \cos\epsilon = \frac{1}{B^{\frac{1}{2}}} V\sigma\rho' p' = \frac{Vw}{B^{\frac{1}{2}}} p',$$

the elimination of p' and q' from these two equations and the arc-relation $Ap'^2 + 2Hp'q' + Bq'^2 = 1$ merely reproducing the value of w.

Again, as in § 141, we have

$$x_s + ky_s + jz_s = \frac{1}{\mathbf{a}^{\frac{1}{2}}} (\mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}q'^2)\rho^2$$

$$= \mathbf{a}^{\frac{1}{2}} p'^2 \rho^2 + 2k\mathbf{b}^{\frac{1}{2}} p'q'\rho^2 + j\mathbf{c}^{\frac{1}{2}} q'^2 \rho^2,$$

so that

$$(x_s - \mathbf{a}^{\frac{1}{2}} p'^2\rho^2) + k(y_s - 2\mathbf{b}^{\frac{1}{2}} p'q'\rho^2) + j(z_s - \mathbf{c}^{\frac{1}{2}} q'^2\rho^2) = 0;$$

and, from the two similar equations,

$$k(x_s - \mathbf{a}^{\frac{1}{2}} p'^2\rho^2) + \ (y_s - 2\mathbf{b}^{\frac{1}{2}} p'q'\rho^2) + i(z_s - \mathbf{c}^{\frac{1}{2}} q'^2\rho^2) = 0,$$

$$j(x_s - \mathbf{a}^{\frac{1}{2}} p'^2\rho^2) + i(y_s - 2\mathbf{b}^{\frac{1}{2}} p'q'\rho^2) + \ (z_s - \mathbf{c}^{\frac{1}{2}} q'^2\rho^2) = 0.$$

The determinant of the coefficients does not vanish, being equal to

$$\frac{\Upsilon}{\mathbf{abc}};$$

hence we have

$$x_s = \mathbf{a}^{\frac{1}{2}} p'^2\rho^2, \quad y_s = 2\mathbf{b}^{\frac{1}{2}} p'q'\rho^2, \quad z_s = \mathbf{c}^{\frac{1}{2}} q'^2\rho^2.$$

We therefore have the five coordinates expressed, explicitly, in terms of the direction-variables p', q', which are subject to the relation

$$Ap'^2 + 2Hp'q' + Bq'^2 = 1.$$

As equations, the combination of which constitutes the eliminant, we have at once

$$-\frac{A^{\frac{1}{2}}(t_s + v_s\cos\epsilon)}{B^{\frac{1}{2}}(v_s + t_s\cos\epsilon)} = \frac{2\mathbf{b}^{\frac{1}{2}} z_s}{\mathbf{c}^{\frac{1}{2}} y_s} = \frac{\mathbf{a}^{\frac{1}{2}} y_s}{2\mathbf{b}^{\frac{1}{2}} x_s},$$

the common value of each fraction being q'/p'; and

$$\frac{A}{\mathbf{a}^{\frac{1}{2}}} x_s + \frac{H}{\mathbf{b}^{\frac{1}{2}}} y_s + \frac{B}{\mathbf{c}^{\frac{1}{2}}} z_s = \rho^2 = x_s^2 + y_s^2 + z_s^2 + 2iy_sz_s + 2jz_sx_s + 2kx_sy_s,$$

effectively three equations of the second order. For the remaining equation, we use

$$\sigma\rho' = -\frac{V}{T_2}\rho^4 Q_5.$$

Here, we have

$$T_2 = \begin{vmatrix} \dfrac{\bar{A}}{\rho}, & \dfrac{\bar{H}}{\rho}, & \dfrac{\bar{B}}{\rho} \\ A, & H, & B \\ \rho^2 q'^2, & -\rho^2 q'p', & \rho^2 p'^2 \end{vmatrix},$$

while

$$x_s + ky_s + jz_s = \frac{1}{\mathrm{a}^{\frac{1}{2}}} \sum \eta_{11}(y_S - y) = \frac{1}{\mathrm{a}^{\frac{1}{2}}}(\mathrm{a}p'^2 + 2\mathrm{h}p'q' + \mathrm{g}q'^2) = \frac{1}{\mathrm{a}^{\frac{1}{2}}} \frac{\bar{A}}{\rho},$$

and similarly for $\bar{H}/\rho$, $\bar{B}/\rho$; thus

$$T_2 = \begin{vmatrix} \mathrm{a}^{\frac{1}{2}}(\ x_s + ky_s + jz_s), & A, & z_s \mathrm{c}^{-\frac{1}{2}} \\ \mathrm{b}^{\frac{1}{2}}(kx_s + \ y_s + iz_s), & H, & -\frac{1}{2} y_s \mathrm{b}^{-\frac{1}{2}} \\ \mathrm{c}^{\frac{1}{2}}(jx_s + \ iy_s + \ z_s), & B, & x_s \mathrm{a}^{-\frac{1}{2}} \end{vmatrix},$$

a homogeneous quadratic expression of the second degree. Also

$$\begin{aligned} \rho^4 Q_5 &= p'\rho^4(\mathrm{e}_0 p'^4 + 4\mathrm{e}_1 p'^3 q' + 6\mathrm{e}_2 p'^2 q'^2 + 4\mathrm{e}_3 p'^3 q' + \mathrm{e}_4 q'^4) \\ &\qquad + q'\rho^4(\mathrm{e}_1 p'^4 + 4\mathrm{e}_2 p'^3 q' + 6\mathrm{e}_3 p'^2 q'^2 + 4\mathrm{e}_4 p' q'^3 + \mathrm{e}_5 q'^4) \\ &= p'\left(\frac{\mathrm{e}_0}{\mathrm{a}} x_s^2 + 2\frac{\mathrm{e}_1}{\mathrm{a}^{\frac{1}{2}}\mathrm{b}^{\frac{1}{2}}} x_s y_s + \frac{\mathrm{e}_2}{\mathrm{b}} y_s^2 + 2\frac{\mathrm{e}_2}{\mathrm{a}^{\frac{1}{2}}\mathrm{c}^{\frac{1}{2}}} x_s z_s + 2\frac{\mathrm{e}_3}{\mathrm{b}^{\frac{1}{2}}\mathrm{c}^{\frac{1}{2}}} y_s z_s + \frac{\mathrm{e}_4}{\mathrm{c}} z_s^2\right) \\ &\qquad + q'\left(\frac{\mathrm{e}_1}{\mathrm{a}} x_s^2 + 2\frac{\mathrm{e}_2}{\mathrm{a}^{\frac{1}{2}}\mathrm{b}^{\frac{1}{2}}} x_s y_s + \frac{\mathrm{e}_3}{\mathrm{b}} y_s^2 + 2\frac{\mathrm{e}_3}{\mathrm{a}^{\frac{1}{2}}\mathrm{c}^{\frac{1}{2}}} x_s z_s + 2\frac{\mathrm{e}_4}{\mathrm{b}^{\frac{1}{2}}\mathrm{c}^{\frac{1}{2}}} y_s z_s + \frac{\mathrm{e}_5}{\mathrm{c}} z_s^2\right) \\ &= p'R_2 + q'S_2, \end{aligned}$$

where R_2 and S_2 denote homogeneous quadratic expressions of the second degree. Hence

$$\sigma\rho' T_2 = -Vp'R_2 + Vq'S_2,$$

and therefore

$$(\sigma\rho')^2 T_2 = -V\sigma\rho' p' R_2 + V\sigma\rho' q' S_2:$$

that is,

$$(v_s^2 + 2v_s t_s \cos\epsilon + t_s^2) T_2 + A^{\frac{1}{2}}(t_s + v_s \cos\epsilon) S_2 + B^{\frac{1}{2}}(v_s + t_s \cos\epsilon) R_2 = 0,$$

a non-homogeneous equation of the fourth degree.

Thus the eliminant is composed of one equation of the fourth degree, and of three linearly independent equations each of the second degree; and no one of the equations can be reduced in degree, nor can equations of lower degree be obtained by combinations of the equations. The required locus lies in the quintuple homaloid; hence it is a skew curve in that homaloid, of degree thirty-two.

145. The converse of the foregoing analysis leads to the parametric representation of the curve and to a verification that the curve is the locus of the

centre of spherical curvature of superficial geodesics through the point. Thus the equation

$$\frac{2\mathrm{b}^{\frac{1}{2}}z_s}{\mathrm{c}^{\frac{1}{2}}y_s}=\frac{\mathrm{a}^{\frac{1}{2}}y_s}{2\mathrm{b}^{\frac{1}{2}}x_s}$$

can be represented by the equations

$$\frac{x_s}{\mathrm{a}^{\frac{1}{2}}p'^2}=\frac{y_s}{2\mathrm{b}^{\frac{1}{2}}p'q'}=\frac{z_s}{\mathrm{c}^{\frac{1}{2}}q'^2}=\phi,$$

the common value of the two earlier fractions being a parameter q'/p'; and then we also have

$$\frac{A^{\frac{1}{2}}(t_s+v_s\cos\epsilon)}{-q'}=\frac{B^{\frac{1}{2}}(v_s+t_s\cos\epsilon)}{p'}=\psi.$$

As only the ratio $p':q'$ has been thus far used, we take

$$Ap'^2+2Hp'q'+Bq'^2=1.$$

Then the equations in v_s and t_s give

$$\psi^2=V^2(t_s^{\,2}+2t_sv_s\cos\epsilon+v_s^{\,2})=V^2w^2;$$

while the equations in x_s, y_s, z_s, give

$$\begin{aligned}r^2&=x_s^{\,2}+y_s^{\,2}+z_s^{\,2}+2iy_sz_s+2jz_sx_s+2kx_sy_s\\&=\phi^2(\mathrm{a}p'^4+4\mathrm{h}p'^3q'+6\mathrm{k}p'^2q'^2+4\mathrm{f}p'q'^3+\mathrm{c}q'^4),\end{aligned}$$

so that

$$\phi=r\rho.$$

The equation of the sphere, being

$$\frac{A}{\mathrm{a}^{\frac{1}{2}}}x_s+\frac{H}{\mathrm{b}^{\frac{1}{2}}}y_s+\frac{B}{\mathrm{c}^{\frac{1}{2}}}z_s=r^2,$$

now gives

$$\phi=r^2,$$

that is, $r=\rho$. Then the final equation

$$(v_s^{\,2}+2v_st_s\cos\epsilon+t_s^{\,2})T_2+A^{\frac{1}{2}}(t_s+v_s\cos\epsilon)S_2+B^{\frac{1}{2}}(v_s+t_s\cos\epsilon)R_2=0,$$

when substitution is made, gives

$$w=\sigma\rho'.$$

Hence, if R denote the radius vector from O to a point on the locus, where the coordinates are t_s, v_s, x_s, y_s, z_s,

$$\begin{aligned}R^2&=x_s^{\,2}+y_s^{\,2}+z_s^{\,2}+2iy_sz_s+2jz_sx_s+2kx_sy_s\\&\qquad+t_s^{\,2}+2t_sv_s\cos\epsilon+v_s^{\,2}\\&=\rho^2+(\sigma\rho')^2,\end{aligned}$$

or R denotes the radius of spherical curvature.

We at once infer the equation which determines the direction of the curves of spherical curvature. They are such as to make R a maximum or a minimum, and therefore (§ 57) they are such as to make the tilt vanish; hence, from the value of $1/\tau$ as obtained on p. 368, the directions of the curves of spherical curvature are given by the equation

$$\bar{\mathfrak{c}}p'^4 - 2\bar{\mathfrak{f}}p'^3q' + (\bar{\mathfrak{b}} + 2\bar{\mathfrak{g}})p'^2q'^2 - 2\bar{\mathfrak{h}}p'q'^3 + \bar{\mathfrak{a}}q'^4 = 0$$

and there are four curves of spherical curvature through each point on the surface.

146. When the plenary space is quadruple, the determinant Y, being

$$\begin{vmatrix} \mathfrak{a}, & \mathfrak{h}, & \mathfrak{g} \\ \mathfrak{h}, & \mathfrak{b}, & \mathfrak{f} \\ \mathfrak{g}, & \mathfrak{f}, & \mathfrak{c} \end{vmatrix},$$

vanishes; and there are relations among its first minors $\bar{\mathfrak{a}}$, $\bar{\mathfrak{b}}$, $\bar{\mathfrak{c}}$, $\bar{\mathfrak{f}}$, $\bar{\mathfrak{g}}$, $\bar{\mathfrak{h}}$, which may be taken * in the form

$$\bar{\mathfrak{f}} = \mathfrak{gh} - \mathfrak{af} = -(\bar{\mathfrak{b}}\bar{\mathfrak{c}})^{\frac{1}{2}}, \quad \bar{\mathfrak{g}} = \mathfrak{hf} - \mathfrak{bg} = (\bar{\mathfrak{c}}\bar{\mathfrak{a}})^{\frac{1}{2}}, \quad \bar{\mathfrak{h}} = \mathfrak{fg} - \mathfrak{ch} = -(\bar{\mathfrak{a}}\bar{\mathfrak{b}})^{\frac{1}{2}}.$$

But now the three directions determined by the quantities η_{11}, η_{12}, η_{22}, lie in the orthogonal plane and no longer can be leading lines of a flat; they are connected, for all the four variables, by the relation

$$\eta_{11}\bar{\mathfrak{a}}^{\frac{1}{2}} - \eta_{12}\bar{\mathfrak{b}}^{\frac{1}{2}} + \eta_{22}\bar{\mathfrak{c}}^{\frac{1}{2}} = 0.$$

For the locus of the centre of spherical curvature, two of the axes are taken in the tangent plane exactly as before; and, again denoting the corresponding coordinates by t_s and v_s, we have

$$t_s + v_s \cos\epsilon = -V\frac{w}{A^{\frac{1}{2}}}q', \quad v_s + t_s\cos\epsilon = V\frac{w}{B^{\frac{1}{2}}}p',$$

where

$$w = \sigma p', \quad w^2 = t_s{}^2 + 2t_sv_s\cos\epsilon + v_s{}^2.$$

In the orthogonal plane of the surface, we take the lines determined in direction by the typical magnitudes η_{11}, η_{22}, at an inclination ω; then, denoting the coordinates of the centre of spherical curvature relative to those axes by x_s and z_s, we have

$$x_s + z_s\cos\omega = \frac{1}{\mathfrak{a}^{\frac{1}{2}}}\sum \eta_{11}(y_S - y) = \frac{1}{\mathfrak{a}^{\frac{1}{2}}}(\mathfrak{a}p'^2 + 2\mathfrak{h}p'q' + \mathfrak{g}q'^2)\rho^2,$$

$$x_s\cos\omega + z_s = \frac{1}{\mathfrak{c}^{\frac{1}{2}}}\sum \eta_{22}(y_S - y) = \frac{1}{\mathfrak{c}^{\frac{1}{2}}}(\mathfrak{g}p'^2 + 2\mathfrak{f}p'q' + \mathfrak{c}q'^2)\rho^2,$$

* *G.F.D.*, vol. i, § 214; for the following discussion, in part, reference may also be made to § 242, *l.c.*

while

$$x_s^2+2x_s z_s\cos\omega+z_s^2=r^2=\rho^2.$$

With these, we combine the permanent equation

$$Ap'^2+2Hp'q'+Bq'^2=1,$$

thus obtaining three equations linear in p'^2, $p'q'$, q'^2. We denote by $\mathbf{I}$ the quantity

$$\mathbf{I}=A\bar{\mathfrak{a}}^{\frac{1}{2}}-H\bar{\mathfrak{b}}^{\frac{1}{2}}+B\bar{\mathfrak{c}}^{\frac{1}{2}},$$

which is an invariant of the surface *; and we find

$$\mathbf{I}p'^2=\bar{\mathfrak{a}}^{\frac{1}{2}}+\frac{1}{r^2}\left\{\frac{x_s}{\mathfrak{a}^{\frac{1}{2}}}(B\bar{\mathfrak{c}}^{\frac{1}{2}}-H\bar{\mathfrak{b}}^{\frac{1}{2}})-\frac{z_s}{\mathfrak{c}^{\frac{1}{2}}}B\bar{\mathfrak{a}}^{\frac{1}{2}}\right\}=P,$$

$$2\mathbf{I}p'q'=-\bar{\mathfrak{b}}^{\frac{1}{2}}+\frac{\bar{\mathfrak{b}}^{\frac{1}{2}}}{r^2}\left\{\frac{x_s}{\mathfrak{a}^{\frac{1}{2}}}A+\frac{z_s}{\mathfrak{c}^{\frac{1}{2}}}B\right\}=2Q,$$

$$\mathbf{I}q'^2=\bar{\mathfrak{c}}^{\frac{1}{2}}+\frac{1}{r^2}\left\{\frac{z_s}{\mathfrak{c}^{\frac{1}{2}}}(A\bar{\mathfrak{a}}^{\frac{1}{2}}-H\bar{\mathfrak{b}}^{\frac{1}{2}})-\frac{x_s}{\mathfrak{a}^{\frac{1}{2}}}A\bar{\mathfrak{b}}^{\frac{1}{2}}\right\}=R,$$

which can also be written in the forms

$$\frac{1}{\bar{\mathfrak{a}}^{\frac{1}{2}}}\left(p'^2-\frac{x_s}{r^2\mathfrak{a}^{\frac{1}{2}}}\right)=-\frac{1}{\bar{\mathfrak{b}}^{\frac{1}{2}}}2p'q'=\frac{1}{\bar{\mathfrak{c}}^{\frac{1}{2}}}\left(q'^2-\frac{z_s}{r^2\mathfrak{c}^{\frac{1}{2}}}\right)$$

$$=\frac{1}{\mathbf{I}}\left\{1-\frac{1}{r^2}\left(\frac{A}{\mathfrak{a}^{\frac{1}{2}}}x_s+\frac{B}{\mathfrak{c}^{\frac{1}{2}}}z_s\right)\right\}.$$

Thus two of the three equations of the curve-locus are

$$-\frac{A^{\frac{1}{2}}(t_s+v_s\cos\epsilon)}{B^{\frac{1}{2}}(v_s+t_s\cos\epsilon)}=\frac{R}{Q}=\frac{Q}{P},$$

the common value of each of the fractions being q'/p'. When the equations are taken, by equating the first fraction to the second fraction and to the third fraction in turn, they are each of the third order in the coordinates.

The parametric value of $\sigma\rho'$ has not yet been used. Being concerned with equations which, except for the relation $Y=0$, are independent of the dimensionality of the plenary space, it is given by the same formula as for the more general case; and therefore

$$-\frac{\rho'}{\rho^3}=p'R_4+q'S_4,$$

where

$$R_4=\mathbf{e}_0p'^4+4\mathbf{e}_1p'^3q'+6\mathbf{e}_2p'^2q'^2+4\mathbf{e}_3p'q'^3+\mathbf{e}_4q'^4,$$
$$S_4=\mathbf{e}_1p'^4+4\mathbf{e}_2p'^3q'+6\mathbf{e}_3p'^2q'^2+4\mathbf{e}_4p'q'^3+\mathbf{e}_5q'^4,$$

* *G.F.D.*, vol. i, §§ 240, 364.

while

$$\frac{V}{\sigma\rho} = \begin{vmatrix} \mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + 3\mathbf{k}p'q'^2 + \mathbf{f}q'^3, & Ap' + Hq' \\ \mathbf{h}p'^3 + 3\mathbf{k}p'^2q' + 3\mathbf{f}p'q'^2 + \mathbf{c}q'^3, & Hp' + Bq' \end{vmatrix} = Q_4,$$

so that R_4, S_4, Q_4, are homogeneous quartics in p', q'. Thus

$$Q_4 \frac{\sigma\rho'}{\rho^2} = -Vq'S_4 - Vp'R_4;$$

and therefore

$$Q_4 \frac{(\sigma\rho')^2}{\rho^2} = -S_4Vq'\sigma\rho' - R_4Vp'\sigma\rho'.$$

Hence

$$\frac{t_s^2 + 2t_sv_s\cos\epsilon + v_s^2}{x_s^2 + 2x_sz_s\cos\omega + z_s^2} = \frac{S_4}{Q_4}A^{\frac{1}{2}}(t_s + v_s\cos\epsilon) - \frac{R_4}{Q_4}B^{\frac{1}{2}}(v_s + t_s\cos\epsilon).$$

But

$$\frac{S_4}{Q_4} = \frac{\mathbf{e}_1P^2 + 4\mathbf{e}_2PQ + \mathbf{e}_3(4Q^2 + 2PR) + 4\mathbf{e}_4QR + \mathbf{e}_5R^2}{(\mathbf{a}H - \mathbf{h}A)P^2 + \ldots\ldots + (\mathbf{f}B - \mathbf{c}H)R^2} = \frac{X_4}{Z_4},$$

and similarly

$$\frac{R_4}{Q_4} = \frac{Y_4}{Z_4},$$

where X_4, Y_4 Z_4 are non-homogeneous expressions of degree four in x_s and y_s; and the equation therefore is

$$\begin{aligned} &Z_4(t_s^2 + 2t_sv_s\cos\epsilon + v_s^2) \\ &\quad = (x_s^2 + 2x_sz_s\cos\omega + z_s^2)\{A^{\frac{1}{2}}X_4(t_s + v_s\cos\epsilon) - B^{\frac{1}{2}}Y_4(v_s + t_s\cos\epsilon)\}, \end{aligned}$$

an equation of the seventh order, as there is apparently no reduction.

Thus the locus in question appears to be a curve of degree sixty-three in the plenary quadruple space.

147. When the plenary space of the surface is triple, much of the foregoing analysis is evanescent; for, not merely does the determinant Y vanish as in quadruple plenary space, but every first minor $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$, $\bar{\mathbf{f}}$, $\bar{\mathbf{g}}$, $\bar{\mathbf{h}}$, also vanishes. It is simple to investigate the locus of the centre of spherical curvature directly.

The normal to the surface at any point is a unique direction at the point, unaffected by any superficial direction through the point. Accordingly, to represent the locus of the spherical centre for all geodesics through the point, we take the two parametric directions in the tangent plane as axes of x and y, and the normal to the surface as the axis of z. Still denoting the angle between the parametric curves by ϵ, and denoting the coordinates of the spherical centre at O, relative to these axes, by x, y, z, we have

$$\begin{aligned} (x + y\cos\epsilon)A^{\frac{1}{2}} &= -q'V\sigma\rho' = -q'Vw, \\ (x\cos\epsilon + y)B^{\frac{1}{2}} &= p'V\sigma\rho' = p'Vw, \end{aligned}$$

while

$$w^2 = x^2 + 2xy\cos\epsilon + y^2.$$

Also we have $z = \rho$, so that

$$\frac{1}{z} = \bar{A}p'^2 + 2\bar{H}p'q' + \bar{B}q'^2,$$

where $\bar{A}$, $\bar{H}$, $\bar{B}$, now are independent of the direction-variables p', q'. Further, it is known * that

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = Pp'^3 + 3Qp'^2q' + 3Rp'q'^2 + Sq'^3,$$

where P, Q, R, S, like $\bar{A}$, $\bar{H}$, $\bar{B}$, are independent of p', q'; and, as usual, we have

$$\begin{aligned}\frac{V}{\sigma} &= (\bar{A}p' + \bar{H}q')(Hp' + Bq') - (\bar{H}p' + \bar{B}q')(Ap' + Hq') \\ &= \alpha p'^2 + 2\beta p'q' + \gamma q'^2,\end{aligned}$$

with obvious significations for α, β, γ, which are independent of p', q'. Hence

$$\frac{w}{\rho^2} = \frac{\sigma\rho'}{\rho^2} = -V\frac{Pp'^3 + 3Qp'^2q' + 3Rp'q'^2 + Sq'^3}{\alpha p'^2 + 2\beta p'q' + \gamma q'^2}.$$

For the curve-locus, two equations are required involving x, y, z, only, among the magnitudes that vary from one geodesic to another. We have

$$\begin{aligned}\frac{V^2}{z}(x^2 + 2xy\cos\epsilon + y^2) &= \bar{A}(p'Vw)^2 + 2\bar{H}(p'Vw)(q'Vw) + \bar{B}(q'Vw)^2 \\ &= \bar{A}B(y + x\cos\epsilon)^2 - 2\bar{H}(AB)^{\frac{1}{2}}(x + y\cos\epsilon)(y + x\cos\epsilon) + \bar{B}A(x + y\cos\epsilon)^2,\end{aligned}$$

being an equation of the third degree. Again,

$$\begin{aligned}&\frac{1}{z^2}(x^2 + 2xy\cos\epsilon + y^2) \\ &\qquad = -Vp'w\frac{Pp'^2 + 2Qp'q' + Rq'^2}{\alpha p'^2 + 2\beta p'q' + \gamma q'^2} - Vq'w\frac{Qp'^2 + 2Rp'q' + Sq'^2}{\alpha p'^2 + 2\beta p'q' + \gamma q'^2},\end{aligned}$$

leading to

$$\frac{1}{z^2}(x^2 + 2xy\cos\epsilon + y^2) = \begin{vmatrix} (x + y\cos\epsilon)A^{\frac{1}{2}}, & \dfrac{Pp'^2 + 2Qp'q' + Rq'^2}{\alpha p'^2 + 2\beta p'q' + \gamma q'^2} \\ (x\cos\epsilon + y)B^{\frac{1}{2}}, & \dfrac{Qp'^2 + 2Rp'q' + Sq'^2}{\alpha p'^2 + 2\beta p'q' + \gamma q'^2} \end{vmatrix}.$$

When, on the right-hand side, we substitute

$$-\frac{A^{\frac{1}{2}}(x + y\cos\epsilon)}{B^{\frac{1}{2}}(x\cos\epsilon + y)}$$

for q'/p', the equation is a non-homogeneous equation of degree five. The locus required is therefore a skew curve of degree fifteen.

* See my *Lectures on Differential Geometry*, § 40.

Centre of globular curvature of a superficial geodesic.

148. The centre of globular curvature of a geodesic is attained by measuring a distance ρ along its prime normal (up to the centre of circular curvature), a distance $\sigma\rho'$ along a line through the centre of circular curvature parallel to the binormal (up to the centre of spherical curvature), and a distance $\tau\dfrac{RR'}{\sigma\rho'}$ along a line through the centre of spherical curvature parallel to the trinormal. If η_g is the typical space-coordinate of the centre of globular curvature, then

$$\eta_g - y = Y\rho + l_3\sigma\rho' + l_4\tau\frac{RR'}{\sigma\rho'}.$$

The locus of the centre is a curve, lying in a quintuple homaloid, leading lines of which are provided, as to two, by the tangent plane, as to the remaining three, by the orthogonal flat of the surface.

For the more definite coordinate-specification of the centre of globular curvature, we choose the same axes of reference in this quintuple homaloid as were chosen in connection with the locus of the centre of spherical curvature. The parametric directions in the tangent plane are chosen as axes of v and t; so that we have

$$\left.\begin{aligned} t + v\cos\epsilon &= \frac{1}{A^{\frac{1}{2}}}\sum\frac{\partial y}{\partial p}(\eta_g - y) = -\frac{1}{A^{\frac{1}{2}}}\sigma\rho' Vq' \\ v + t\cos\epsilon &= \frac{1}{B^{\frac{1}{2}}}\sum\frac{\partial y}{\partial q}(\eta_g - y) = \quad\frac{1}{B^{\frac{1}{2}}}\sigma\rho' Vp' \end{aligned}\right\},$$

while

$$(\sigma\rho')^2 = t^2 + 2tv\cos\epsilon + v^2.$$

The three axes of reference in the orthogonal flat are the lines determined, as to direction, by the magnitudes η_{11}, η_{12}, η_{22}; hence, when the coordinates of the centre of globular curvature (relative to these lines) are denoted by x_g, y_g, z_g, we have, by taking projections along the axes, the relations

$$\begin{aligned} x_g + ky_g + jz_g &= \frac{1}{\mathrm{a}^{\frac{1}{2}}}\sum\eta_{11}(\eta_g - y), \\ kx_g + \ y_g + iz_g &= \frac{1}{\mathrm{b}^{\frac{1}{2}}}\sum\eta_{12}(\eta_g - y), \\ jx_g + iy_g + \ z_g &= \frac{1}{\mathrm{c}^{\frac{1}{2}}}\sum\eta_{22}(\eta_g - y), \end{aligned}$$

with the former values

$$i = \frac{\mathrm{f}}{(\mathrm{bc})^{\frac{1}{2}}},\quad j = \frac{\mathrm{g}}{(\mathrm{ca})^{\frac{1}{2}}},\quad k = \frac{\mathrm{h}}{(\mathrm{ab})^{\frac{1}{2}}}.$$

Thus, when the values of the quantities $\eta_g - y_0$ are substituted,

$$x_g + ky_g + jz_g = \frac{1}{\mathbf{a}^{\frac{1}{2}}}\left\{(\mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}q'^2)\rho^2 + A_4\tau\frac{RR'}{\sigma\rho'}\right\},$$

$$kx_g + \ y_g + iz_g = \frac{1}{\mathbf{b}^{\frac{1}{2}}}\left\{(\mathbf{h}p'^2 + 2\mathbf{b}p'q' + \mathbf{f}q'^2)\rho^2 + H_4\tau\frac{RR'}{\sigma\rho'}\right\},$$

$$jx_g + \ iy_g + \ z_g = \frac{1}{\mathbf{c}^{\frac{1}{2}}}\left\{(\mathbf{g}p'^2 + 2\mathbf{f}p'q' + \mathbf{c}q'^2)\rho^2 + B_4\tau\frac{RR'}{\sigma\rho'}\right\}.$$

Now

$$\mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}q'^2 = \mathbf{a}^{\frac{1}{2}}(\ \mathbf{a}^{\frac{1}{2}}p'^2 + k2\mathbf{b}^{\frac{1}{2}}p'q' + j\mathbf{c}^{\frac{1}{2}}q'^2),$$

$$\mathbf{h}p'^2 + 2\mathbf{b}p'q' + \mathbf{f}q'^2 = \mathbf{b}^{\frac{1}{2}}(k\mathbf{a}^{\frac{1}{2}}p'^2 + \ 2\mathbf{b}^{\frac{1}{2}}p'q' + i\mathbf{c}^{\frac{1}{2}}q'^2),$$

$$\mathbf{g}p'^2 + 2\mathbf{f}p'q' + \mathbf{c}q'^2 = \mathbf{c}^{\frac{1}{2}}(j\mathbf{a}^{\frac{1}{2}}p'^2 + i2\mathbf{b}^{\frac{1}{2}}p'q' + \ \mathbf{c}^{\frac{1}{2}}q'^2);$$

and therefore

$$(x_g - \mathbf{a}^{\frac{1}{2}}p'^2\rho^2) + k(y_g - 2\mathbf{b}^{\frac{1}{2}}p'q'\rho^2) + j(z_g - \mathbf{c}^{\frac{1}{2}}q'^2\rho^2) = \frac{A_4}{\mathbf{a}^{\frac{1}{2}}}\tau\frac{RR'}{\sigma\rho'},$$

$$k(x_g - \mathbf{a}^{\frac{1}{2}}p'^2\rho^2) + \ (y_g - 2\mathbf{b}^{\frac{1}{2}}p'q'\rho^2) + i(z_g - \mathbf{c}^{\frac{1}{2}}q'^2\rho^2) = \frac{H_4}{\mathbf{b}^{\frac{1}{2}}}\tau\frac{RR'}{\sigma\rho'},$$

$$j(x_g - \mathbf{a}^{\frac{1}{2}}p'^2\rho^2) + i(y_g - 2\mathbf{b}^{\frac{1}{2}}p'q'\rho^2) + \ (z_g - \mathbf{c}^{\frac{1}{2}}q'^2\rho^2) = \frac{B_4}{\mathbf{c}^{\frac{1}{2}}}\tau\frac{RR'}{\sigma\rho'}.$$

The values of A_4, H_4, B_4, can be taken (p. 368) in the forms

$$\frac{A_4}{\rho\tau} = \frac{1}{V}\{v_1(\mathbf{h}p' + \mathbf{g}q') - v_2(\mathbf{a}p' + \mathbf{h}q')\},$$

$$\frac{H_4}{\rho\tau} = \frac{1}{V}\{v_1(\mathbf{b}p' + \mathbf{f}q') - v_2(\mathbf{h}p' + \mathbf{b}q')\},$$

$$\frac{B_4}{\rho\tau} = \frac{1}{V}\{v_1(\mathbf{f}p' + \mathbf{c}q') - v_2(\mathbf{g}p' + \mathbf{f}q')\};$$

and therefore

$$A_4\bar{\mathfrak{a}} + H_4\bar{\mathfrak{h}} + B_4\bar{\mathfrak{g}} = -\frac{\rho\tau}{V}Yv_2p',$$

with like expressions for $A_4\bar{\mathfrak{h}} + H_4\bar{\mathfrak{b}} + B_4\bar{\mathfrak{f}}$, $A_4\bar{\mathfrak{g}} + H_4\bar{\mathfrak{f}} + B_4\bar{\mathfrak{c}}$. When the equations for x_g, y_g, z_g, are resolved, we find

$$\frac{Y}{\mathbf{abc}}(x_g - \mathbf{a}^{\frac{1}{2}}p'^2\rho^2) = \frac{\tau RR'}{\sigma\rho'}\left\{\frac{A_4}{\mathbf{a}^{\frac{1}{2}}}(1 - i^2) + \frac{H_4}{\mathbf{b}^{\frac{1}{2}}}(ij - k) + \frac{B_4}{\mathbf{c}^{\frac{1}{2}}}(ki - j)\right\}$$

$$= \frac{\tau RR'}{\sigma\rho'}\frac{1}{\mathbf{a}^{\frac{1}{2}}\mathbf{bc}}(A_4\bar{\mathfrak{a}} + H_4\bar{\mathfrak{h}} + B_4\bar{\mathfrak{g}})$$

$$= -\frac{\tau RR'}{\sigma\rho'}\frac{Y}{\mathbf{a}^{\frac{1}{2}}\mathbf{bc}}\frac{\rho\tau}{V}v_2p'.$$

Similarly for y_g and z_g; the values are

$$\left.\begin{aligned}
x_g - \mathrm{a}^{\frac{1}{2}}p'^2\rho^2 &= \mathrm{a}^{\frac{1}{2}}\frac{\rho\tau^2 RR'}{V\sigma\rho'}(-v_2 p') \\
y_g - 2\mathrm{b}^{\frac{1}{2}}p'q'\rho^2 &= \mathrm{b}^{\frac{1}{2}}\frac{\rho\tau^2 RR'}{V\sigma\rho'}(v_1 p' - v_2 q') \\
z_g - \mathrm{c}^{\frac{1}{2}}q'^2\rho^2 &= \mathrm{c}^{\frac{1}{2}}\frac{\rho\tau^2 RR'}{V\sigma\rho'}\,v_1 q'.
\end{aligned}\right\}.$$

Thus the five coordinates are expressed in terms of the direction-variables p', q', subject to the relation $Ap'^2 + 2Hp'q' + Bq'^2 = 1$; for expressions for ρ, $\sigma\rho'$, τ, in terms of p' and q' are known, while

$$\frac{RR'}{\rho^2\sigma^2\rho'} = -\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) + \frac{1}{\sigma^2\rho} + 2\frac{\rho'^2}{\rho^3} + \frac{\sigma'}{\sigma}\frac{\rho'}{\rho^2},$$

so that the parametric value of RR' is known. But there seems no simple form of eliminant, in the shape of four equations between the coordinates t, v, x_g, y_g, z_g, for the expression of the curve free from the direction-variables p', q'.

It is easy to verify the relations

$$\left.\begin{aligned}
&\frac{Ax_g}{\mathrm{a}^{\frac{1}{2}}} + \frac{Hy_g}{\mathrm{b}^{\frac{1}{2}}} + \frac{Bz_g}{\mathrm{c}^{\frac{1}{2}}} = \rho^2 + \frac{\rho\tau}{\sigma}\,\tau\frac{RR'}{\sigma\rho'} \\
&\frac{\bar{A}x_g}{\mathrm{a}^{\frac{1}{2}}} + \frac{\bar{H}y_g}{\mathrm{b}^{\frac{1}{2}}} + \frac{\bar{B}z_g}{\mathrm{c}^{\frac{1}{2}}} = \rho, \\
&\frac{A_4x_g}{\mathrm{a}^{\frac{1}{2}}} + \frac{H_4y_g}{\mathrm{b}^{\frac{1}{2}}} + \frac{B_4z_g}{\mathrm{c}^{\frac{1}{2}}} = \tau\frac{RR'}{\sigma\rho'},
\end{aligned}\right\}.$$

CHAPTER XIII

Parametric Curves on a Surface

Parametric definition of a curve on a surface.

149. Thus far, the only curves which have been considered in their analytical relation to a surface are its organic geodesics; we now proceed to consider some properties of any non-geodesic curve on a surface. It can be represented analytically by an equation

$$\theta(p, q)=0$$

between the superficial parameters, the function θ being unrestricted. Its tangent at any point is a tangent line of the surface; and the direction-variables p', q', of its tangent satisfy the equation

$$\theta_1 p' + \theta_2 q' = 0,$$

where θ_1 and θ_2 denote the first derivatives of θ with regard to p and q respectively, so that

$$\frac{p'}{\theta_2}=\frac{q'}{-\theta_1}=(A\theta_2{}^2-2H\theta_1\theta_2+B\theta_1{}^2)^{-\frac{1}{2}}=\kappa'.$$

Thus differentiation, taken along the curve, of magnitudes which are functions of position alone, such as A, H, B, V, can be expressed as parametric differentiation by means of the equivalent operators

$$\frac{d}{ds}, \quad \frac{1}{\kappa'}\left(\theta_2\frac{\partial}{\partial p}-\theta_1\frac{\partial}{\partial q}\right).$$

But this operation cannot be repeated, without modification; and it cannot be performed upon a magnitude which is a function of direction as well as of position. For second and higher derivatives in the surface, both in significance and in analytical expression, there is a difference according as they continue to be taken along the curve or as they are to be taken along a superficial geodesic touching the curve. To represent geodesic differentiation, the former notation

$$p'', \quad \frac{d^2p}{ds^2}, \quad q'', \quad \frac{d^2q}{ds^2},$$

and similarly for higher derivatives, will be used; for differentiation along the curve (which still is differentiation on the surface), the notation

$$p_t'', \quad \frac{d^2p}{dt^2}, \quad q_t'', \quad \frac{d^2q}{dt^2},$$

and similarly for higher derivatives, will be used.

For the representation of repeated parametric differentiation of θ, we shall write

$$\theta_{11}=\frac{\partial^2\theta}{\partial p^2},\quad \theta_{12}=\frac{\partial^2\theta}{\partial p\,\partial q},\quad \theta_{22}=\frac{\partial^2\theta}{\partial q^2}.$$

Hence along the curve $\theta=0$, we have

$$\theta_1 p_t''+\theta_2 q_t''+\theta_{11}p'^2+2\theta_{12}p'q'+\theta_{22}q'^2=0\,;$$

and therefore, when we take symbols ϑ_{11}, ϑ_{12}, ϑ_{22}, analogous to η_{11}, η_{12}, η_{22}, such that

$$\left.\begin{aligned}\vartheta_{11}&=\theta_{11}-\theta_1\Gamma_{11}-\theta_2\Delta_{11}\\ \vartheta_{12}&=\theta_{12}-\theta_1\Gamma_{12}-\theta_2\Delta_{12}\\ \vartheta_{22}&=\theta_{22}-\theta_1\Gamma_{22}-\theta_2\Delta_{22}\end{aligned}\right\},$$

as well as a symbol Θ such that

$$\Theta=\vartheta_{11}p'^2+2\vartheta_{12}p'q'+\vartheta_{22}q'^2,$$

the last equation becomes

$$\theta_1(p_t''+\Gamma_{11}p'^2+2\Gamma_{12}p'q'+\Gamma_{22}q'^2)+\theta_2(q_t''+\Delta_{11}p'^2+2\Delta_{12}p'q'+\Delta_{22}q'^2)=-\Theta.$$

But when we take a differentiation of the permanent relation

$$Ap'^2+2Hp'q'+Bq'^2=1,$$

merely in the surface along the arbitrary curve and not specially along a geodesic, we have

$$(Ap'+Hq')(p_t''+\Gamma_{11}p'^2+2\Gamma_{12}p'q'+\Gamma_{22}q'^2)$$
$$+(Hp'+Bq')(q_t''+\Delta_{11}p'^2+2\Delta_{12}p'q'+\Delta_{22}q'^2)=0.$$

Also

$$p''=-(\Gamma_{11}p'^2+2\Gamma_{12}p'q'+\Gamma_{22}q'^2),\quad q''=-(\Delta_{11}p'^2+2\Delta_{12}p'q'+\Delta_{22}q'^2),$$

along the geodesic which touches the curve ; hence the foregoing equations give

$$\frac{p_t''-p''}{Hp'+Bq'}=\frac{q_t''-q''}{-(Ap'+Hq')}=\frac{\Theta}{C},$$

where

$$C=\theta_1(Hp'+Bq')-\theta_2(Ap'+Hq')=-\frac{1}{\kappa'},$$

on substituting the values of p' and q'. Thus

$$\left.\begin{aligned}p_t''-p''&=\kappa'(Hp'+Bq')\Theta\\ q_t''-q''&=-\kappa'(Ap'+Hq')\Theta\end{aligned}\right\}.$$

Again, for the second variations of the typical variable y, along the curve and along the geodesic touching the curve, we have

$$y_t''=\frac{\partial y}{\partial p}p_t''+\frac{\partial y}{\partial q}q_t''+\frac{\partial^2 y}{\partial p^2}p'^2+2\frac{\partial^2 y}{\partial p\,\partial q}p'q'+\frac{\partial^2 y}{\partial q^2}q'^2,$$

$$y''=\frac{\partial y}{\partial p}p''+\frac{\partial y}{\partial q}q''+\frac{\partial^2 y}{\partial p^2}p'^2+2\frac{\partial^2 y}{\partial p\,\partial q}p'q'+\frac{\partial^2 y}{\partial q^2}q'^2\,;$$

and therefore

$$\begin{aligned} y_t'' - y'' &= \frac{\partial y}{\partial p}(p_t'' - p'') + \frac{\partial y}{\partial q}(q_t'' - q'') \\ &= \kappa'\Theta\left\{(Hp' + Bq')\frac{\partial y}{\partial p} - (Ap' + Hq')\frac{\partial y}{\partial q}\right\} \\ &= -\kappa'\Theta V l_3, \end{aligned}$$

where l_3 is the typical direction-cosine of the binormal of the superficial geodesic (§ 371), being a line in the tangent plane of the surface at right angles to the common tangent of the curve and the geodesic.

Now let ρ_c denote the radius of circular curvature of the curve and Y_c denote the typical direction-cosine of its prime normal, these quantities corresponding to ρ and Y for the geodesic; thus

$$\rho_c y_t'' = Y_c, \quad \rho y'' = Y,$$

so that the foregoing relation becomes

$$\frac{Y_c}{\rho_c} - \frac{Y}{\rho} = -\kappa'\Theta V l_3.$$

In considering the relation of the curve $\theta = 0$ to the surface, one obvious measure is its arc-rate of deviation from the geodesic tangent; we shall call this arc-rate the *flexure* (or, less briefly, the superficial flexure) of the curve, being a quantity similar to the arc-rate of deviation of the curve from its linear tangent in the plenary space, that is, similar to the circular curvature of the curve. This flexure is denoted by $1/\gamma$; and it will appear that there is a radius of flexure, equal to γ, as well as a direction of that radius, being that of the binormal of the geodesic tangent.

A typical direction-cosine of the geodesic tangent is y'. At a consecutive point along the curve, distant δ from O, the same typical direction-cosine of the tangent to the curve is $(\lambda_c =)\ y' + y_t''\delta$; while at a consecutive point along the geodesic, at an equal distance δ from O, the same typical direction-cosine of the tangent to the geodesic is $(\lambda =)\ y' + y''\delta$. If then i denote the same angle between these tangents at consecutive points, we have

$$\cos i = \sum \lambda\lambda_c,$$

and therefore

$$2(1 - \cos i) = \sum(\lambda_c - \lambda)^2 = \delta^2 \sum (y_t'' - y'')^2,$$

that is,

$$i^2 = \delta^2 \kappa'^2 \Theta^2 V^2,$$

to the most significant power of δ. But the arc-rate of variation of the deviation of the curve from its geodesic tangent is the limit of i/δ, as δ tends towards zero; hence

$$\frac{1}{\gamma} = -\kappa'\Theta V,$$

the negative sign being chosen by reference to the earlier relations. Thus the magnitude of the flexure

$$= -V\Theta\kappa'$$

$$= -V\frac{\vartheta_{11}\theta_2^2 - 2\vartheta_{12}\theta_1\theta_2 + \vartheta_{11}\theta_1^2}{(A\theta_2^2 - 2H\theta_2\theta_1 + B\theta_1^2)^{\frac{3}{2}}};$$

but the first form

$$\frac{1}{\gamma} = -\kappa' V(\vartheta_{11}p'^2 + 2\vartheta_{12}p'q' + \vartheta_{22}q'^2)$$

will be the more useful. And now, with this interpretation, the former relation connecting the circular curvatures of the curve and its geodesic tangent becomes

$$\frac{Y_c}{\rho_c} - \frac{Y}{\rho} = \frac{l_3}{\gamma},$$

with the selected sign in the value of $1/\gamma$.

We can obtain the specific result in a different manner. Let y_Q be the typical space-coordinate at the point Q on the curve distant δ from O, and let y_P denote the corresponding space-coordinate at the point P on the geodesic distant δ from O. The common tangent of the curve and the geodesic is

$$\frac{\bar{y}_1 - y_1}{y_1'} = \frac{\bar{y}_2 - y_2}{y_2'} = \dots;$$

we denote the foot of the perpendicular from Q on this line by the typical coordinate

$$y + \mu_Q y'$$

and the foot of the perpendicular from P on this line by

$$y + \mu_P y'.$$

Let l_c and Π_c denote a typical direction-cosine and the length of the former perpendicular; and let l, Π, have the corresponding significance for the latter perpendicular: so that

$$l_c\Pi_c = y_Q - (y + \mu_Q y'), \quad l\Pi = y_P - (y + \mu_P y').$$

For perpendicularity in the respective instances, we must have each of the quantities

$$\sum\{y_Q - (y + \mu_Q y')\}^2, \quad \sum\{y_P - (y + \mu_P y')\}^2,$$

a minimum.

For the first instance, the critical equation is

$$\sum y'\{y_Q - (y + \mu_Q y')\} = 0,$$

the variable parameter being μ_Q; and therefore

$$\begin{aligned}\mu_Q &= \sum y'(y_Q - y) \\ &= \sum y'(y'\delta + \tfrac{1}{2}\delta^2 y_t'' + \tfrac{1}{6}\delta^3 y_t''' + \ldots).\end{aligned}$$

Now

$$\sum y'^2 = 1, \quad \sum y' y_t'' = 0, \quad \sum y_t''^2 = \frac{1}{\rho_c^2}, \quad \sum y' y_t''' = -\sum y_t''^2 = -\frac{1}{\rho_c^2};$$

and therefore

$$\mu_Q = \delta - \tfrac{1}{6}\frac{\delta^3}{\rho_c^2} + \text{higher powers of } \delta.$$

Similarly, in the second instance,

$$\mu_P = \delta - \tfrac{1}{6}\frac{\delta^3}{\rho^2} + \text{higher powers of } \delta.$$

Accordingly, up to the second power of δ inclusive, we have

$$\mu_Q = \mu_P = \delta.$$

Hence

$$\begin{aligned}l_c \Pi_c &= y_Q - (y + \delta y') \\ &= \tfrac{1}{2} y_t'' \delta^2,\end{aligned}$$

accurately up to the second power of δ inclusive; and therefore

$$\frac{1}{\rho_c} = \frac{2\Pi_c}{\delta^2}, \quad l_c = \rho_c y_t'' = Y_c.$$

Similarly,

$$\frac{1}{\rho} = \frac{2\Pi}{\delta^2}, \quad l = \rho y'' = Y.$$

Now the component of the deviation QP, between the point on the curve and the point on the geodesic, measured away from the geodesic, is

$$y_Q - y_P$$

along the typical spatial axis; and

$$\begin{aligned}y_Q - y_P &= (y_Q - y) - (y_P - y) \\ &= \tfrac{1}{2}(y_t'' - y'')\delta^2 = -\tfrac{1}{2} l_3 \kappa' \Theta V \delta^2,\end{aligned}$$

up to the second power of δ inclusive. Consequently, the typical direction-cosine of this deviation of the curve, measured away from the geodesic in the positive direction of the spatial axis, is l_3; and if $\bar{\Pi}$ denotes this deviation,

$$\bar{\Pi} = -\tfrac{1}{2}\kappa' \Theta V \delta^2.$$

If then γ is the radius of the circle in the tangent plane of the surface, drawn so as to have the closest contact with the curve in that plane, the direction of that

radius is along the line with l_3 for its typical direction-cosine (that is, it lies along the binormal), and the magnitude of that radius is given by

$$\frac{1}{\gamma}=\frac{2\bar{\Pi}}{\delta^2}=-\kappa'\Theta V.$$

Thus there is a radius of flexure, γ, for the curve; the direction of that radius of flexure is the binormal of the geodesic; and the directions and the magnitudes, of the circular curvatures and of the flexure, are connected by the relation

$$\frac{Y_c}{\rho_c}-\frac{Y}{\rho}=\frac{l_3}{\gamma}.$$

Relations of circular curvature and flexure of a curve.

150. From this relation, it follows that the equations

$$\| \; Y_c, \quad Y, \quad l_3 \; \|=0$$

are satisfied; and therefore the three directions, typified by Y_c, Y, l_3, lie in one plane. Now two directions determine a plane; and two of the three lines are the prime normal and the binormal of the geodesic tangent, both being perpendicular to the tangent line. Hence the radius of flexure of the curve lies in this plane, and it is perpendicular to the tangent.

In the diagram, let OG represent the radius of circular curvature of the geodesic tangent of the curve, OF represent the radius of flexure of the curve (so that OF is the direction of the binormal of the geodesic). Then if OC represents the radius of circular curvature of the curve, OC lies in the plane GOF, the tangent of the curve and the geodesic being perpendicular to that plane: while the osculating flat has this tangent, together with OF and OG, for guiding lines. The angle GOF is a right angle; let ψ denote the angle GOC, the angle between the prime normals of the curve and the geodesic, and also the angle between the osculating planes of the curve and the geodesic. Now

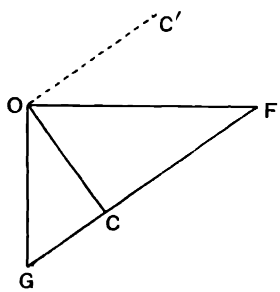

FIG. 15.

$$\sum Y^2=1, \quad \sum Y_c^2=1, \quad \sum l_3^2=1, \quad \sum Yl_3=0,$$

while

$$\sum YY_c=\cos\psi, \quad \sum l_3 Y_c=\sin\psi.$$

Hence, multiplying the relation by Y, and adding for all space-variables, we have

$$\frac{\cos\psi}{\rho_c}=\frac{1}{\rho};$$

multiplying by l_3 and adding for all space-variables, we have

$$\frac{\sin\psi}{\rho_c}=\frac{1}{\gamma};$$

and multiplying by Y_c and adding for all space-variables, we have

$$\frac{1}{\rho_c}=\frac{\cos\psi}{\rho}+\frac{\sin\psi}{\gamma}.$$

Thus there are the further equations

$$Y_c=Y\cos\psi+l_3\sin\psi,$$

$$\frac{1}{{\rho_c}^2}=\frac{1}{\rho^2}+\frac{1}{\gamma^2}.$$

Also there is a direction OC' through O parallel to GF; its typical direction-cosine L_3 is given by

$$L_3=l_3\cos\psi-Y\sin\psi.$$

Now

$$\rho_c=\rho\cos\psi=OG\cos GOC$$
$$=\gamma\sin\psi=OF\cos FOC,$$

and ρ_c lies along OC; hence, if OC be drawn perpendicular to GF, its direction is that of the prime normal of the curve, and its magnitude is the radius of circular curvature of the curve.

Ex. The simplest example is provided by a small circle of angular radius α on a unit sphere. Then $\psi=\frac{1}{2}\pi-\alpha$; the radius of circular curvature is $\sin\alpha$; the radius of flexure is $\tan\alpha$.

The preceding analysis can be illustrated * geometrically. At O let the tangent line, common to the curve and the geodesic, be TOD; from Q, the consecutive point (as on p. 417) on the curve, let a perpendicular be drawn on TOD, meeting it at a distance μ_Q from O; and from P, the consecutive point (as on p. 417) on the geodesic, let a perpendicular be drawn also on TOD, meeting it at a distance μ_P from O. Now μ_Q and μ_P differ from one another by third powers of δ, the equal lengths of OP and OQ; and their common value is δ, accurately up to the second order inclusive. If then we take a length OD equal to δ, the point D can be regarded as the common foot of the perpendiculars from Q and from P upon the tangent.

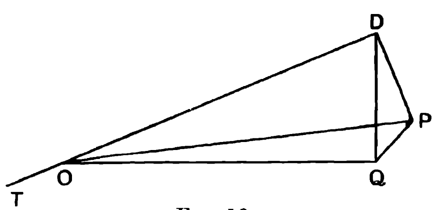

FIG. 16.

* The construction, for a surface in triple space, is due to J. Liouville; see his edition of Monge's *Applications de l'analyse à la géométrie*, (1850), p. 575.

Thus OD is at right angles to the plane DPQ at D; the plane ODP is the osculating plane of the geodesic, ODQ is the osculating plane of the curve. The line DQ is parallel to the prime normal of the curve; the line DP is parallel to the prime normal of the geodesic; and the line PQ is parallel to the binormal of the geodesic. Thus

$$2\rho_c \,.\, QD = OD^2, \quad 2\rho \,.\, PD = OD^2, \quad 2\gamma \,.\, QP = OP^2 = OD^2,$$

accurately up to the second order inclusive. Also, the angle QDP is ψ, and

$$PD = QD \cos\psi, \quad QP = QD \sin\psi;$$

and therefore

$$\rho \cos\psi = \rho_c, \quad \gamma \sin\psi = \rho_c.$$

Finally, taking the equal projections of the line QD and of the broken line QPD upon the spatial coordinate-axes, we have, as their equality upon the typical axis, the equation

$$Y_c \,.\, QD = Y \,.\, PD + l_3 \,.\, QP,$$

leading to the two relations

$$Y_c = Y \cos\psi + l_3 \sin\psi, \quad \frac{Y_c}{\rho_c} = \frac{Y}{\rho} + \frac{l_3}{\gamma},$$

which are the fundamental equations for the directions and the magnitudes of the circular curvature and the flexure.

151. Some preliminary lemmas are required for the evaluation, in relation to the surface, of second and higher differentiations along the curve, in so far as these are connected with the curvatures of the curve. For the purpose of curve-differentiation we shall use the symbol $\frac{d}{dt}$; so that, if f denote any function, we shall require the value of

$$\frac{df}{dt} - \frac{df}{ds},$$

the value being zero only when f is independent of any direction; and to obtain such value in all other cases, the formulæ

$$\left.\begin{aligned} \frac{d^2p}{dt^2} - \frac{d^2p}{ds^2} &= \kappa'\Theta(Hp' + Bq') \\ \frac{d^2q}{dt^2} - \frac{d^2q}{ds} &= -\kappa'\Theta(Ap' + Hq') \end{aligned}\right\},$$

will be used.

(i) We have

$$\frac{1}{\rho^2} = \mathbf{a}p'^4 + 4\mathbf{h}p'^3q' + 6\mathbf{k}p'^2q'^2 + 4\mathbf{f}p'q'^3 + \mathbf{c}q'^4,$$

$$\frac{1}{\rho}(\bar{A}p' + \bar{H}q') = \mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + 3\mathbf{k}p'q'^2 + \mathbf{f}q'^3,$$

$$\frac{1}{\rho}(\bar{H}p' + \bar{B}q') = \mathbf{h}p'^3 + 3\mathbf{k}p'^2q + 3\mathbf{f}p'q'^2 + \mathbf{c}q'^3;$$

and therefore

$$\frac{d}{dt}\left(\frac{1}{\rho^2}\right)-\frac{d}{ds}\left(\frac{1}{\rho^2}\right)=\frac{4}{\rho}\left\{(\bar{A}p'+\bar{H}q')\left(\frac{d^2p}{dt^2}-\frac{d^2p}{ds^2}\right)+(\bar{H}p'+\bar{B}q')\left(\frac{d^2q}{dt^2}-\frac{d^2q}{ds^2}\right)\right\}$$

$$=\frac{4}{\rho}\kappa'\Theta\{(\bar{A}p'+\bar{H}q')(Hp'+Bq')-(\bar{H}p'+\bar{B}q')(Ap'+Hq')\}$$

$$=4\kappa'\frac{V\Theta}{\rho\sigma},$$

by the expression (§ 106) for the torsion of a geodesic. Hence

$$\frac{d}{dt}\left(\frac{1}{\rho}\right)-\frac{d}{ds}\left(\frac{1}{\rho}\right)=2\kappa'\frac{V\Theta}{\sigma}=-\frac{2}{\gamma\sigma}.$$

(ii) Again, in the relation

$$\frac{Y}{\rho}=\eta_{11}p'^2+2\eta_{12}p'q'+\eta_{22}q'^2,$$

the quantities η_{11}, η_{12}, η_{22}, are functions of position only; and therefore

$$\frac{d}{dt}\left(\frac{Y}{\rho}\right)-\frac{d}{ds}\left(\frac{Y}{\rho}\right)=2\left\{(\eta_{11}p'+\eta_{12}q')\left(\frac{d^2p}{dt^2}-\frac{d^2p}{ds^2}\right)+(\eta_{12}p'+\eta_{22}q')\left(\frac{d^2q}{dt^2}-\frac{d^2q}{ds^2}\right)\right\}$$

$$=2\kappa'\Theta\{(\eta_{11}p'+\eta_{12}q')(Hp'+Bq')-(\eta_{12}p'+\eta_{22}q')(Ap'+Hq')\}.$$

But (§ 132) we have

$$\eta_{11}p'+\eta_{12}q'=Y(\bar{A}p'+\bar{H}q')-l_4\frac{Vq'}{\tau},$$

$$\eta_{12}p'+\eta_{22}q'=Y(\bar{H}p'+\bar{B}q')+l_4\frac{Vp'}{\tau};$$

and therefore

$$\begin{vmatrix}\eta_{11}p'+\eta_{12}q', & \eta_{12}p'+\eta_{22}q' \\ Ap'+Hq', & Hp'+Bq'\end{vmatrix}=V\begin{vmatrix}\bar{A}p'+\bar{H}q', & \bar{H}p'+\bar{B}q' \\ Ap'+Hq', & Hp'+Bq'\end{vmatrix}-l_4\frac{V}{\tau}$$

$$=V\left(\frac{Y}{\sigma}-\frac{l_4}{\tau}\right).$$

Consequently

$$\frac{d}{dt}\left(\frac{Y}{\rho}\right)-\frac{d}{ds}\left(\frac{Y}{\rho}\right)=2V\Theta\kappa'\left(\frac{Y}{\sigma}-\frac{l_4}{\tau}\right)=\frac{2}{\gamma}\left(\frac{l_4}{\tau}-\frac{Y}{\sigma}\right).$$

The left-hand side

$$=\frac{1}{\rho}\left(\frac{dY}{dt}-\frac{dY}{ds}\right)+Y\left\{\frac{d}{dt}\left(\frac{1}{\rho}\right)-\frac{d}{ds}\left(\frac{1}{\rho}\right)\right\}$$

$$=\frac{1}{\rho}\left(\frac{dY}{dt}-\frac{dY}{ds}\right)-\frac{2Y}{\gamma\sigma};$$

and therefore

$$\frac{dY}{dt} - \frac{dY}{ds} = 2\frac{\rho}{\gamma\tau} l_4,$$

so that, finally,

$$\frac{dY}{dt} = -\frac{y'}{\rho} + \frac{l_3}{\sigma} + 2\frac{\rho}{\gamma\tau} l_4.$$

(iii) Next, in the relation

$$Vl_3 = (Ap' + Hq')\frac{\partial y}{\partial q} - (Hp' + Bq')\frac{\partial y}{\partial p},$$

the quantities $V, A, H, B, \frac{\partial y}{\partial p}, \frac{\partial y}{\partial q}$, are functions of position only; hence, as before,

$$\begin{aligned}V\left(\frac{dl_3}{dt} - \frac{dl_3}{ds}\right) &= \left(A\frac{\partial y}{\partial q} - H\frac{\partial y}{\partial p}\right)\left(\frac{d^2p}{dt^2} - \frac{d^2p}{ds^2}\right) + \left(H\frac{\partial y}{\partial q} - B\frac{\partial y}{\partial p}\right)\left(\frac{d^2q}{dt^2} - \frac{d^2q}{ds^2}\right)\\ &= \kappa'\Theta\left\{\left(A\frac{\partial y}{\partial q} - H\frac{\partial y}{\partial p}\right)(Hp' + Bq') - \left(H\frac{\partial y}{\partial q} - B\frac{\partial y}{\partial p}\right)(Ap' + Hq')\right\}\\ &= \kappa'\Theta V^2 y',\end{aligned}$$

on reduction. Consequently

$$\frac{dl_3}{dt} - \frac{dl_3}{ds} = -\frac{y'}{\gamma};$$

and, after substitution from the Frenet equations,

$$\frac{dl_3}{dt} = -\frac{y'}{\gamma} - \frac{Y}{\sigma} + \frac{l_4}{\tau}.$$

(iv) With the denotations

$$u_1 = Ap' + Hq', \quad u_2 = Hp' + Bq',$$

where A, H, B, are functions of position only, we have

$$\begin{aligned}\frac{du_1}{dt} - \frac{du_1}{ds} &= A\left(\frac{d^2p}{dt^2} - \frac{d^2p}{ds^2}\right) + H\left(\frac{d^2q}{dt^2} - \frac{d^2q}{ds^2}\right)\\ &= \kappa' V^2\Theta q' = -\frac{V}{\gamma} q';\end{aligned}$$

and similarly

$$\frac{du_2}{dt} - \frac{du_2}{ds} = \frac{V}{\gamma} p'.$$

(v) Next, $\bar{A}, \bar{H}, \bar{B}$, are functions of direction as well as of position, given by relations

$$\frac{\bar{A}}{\rho} = \mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}q'^2,$$

where a, h, g, and other coefficients in $\bar{H}/\rho$ and $\bar{B}/\rho$ are functions of position only. Therefore

$$\frac{d}{dt}\left(\frac{\bar{A}}{\rho}\right)-\frac{d}{ds}\left(\frac{\bar{A}}{\rho}\right)=2\{(\mathbf{a}p'+\mathbf{h}q')\kappa'\Theta u_2-(\mathbf{h}p'+\mathbf{g}q')\kappa'\Theta u_1\}$$

$$=-2\kappa'\Theta\{u_1(\mathbf{h}p'+\mathbf{g}q')-u_2(\mathbf{a}p'+\mathbf{h}q')\};$$

also

$$\frac{d}{dt}\left(\frac{1}{\rho}\right)-\frac{d}{ds}\left(\frac{1}{\rho}\right)=-\frac{2}{\gamma\sigma};$$

hence

$$\frac{d\bar{A}}{dt}-\frac{d\bar{A}}{ds}=2\frac{\rho}{\gamma\sigma}\bar{A}+2\frac{\rho}{\gamma V}\{(\mathbf{h}p'+\mathbf{g}q')u_1-(\mathbf{a}p'+\mathbf{h}q')u_2\}.$$

Similarly

$$\frac{d\bar{H}}{dt}-\frac{d\bar{H}}{ds}=2\frac{\rho}{\gamma\sigma}\bar{H}+2\frac{\rho}{\gamma V}\{(\mathbf{b}p'+\mathbf{f}q')u_1-(\mathbf{h}p'+\mathbf{b}q')u_2\},$$

$$\frac{d\bar{B}}{dt}-\frac{d\bar{B}}{ds}=2\frac{\rho}{\gamma\sigma}\bar{B}+2\frac{\rho}{\gamma V}\{(\mathbf{f}p'+\mathbf{c}q')u_1-(\mathbf{g}p'+\mathbf{f}q')u_2\}.$$

These, by the result in § 134, *Ex.* 2, can also be expressed in the form

$$\frac{d\bar{A}}{dt}-\frac{d\bar{A}}{ds}=2\frac{\rho}{\gamma\tau}A_4,$$

$$\frac{d\bar{H}}{dt}-\frac{d\bar{H}}{ds}=2\frac{\rho}{\gamma\tau}H_4,$$

$$\frac{d\bar{B}}{dt}-\frac{d\bar{B}}{ds}=2\frac{\rho}{\gamma\tau}B_4.$$

(vi) With the denotations

$$v_1=\bar{A}p'+\bar{H}q',\quad v_2=\bar{H}p'+\bar{B}q',$$

we have

$$\frac{dv_1}{dt}-\frac{dv_1}{ds}=\bar{A}\left(\frac{d^2p}{dt^2}-\frac{d^2p}{ds^2}\right)+\bar{H}\left(\frac{d^2q}{dt^2}-\frac{d^2q}{ds^2}\right)+p'\left(\frac{d\bar{A}}{dt}-\frac{d\bar{A}}{ds}\right)+q'\left(\frac{d\bar{H}}{dt}-\frac{d\bar{H}}{ds}\right)$$

$$=\kappa'\Theta(\bar{A}u_2-\bar{H}u_1)+\frac{2\rho}{\gamma\tau}(A_4p'+H_4q')$$

$$=-\frac{1}{\gamma V}\left(\bar{A}u_2-\bar{H}u_1+2\frac{V^2\rho}{\tau^2}q'\right);$$

and similarly

$$\frac{dv_2}{dt}-\frac{dv_2}{ds}=-\frac{1}{\gamma V}\left(\bar{H}u_2-\bar{B}u_1-2\frac{V^2\rho}{\tau^2}p'\right).$$

(vii) The torsion of the geodesic is given (§ 106) by the relation

$$\frac{V}{\sigma}=v_1u_2-v_2u_1\,;$$

and therefore, V being a function of position only,

$$V\left\{\frac{d}{dt}\left(\frac{1}{\sigma}\right)-\frac{d}{ds}\left(\frac{1}{\sigma}\right)\right\}=v_1\left(\frac{du_2}{dt}-\frac{du_2}{ds}\right)-v_2\left(\frac{du_1}{dt}-\frac{du_1}{ds}\right)$$
$$+u_2\left(\frac{dv_1}{dt}-\frac{dv_1}{ds}\right)-u_1\left(\frac{dv_2}{dt}-\frac{dv_1}{ds_1}\right).$$

The first line on the right-hand side

$$=\frac{V}{\gamma}(v_1p'+v_2q')=\frac{V}{\gamma\rho}.$$

The second line on that right-hand side

$$=-\frac{1}{\gamma V}\left(\bar{A}u_2{}^2-2\bar{H}u_1u_2+\bar{B}u_1{}^2+2\frac{V^2\rho}{\tau^2}\right);$$

and

$$\bar{A}u_2{}^2-2\bar{H}u_1u_2+\bar{B}u_1{}^2=\bar{A}(B-V^2p'^2)-2\bar{H}(H+V^2p'q')+\bar{B}(A-V^2q'^2)$$
$$=\bar{A}B-2\bar{H}H+\bar{B}A-\frac{V^2}{\rho}.$$

Hence

$$V\left\{\frac{d}{dt}\left(\frac{1}{\sigma}\right)-\frac{d}{ds}\left(\frac{1}{\sigma}\right)\right\}=2\frac{V}{\gamma\rho}-\frac{1}{\gamma V}(\bar{A}B-2\bar{H}H+\bar{B}A)-2\frac{V\rho}{\gamma\tau^2},$$

that is,

$$\frac{d}{dt}\left(\frac{1}{\sigma}\right)-\frac{d}{ds}\left(\frac{1}{\sigma}\right)=\frac{2}{\gamma\rho}-2\frac{\rho}{\gamma\tau^2}-\frac{1}{\gamma V^2}(A\bar{B}-2H\bar{H}+B\bar{A}).$$

The value of the concomitant $A\bar{B}-2H\bar{H}+B\bar{A}$ is known (p. 367); but the form here stated for the right-hand side is convenient.

(viii) With the first set of values obtained in (v) for derivatives of $\bar{A}$, $\bar{H}$, $\bar{B}$, we have

$$\frac{d}{dt}(\bar{A}\bar{B}-\bar{H}^2)-\frac{d}{ds}(\bar{A}\bar{B}-\bar{H}^2)$$
$$=\frac{2\rho}{\gamma\sigma}2(\bar{A}\bar{B}-\bar{H}^2)+\frac{2\rho}{\gamma V}(u_1\eta_0-u_2\xi_0)$$
$$=\frac{4V^2\rho}{\gamma\sigma}\left(\frac{1}{\tau^2}+K\right)+\frac{2\rho}{\gamma V}(u_1\eta_0-u_2\xi_0),$$

where

$$\xi_0=\bar{B}(\mathrm{a}p'+\mathrm{h}q')-2\bar{H}(\mathrm{h}p'+\mathrm{b}q')+\bar{A}(\mathrm{g}p'+\mathrm{f}q'),$$
$$\eta_0=\bar{B}(\mathrm{a}p'+\mathrm{g}q')-2\bar{H}(\mathrm{b}p'+\mathrm{f}q')+\bar{A}(\mathrm{f}p'+\mathrm{c}q').$$

When the values of $\bar{A}$, $\bar{H}$, $\bar{B}$, are inserted in ξ_0 and η_0, we find

$$\begin{aligned}\frac{\xi_0}{\rho} &= (\mathbf{a}p' + \mathbf{h}q')(\mathbf{g}p'^2 + 2\mathbf{f}p'q' + \mathbf{c}q'^2) - 2(\mathbf{h}p' + \mathbf{b}q')(\mathbf{h}p'^2 + 2\mathbf{b}p'q' + \mathbf{f}q'^2) \\ &\qquad + (\mathbf{g}p' + \mathbf{f}q')(\mathbf{a}p'^2 + 2\mathbf{h}p'q' + \mathbf{g}q'^2) \\ &= p'^3(2\bar{\mathbf{c}} + 2\mathbf{a}V^2K) + p'^2q'(-3\bar{\mathbf{f}} + 6\mathbf{h}V^2K) \\ &\qquad + p'q'^2\{\bar{\mathbf{b}} + 2\bar{\mathbf{g}} + 2(\mathbf{g} + 2\mathbf{b})V^2K\} + q'^3(-\bar{\mathbf{h}} + 2\mathbf{f}V^2K) \\ &= 2\bar{\mathbf{c}}p'^3 - 3\bar{\mathbf{f}}p'^2q' + (\bar{\mathbf{b}} + 2\bar{\mathbf{g}})p'q'^2 - \bar{\mathbf{h}}q'^3 \\ &\qquad + 2V^2K(\mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + 3\mathbf{k}p'q'^2 + \mathbf{f}q'^3) \\ &= -\frac{V}{\rho^2\tau}\frac{A_4}{q'} + 2\frac{V^2K}{\rho}v_1,\end{aligned}$$

by the results in § 133. Similarly

$$\frac{\eta_0}{\rho} = \frac{V}{\rho^2\tau}\frac{B_4}{p'} + 2\frac{V^2K}{\rho}v_2.$$

Now

$$\frac{1}{q'}A_4u_2 + \frac{1}{p'}B_4u_1 = A_4B + H\left(A_4\frac{p'}{q'} + B_4\frac{q'}{p'}\right) + B_4A = A_4B - 2H_4H + B_4A,$$

because of the relation (§ 129)

$$A_4p'^2 + 2H_4p'q' + B_4q'^2 = 0\,;$$

and therefore

$$\frac{1}{\rho}(u_1\eta_0 - u_2\xi_0) = \frac{V}{\rho^2\tau}(A_4B - 2H_4H + B_4A) + 2\frac{V^2K}{\rho}(u_1v_2 - u_2v_1),$$

that is,

$$u_1\eta_0 - u_2\xi_0 = \frac{V}{\rho\tau}(A_4B - 2H_4H + B_4A) - 2\frac{V^3K}{\sigma}.$$

When this value of $u_1\eta_0 - u_2\xi_0$ is substituted, we have

$$\frac{d}{dt}(\bar{A}\bar{B} - \bar{H}^2) - \frac{d}{ds}(\bar{A}\bar{B} - \bar{H}^2) = \frac{4V^2\rho}{\gamma\sigma\tau^2} + \frac{2}{\gamma\tau}(A_4B - 2H_4H + B_4A).$$

But

$$\bar{A}\bar{B} - \bar{H}^2 = V^2\left(\frac{1}{\tau^2} + K\right),$$

and V, K, are functions of position only; hence

$$\frac{d}{dt}\left(\frac{1}{\tau}\right) - \frac{d}{ds}\left(\frac{1}{\tau}\right) = \frac{2\rho}{\gamma\sigma\tau} + \frac{1}{V^2\gamma}(A_4B - 2H_4H + B_4A).$$

Another form can be given to these expressions. With the second set of values obtained in (v) for the derivatives of $\bar{A}$, $\bar{H}$, $\bar{B}$, we have

$$\frac{d}{dt}(\bar{A}\bar{B} - \bar{H}^2) - \frac{d}{ds}(\bar{A}\bar{B} - \bar{H}^2) = \frac{2\rho}{\gamma\tau}(A_4\bar{B} - 2H_4\bar{H} + B_4\bar{A}) = \frac{2\rho^4}{V\gamma}\Phi,$$

where Φ is the cubicovariant in § 138, its expression being

$$\Phi = (\mathbf{a}^2\mathbf{f} - 3\mathbf{ahk} + 2\mathbf{h}^3)p'^6 + \dots ;$$

and then we have

$$\frac{d}{dt}\left(\frac{1}{\tau}\right) - \frac{d}{ds}\left(\frac{1}{\tau}\right) = \frac{\rho^4\tau}{V^3\gamma}\Phi.$$

When the two values of the derivative-differences of $\bar{A}\bar{B} - \bar{H}^2$ are compared, they require the relation

$$\frac{2V^2}{\sigma\tau} + \frac{1}{\rho}(A_4B - 2H_4H + B_4A) = A_4\bar{B} - 2H_4\bar{H} + B_4\bar{A},$$

which is verified by means of the relations

$$A_4p' + H_4q' = -V\frac{q'}{\tau}, \quad H_4p' + B_4q' = V\frac{p'}{\tau},$$

$$\frac{V}{\sigma} = (\bar{A}p' + \bar{H}q')(Hp' + Bq') - (\bar{H}p' + \bar{B}q')(Ap' + Hq'),$$

$$\frac{A}{\rho} - \bar{A} = 2(A\bar{H} - H\bar{A})p'q' + (A\bar{B} - B\bar{A})q'^2,$$

$$\frac{H}{\rho} - \bar{H} = (H\bar{A} - A\bar{H})p'^2 + (H\bar{B} - B\bar{H})q'^2,$$

$$\frac{B}{\rho} - \bar{B} = (B\bar{A} - A\bar{B})p'^2 + 2(B\bar{H} - H\bar{B})p'q'.$$

This relation, in a slightly modified form, occurs in § 142.

(ix) We have had, by (ii) and (vii),

$$\frac{dY}{dt} - \frac{dY}{ds} = \frac{2\rho}{\gamma\tau}l_4,$$

$$\frac{d}{dt}\left(\frac{1}{\sigma}\right) - \frac{d}{ds}\left(\frac{1}{\sigma}\right) = \frac{2}{\gamma\rho} - \frac{2\rho}{\gamma\tau^2} - \frac{1}{\gamma V^2}(A\bar{B} - 2H\bar{H} + B\bar{A}),$$

and therefore

$$\frac{d}{dt}\left(\frac{Y}{\sigma}\right) - \frac{d}{ds}\left(\frac{Y}{\sigma}\right) = \frac{2\rho}{\gamma\sigma\tau}l_4 + Y\left\{\frac{2}{\gamma\rho} - \frac{2\rho}{\gamma\tau^2} - \frac{1}{\gamma V^2}(A\bar{B} - 2H\bar{H} + B\bar{A})\right\}.$$

(x) Finally, for the present, there is the relation (§ 132)

$$\frac{l_4}{\tau} - \frac{Y}{\sigma} = -\frac{1}{V}(\xi_1u_2 - \xi_2u_1);$$

and therefore

$$\frac{d}{dt}\left(\frac{l_4}{\tau}\right)-\frac{d}{ds}\left(\frac{l_4}{\tau}\right)-\left\{\frac{d}{dt}\left(\frac{Y}{\sigma}\right)-\frac{d}{ds}\left(\frac{Y}{\sigma}\right)\right\}$$

$$=-\frac{1}{V}\left\{\xi_1\left(\frac{du_2}{dt}-\frac{du_2}{ds}\right)-\xi_2\left(\frac{du_1}{dt}-\frac{du_1}{ds}\right)\right\}$$

$$-\frac{1}{V}\left\{(\eta_{11}u_2-\eta_{12}u_1)\left(\frac{d^2p}{dt^2}-\frac{d^2p}{ds^2}\right)+(\eta_{12}u_2-\eta_{22}u_1)\left(\frac{d^2q}{dt^2}-\frac{d^2q}{ds^2}\right)\right\}.$$

The first line on the right-hand side

$$=-\frac{1}{V}\cdot\frac{V}{\gamma}(\xi_1p'+\xi_2q')=-\frac{Y}{\gamma\rho}.$$

The second line on that right-hand side

$$=\frac{1}{V^2\gamma}(\eta_{11}u_2{}^2-2\eta_{12}u_1u_2+\eta_{22}u_1{}^2)$$

$$=\frac{1}{V^2\gamma}\{\eta_{11}(B-V^2p'^2)-2\eta_{12}(H+V^2p'q')+\eta_{22}(A-V^2q'^2)\}$$

$$=\frac{1}{V^2\gamma}(\eta_{11}B-2\eta_{12}H+\eta_{22}A)-\frac{Y}{\gamma\rho}.$$

Hence, when we use the derivative-differences of Y/σ as given in (ix), we have

$$\frac{d}{dt}\left(\frac{l_4}{\tau}\right)-\frac{d}{ds}\left(\frac{l_4}{\tau}\right)$$

$$=\frac{2\rho}{\gamma\sigma\tau}l_4-\frac{2\rho}{\gamma\tau^2}Y+\frac{1}{V^2\gamma}\{B(\eta_{11}-Y\bar{A})-2H(\eta_{12}-Y\bar{H})+A(\eta_{22}-Y\bar{B})\}.$$

We had the relation

$$\frac{d}{dt}\left(\frac{1}{\tau}\right)-\frac{d}{ds}\left(\frac{1}{\tau}\right)=\frac{2\rho}{\gamma\sigma\tau}+\frac{1}{V^2\gamma}(A_4B-2H_4H+B_4A);$$

and therefore

$$\frac{1}{\tau}\left(\frac{dl_4}{dt}-\frac{dl_4}{ds}\right)+\frac{2\rho}{\gamma\tau^2}Y$$

$$=\frac{1}{V^2\gamma}\{B(\eta_{11}-Y\bar{A}-l_4A_4)-2H(\eta_{12}-Y\bar{H}-l_4H_4)+A(\eta_{22}-Y\bar{B}-l_4B_4)\}$$

$$=\frac{1}{V^2\gamma}lT(Bq'^2+2Hq'p'+Ap'^2)=\frac{T}{V^2\gamma}l,$$

by the relations on p. 359. Also, by § 133,

$$T=\frac{Y^{\frac{1}{2}}}{V}\rho\tau,$$

while the value of l, the typical direction-cosine of the line in the orthogonal flat of the surface perpendicular to the prime normal and the trinormal, is given on p. 359. Hence, we have

$$\frac{dl_4}{dt}-\frac{dl_4}{ds}=-\frac{2\rho}{\gamma\tau}Y+\frac{\rho\tau^2 Y^{\frac{1}{2}}}{\gamma V^3}l.$$

Binormal of a curve: the spatial torsion.

152. The binormal of a curve is at right angles to its tangent and its prime normal; hence the binormal of the curve $\theta(p, q)=0$ is at right angles to the line with a typical direction-cosine y' and the line with a typical direction-cosine Y_c, where

$$Y_c = Y\cos\psi + l_3 \sin\psi.$$

The direction with a typical direction-cosine L_3, where

$$L_3 = l_3\cos\psi - Y\sin\psi,$$

is at right angles to the prime normal of the curve; also, because it lies in the plane determined by the prime normal and the binormal of the geodesic, this L_3-direction is at right angles to the tangent of the curve. Now an orthogonal frame in the plenary space is constituted by the organic lines of the geodesic, with typical direction-cosines y', Y, l_3, l_4, l_5, ...; and thus another orthogonal frame in the plenary space is constituted by lines with typical direction-cosines y', Y_c, L_3, l_4, l_5, ... Let λ_3 denote a typical direction-cosine of the binormal to the curve, which is a direction in the frame; hence there is a typical relation

$$\lambda_3 = \mu_1 y' + \mu_2 Y_c + \mu_3 L_3 + \mu_4 l_4 + \mu_5 l_5 + \dots .$$

Also we have, for the curve,

$$\sum \lambda_3 y' = 0, \quad \sum \lambda_3 Y_c = 0,$$

so that $\mu_1=0$, $\mu_2=0$; the remaining quantities μ are to be determined.

Let $1/\sigma_c$ denote the torsion of the curve, the arc of which will continue to be denoted by dt; then, by the Frenet equations for the orthogonal frame of the curve, we have

$$\frac{dy'}{dt}=\frac{Y_c}{\rho_c}, \quad \frac{dY_c}{dt}=\frac{\lambda_3}{\sigma_c}-\frac{y'}{\rho_c}.$$

With the foregoing value of Y_c, we have

$$\frac{dY_c}{dt}=\frac{dY}{dt}\cos\psi+\frac{dl_3}{dt}\sin\psi+(l_3\cos\psi - Y\sin\psi)\frac{d\psi}{dt}.$$

But (pp. 422, 423)

$$\frac{dY}{dt}=\frac{dY}{ds}+2\frac{\rho}{\gamma\tau}l_4=-\frac{y'}{\rho}+\frac{l_3}{\sigma}+2\frac{\rho}{\gamma\tau}l_4,$$

$$\frac{dl_3}{dt}=\frac{dl_3}{ds}-\frac{y'}{\gamma}=-\frac{y'}{\gamma}-\frac{Y}{\sigma}+\frac{l_4}{\tau};$$

and also

$$\frac{\cos\psi}{\rho}+\frac{\sin\psi}{\gamma}=\frac{1}{\rho_c},\quad \frac{\rho}{\gamma}\cos\psi=\sin\psi\,;$$

therefore

$$\frac{dY_c}{dt}=-\frac{y'}{\rho_c}+L_3\left(\frac{1}{\sigma}+\frac{d\psi}{dt}\right)+3\frac{l_4}{\tau}\sin\psi.$$

Equating this value to the value from the Frenet equations, we have

$$\frac{\lambda_3}{\sigma_c}=L_3\left(\frac{1}{\sigma}+\frac{d\psi}{dt}\right)+3\frac{l_4}{\tau}\sin\psi.$$

Consequently, when λ_3 is expressed in the form

$$\lambda_3=\mu_3L_3+\mu_4l_4+\mu_5l_5+\dots,$$

we have

$$\mu_5=0,\quad \mu_6=0,\ \dots,$$

$$\frac{\mu_3}{\sigma_c}=\frac{1}{\sigma}+\frac{d\psi}{dt},\quad \frac{\mu_4}{\sigma_c}=\frac{3}{\tau}\sin\psi.$$

The binormal of the curve therefore lies in the plane, which is determined by the trinormal of the geodesic tangent and the line having L_3 for its typical direction-cosine. Let χ denote the angle between the binormal of the curve and this L_3-direction; we can take $\mu_3=\cos\chi$, $\mu_4=\sin\chi$. Thus the equations for the direction of the binormal of the curve and for its torsion become

$$\lambda_3=(l_3\cos\psi-Y\sin\psi)\cos\chi+l_4\sin\chi,$$

$$\frac{\cos\chi}{\sigma_c}=\frac{1}{\sigma}+\frac{d\psi}{dt},\quad \frac{\sin\chi}{\sigma_c}=\frac{3}{\tau}\sin\psi.$$

Obviously

$$\tan\chi=\frac{\frac{3}{\tau}\sin\psi}{\frac{1}{\sigma}+\frac{d\psi}{dt}}=\frac{3\frac{\rho_c}{\gamma\tau}}{\frac{1}{\sigma}+\frac{d\psi}{dt}},$$

$$\frac{1}{\sigma_c^2}=\left(\frac{1}{\sigma}+\frac{d\psi}{dt}\right)^2+9\frac{\rho_c^2}{\gamma^2\tau^2}.$$

The equations involve the curve-derivative of ψ. Because

$$\tan\psi=\frac{\rho}{\gamma},$$

it follows that

$$\left(\frac{1}{\rho^2}+\frac{1}{\gamma^2}\right)\frac{d\psi}{dt}=\frac{1}{\rho}\frac{d}{dt}\left(\frac{1}{\gamma}\right)-\frac{1}{\gamma}\frac{d}{dt}\left(\frac{1}{\rho}\right),$$

that is,

$$\frac{1}{\rho_c^2}\frac{d\psi}{dt}=\frac{1}{\rho}\frac{d}{dt}\left(\frac{1}{\gamma}\right)-\frac{1}{\gamma}\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-\frac{2}{\gamma\sigma}\right\}.$$

It is to be noted that, when the plenary space is triple, there is no tilt, so that analytically $\chi=0$; and the equation for the torsion of the curve becomes

$$\frac{1}{\sigma_c}=\frac{1}{\sigma}+\frac{d\psi}{dt},$$

a known result *.

For a general plenary space, the positions of the prime normal and the binormal of the curve, relative to the orthogonal frame of the geodesic tangent, can be indicated by a diagram on a spherical surface, of radius unity, and centre O. Consider the flat the leading lines of which are the prime normal OY of the geodesic, the binormal OB of the geodesic, and the trinormal OT of the geodesic; the tangent, of the curve and the geodesic, is orthogonal to this flat. Let the direction typified by L_3 be OB', which must lie on the great circle YB because

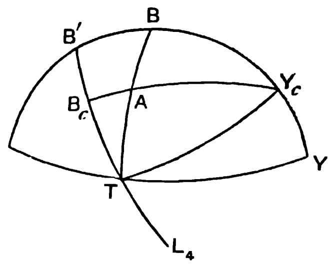

FIG. 17.

$$L_3=l_3\cos\psi-Y\sin\psi.$$

Because of the earlier result

$$Y_c=Y\cos\psi+l_3\sin\psi,$$

the prime normal of the curve lies in the plane OYB and makes an angle ψ with OY; accordingly, if $YY_c=\psi$, then OY_c is the prime normal of the curve. Also, as the L_3-direction is at right angles to this prime normal, the arc Y_cB' is $\frac{1}{2}\pi$; and therefore $BB'=\psi$. Also, the binormal of the curve (typified by λ_3) lies in the plane TOB', which contains OB' (typified by L_3) and OT (typified by l_4). We take $B'B_c=\chi$; and now OB_c is the binormal of the curve.

The inclination of the curve-binormal to the geodesic-binormal is BB_c; so that, as the angle at B' is a right angle,

$$\cos BB_c=\cos BB'\cos B'B_c=\cos\psi\cos\chi,$$

a result in accordance with the analysis, which gives

$$\cos BB_c=\sum l_3\lambda_3=(\sum l_3L_3)\cos\chi=\cos\psi\cos\chi.$$

The inclination of the plane OB_cY_c, through the prime normal and the binormal of the curve, to the plane OBY, through the prime normal and the binormal of the geodesic, is the angle B_cY_cB; being measured by the arc $B'B_c$, because Y_c is the pole of $B'T$, this inclination is χ.

* *Lectures on Differential Geometry*, § 106, where the angle corresponding to ψ is measured in the opposite sense to ψ.

Let ω denote the inclination of the same plane OB_cY_c to the plane OBT, through the binormal and the trinormal of the geodesic, the intersection of these planes being OA, so that

$$\omega = TAB_c\,;$$

then, as the angle AB_cT is a right angle, while $ATB_c=\psi$, we have

$$\cos\omega = \sin\psi \sin\chi.$$

It is easy to verify the results :

$$\frac{\sin AT}{\cos\chi} = \frac{\cos AT}{\sin\chi\cos\psi} = -\frac{\sin AY_c}{\cos\psi} = \frac{\cos AY_c}{\sin\psi\cos\chi} = -\frac{1}{\sin\omega}.$$

Finally, if $B'T$ is produced to L_4 so that $TL_4=B'B_c=\chi$, the typical direction-cosine of OL_4 is

$$L_4 = l_4\cos\chi - L_3\sin\chi.$$

Trinormal of the curve : the spatial tilt.

153. The trinormal of the curve $\theta=0$, being at right angles to the tangent, the prime normal, and the binormal of the curve, is at right angles to the lines which have y', Y_c, λ_3, for typical directions, where

$$Y_c = Y\cos\psi + l_3\sin\psi, \quad L_3 = l_3\cos\psi - Y\sin\psi, \quad \lambda_3 = L_3\cos\chi + l_4\sin\chi.$$

There is an associated direction with a typical direction-cosine L_4, where

$$L_4 = l_4\cos\chi - L_3\sin\chi.$$

This associated direction, in the plane through the trinormal of the geodesic tangent and the L_3-direction, is perpendicular to all lines to which the trinormal and the L_3-direction are themselves perpendicular. Among such lines are (i), the tangent and the prime normal of the curve with typical direction-cosines y' and Y_c, and (ii), all the principal lines of the geodesic tangent of rank subsequent to the trinormal. Also, we have

$$\sum \lambda_3 L_4 = 0.$$

Thus the directions λ_3 and L_4, perpendicular to one another, can take the place of directions l_4 and L_3 in the orthogonal frame of a geodesic and still leave an orthogonal frame; and so there is an orthogonal frame, with lines having y', Y_c, λ_3, L_4, l_5, l_6, ... as their typical direction-cosines.

Let the typical direction-cosine of the trinormal of the curve $\theta=0$ be denoted by λ_4; it is a direction in any orthogonal frame of the plenary space, and therefore there are relations typified by

$$\lambda_4 = \eta_1 y' + \eta_2 Y_c + \eta_3\lambda_3 + \eta_4 L_4 + \eta_5 l_5 + \eta_6 l_6 + \dots .$$

But because it is the trinormal of the curve, it is orthogonal to the flat determined by y', Y_c, λ_3, and therefore

$$\lambda_4 = \eta_4 L_4 + \eta_5 l_5 + \eta_6 l_6 + \dots,$$

where the quantities η_4, η_5, η_6, ... remain for determination.

The Frenet equations, giving the direction of the trinormal and the magnitude of the tilt of the curve, are typified by

$$\frac{d\lambda_3}{dt} = \frac{\lambda_4}{\tau_c} - \frac{Y_c}{\sigma_c},$$

where $1/\tau_c$ denotes the tilt of the curve. The foregoing value of λ_3 is

$$\lambda_3 = L_3 \cos\chi + l_4 \sin\chi;$$

and therefore

$$\begin{aligned}\frac{d\lambda_3}{dt} &= (l_4 \cos\chi - L_3 \sin\chi)\frac{d\chi}{dt} + \frac{dL_3}{dt}\cos\chi + \frac{dl_4}{dt}\sin\chi \\ &= L_4 \frac{d\chi}{dt} + \frac{dL_3}{dt}\cos\chi + \frac{dl_4}{dt}\sin\chi.\end{aligned}$$

Now

$$\begin{aligned}\frac{dL_3}{dt} &= (-L_3 \sin\psi - Y\cos\psi)\frac{d\psi}{dt} + \frac{dl_3}{dt}\cos\psi - \frac{dY}{dt}\sin\psi \\ &= -Y_c \frac{d\psi}{dt} + \left(\frac{dl_3}{ds} - \frac{y'}{\gamma}\right)\cos\psi - \left(\frac{dY}{ds} + 2\frac{\rho}{\gamma\tau}l_4\right)\sin\psi \\ &= -Y_c \frac{d\psi}{dt} + \left(\frac{l_4}{\tau} - \frac{Y}{\sigma} - \frac{y'}{\gamma}\right)\cos\psi - \left(\frac{l_3}{\sigma} - \frac{y'}{\rho} + 2\frac{\rho}{\gamma\tau}l_4\right)\sin\psi.\end{aligned}$$

On the right-hand side, the total coefficient of $-\dfrac{1}{\sigma}$

$$= Y\cos\psi + l_3 \sin\psi = Y_c;$$

and the total coefficient of y'

$$= -\frac{\cos\psi}{\gamma} + \frac{\sin\psi}{\rho} = 0;$$

therefore

$$\begin{aligned}\frac{dL_3}{dt} &= -Y_c\left(\frac{1}{\sigma} + \frac{d\psi}{dt}\right) + l_4 \frac{1}{\tau}\left(\cos\psi - 2\frac{\rho}{\gamma}\sin\psi\right) \\ &= -Y_c\left(\frac{1}{\sigma} + \frac{d\psi}{dt}\right) + l_4 \frac{1 - 3\sin^2\psi}{\tau\cos\psi}.\end{aligned}$$

Also, by (x) in § 151,

$$\frac{dl_4}{dt} = \frac{dl_4}{ds} - \frac{2\rho}{\gamma\tau}Y + \frac{\rho\tau^2 Y^{\frac{1}{2}}}{\gamma V^3}l = \frac{l_5}{\kappa} - \frac{l_3}{\tau} - 2\frac{\rho}{\gamma\tau}Y + \frac{\rho\tau^2 Y^{\frac{1}{2}}}{\gamma V^3}l.$$

Accordingly

$$\frac{d\lambda_3}{dt}=L_4\frac{d\chi}{dt}+\left(\frac{l_5}{\kappa}+\frac{\rho\tau^2}{\gamma V^3}Y^{\frac{1}{2}}l\right)\sin\chi$$
$$+\cos\chi\left\{-Y_c\left(\frac{1}{\sigma}+\frac{d\psi}{dt}\right)+l_4\frac{1-3\sin^2\psi}{\tau\cos\psi}\right\}-\left(\frac{l_3}{\tau}+2\frac{\rho}{\gamma\tau}Y\right)\sin\chi.$$

By using the formulæ

$$\frac{\cos\chi}{\sigma_c}=\frac{1}{\sigma}+\frac{d\psi}{dt},\quad \frac{\sin\chi}{\sigma_c}=\frac{3}{\tau}\sin\psi,\quad \frac{\rho}{\gamma}=\tan\psi,$$
$$L_4=l_4\cos\chi-L_3\sin\chi$$
$$=l_4\cos\chi-(l_3\cos\psi-Y\sin\psi)\sin\chi,$$

the second line in the last expression can be transformed to

$$L_4\frac{1-3\sin^2\psi}{\tau\cos\psi}-\frac{Y_c}{\sigma_c};$$

and therefore

$$\frac{d\lambda_3}{dt}=L_4\left(\frac{d\chi}{dt}+\frac{1-3\sin^2\psi}{\tau\cos\psi}\right)+\left(\frac{l_5}{\kappa}+\frac{\rho\tau^2}{\gamma V^3}Y^{\frac{1}{2}}l\right)\sin\chi-\frac{Y_c}{\sigma_c}.$$

When this value of $\frac{d\lambda_3}{dt}$ is compared with the value given by the cited Frenet equation, we have

$$\frac{\lambda_4}{\tau_c}=L_4\left(\frac{d\chi}{dt}+\frac{1-3\sin^2\psi}{\tau\cos\psi}\right)+\left(\frac{l_5}{\kappa}+\frac{\rho\tau^2}{\gamma V^3}Y^{\frac{1}{2}}l\right)\sin\chi$$
$$=L_4\left(\frac{d\chi}{dt}+\frac{1-3\sin^2\psi}{\tau\cos\psi}\right)+\left(\frac{l_5}{\kappa}+\frac{\tau}{\gamma V^2}Tl\right)\sin\chi$$
$$=L_4\left(\frac{\cos\psi}{\tau}-2\frac{\rho}{\gamma\tau}\sin\psi+\frac{d\chi}{dt}\right)+\left(\frac{l_5}{\kappa}+\frac{\tau}{\gamma V^2}Tl\right)\sin\chi,$$

where T is a covariant already known (§ 133).

The necessary relations

$$\sum\lambda_4 Y_c=0,\quad \sum\lambda_4\lambda_3=0$$

are at once verified.

We take a magnitude U and a typical direction-cosine λ, such that

$$\frac{l_5}{\kappa}+l\frac{\tau T}{\gamma V^2}=\lambda U,$$

this new direction lying in a plane having, as its leading lines, (i) the quartinormal of the geodesic tangent of the curve and (ii) the line in the orthogonal flat of that

geodesic which is at right angles to the prime normal and the trinormal of the geodesic. Regarding these lines as oblique axes in the plane, we have

$$U^2 = \frac{1}{\kappa^2} + 2\,\frac{\tau T}{\gamma\kappa V^2}\cos\omega + \frac{\tau^2 T^2}{\gamma^2 V^4},$$

where ω, the angle between the lines, is given by

$$\cos\omega = \sum l l_5.$$

Take a sphere, centre O, in a flat, which contains the foregoing plane and the line OD having the quantity L_4 for its typical direction-cosine; let OQ be the direction of the quartinormal l_5, and OL be the direction of the line with typical direction-cosine l; then

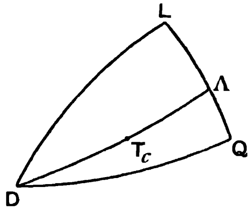

FIG. 18.

$$QL = \omega.$$

Now

$$\cos DQ = \sum L_4 l_5 = \sum l_5 \{l_4 \cos\chi - (l_3 \cos\psi - Y \sin\psi)\sin\chi\} = 0,$$
$$\cos DL = \sum L_4 l \;= \sum l\;\{l_4 \cos\chi - (l\;\cos\psi - Y\sin\psi)\sin\chi\} = 0,$$

so that, on the spherical surface, D is the pole of the arc QL. The line $O\Lambda$, with λ for its typical direction-cosine, then meets QL in some point Λ; and, if $Q\Lambda = \omega_1$, $\Lambda L = \omega_2$, so that $\omega_1 + \omega_2 = \omega$, we have

$$U\cos\omega_1 = U\sum \lambda l_5 = \frac{1}{\kappa} + \frac{\tau T}{\gamma V^2}\cos\omega,$$

$$U\cos\omega_2 = U\sum \lambda l \;= \frac{1}{\kappa}\cos\omega + \frac{\tau T}{\gamma V^2}.$$

The equation for the trinormal of the curve now is

$$\frac{\lambda_4}{\tau_c} = L_4\left(\frac{\cos\psi}{\tau} - 2\,\frac{\rho}{\gamma\tau}\sin\psi + \frac{d\chi}{dt}\right) + \lambda U \sin\chi.$$

The line $O\Lambda$, in the plane OQL, is perpendicular to the line OD; and therefore we can take

$$\lambda_4 = L_4\cos\phi + \lambda\sin\phi,$$

$$\frac{\cos\phi}{\tau_c} = \frac{\cos\psi}{\tau} - 2\,\frac{\rho}{\gamma\tau}\sin\psi + \frac{d\chi}{dt},$$

$$\frac{\sin\phi}{\tau_c} = U\sin\chi,$$

which determine ϕ, and give

$$\frac{1}{\tau_c{}^2} = \left(\frac{\cos\psi}{\tau} - 2\,\frac{\rho}{\gamma\tau}\sin\psi + \frac{d\chi}{dt}\right)^2 + \left(\frac{1}{\kappa^2} + 2\,\frac{\tau T}{\gamma\kappa V^2}\cos\omega + \frac{\tau^2 T^2}{\gamma^2 V^2}\right)\sin^2\chi.$$

Finally, if on the arc $D\Lambda$ we take a point T_c such that

$$DT_c = \phi, \quad T_c\Lambda = \tfrac{1}{2}\pi - \phi,$$

then OT_c is the direction of the trinormal of the curve and the quantity $1/\tau_c$ is the magnitude of its tilt; and the inclination of the trinormal of the curve to the trinormal of the geodesic tangent is

$$\cos^{-1}(\sum l_4\lambda_4)=\cos^{-1}(\cos\phi\cos\chi).$$

The last result follows also from the fact that, if we take a globe with OT (from Figure 17, p. 431) and OL, OD, OQ, as radii, so that OL, OT, OQ are orthogonal, the arc $DT=\chi$ and the arc $DT_c=\phi$.

Superficial normal to the curve.

154. Thus far, in the discussion of the curve $\theta(p, q)=0$, account has been taken solely of its successive curvatures; and the only arc-directions considered were along its course. But it can have other relations within the range of the surface in which it lies, such relations implying surface-deviations from the curve. Thus the curve might be a member of a family, distinguished among themselves by a parameter: the simplest form occurs as an equation

$$\theta(p, q)=\theta,$$

where, on the right-hand side, θ is a parameter unvarying along the curve and therefore not entering into the foregoing relations, but varying from one curve of the family to another.

We require variations of such a parameter within the range of the surface: and, in particular, we consider directions which, lying in the surface, are normal to the curve. We denote the surface-direction at O, which is at right angles to the curve, by the direction-variables $\frac{dp}{dn}, \frac{dq}{dn}$, where dn is taken to be an element of arc dn in that normal direction, so that

$$A\left(\frac{dp}{dn}\right)^2+2H\frac{dp}{dn}\frac{dq}{dn}+B\left(\frac{dq}{dn}\right)^2=1.$$

For repeated arc-differentiation along the curve, we shall continue to use dt as the element of arc, the distinction from ds arising in second differentiations; thus

$$\frac{dp}{dt}=\frac{dp}{ds}=p', \quad \frac{dq}{dt}=\frac{dq}{ds}=q',$$

while

$$\frac{d^2p}{dt^2}-\frac{d^2p}{ds^2}, \quad \frac{d^2q}{dt^2}-\frac{d^2q}{ds^2},$$

are different from zero.

The perpendicularity of the two directions is secured analytically by the relation

$$(Ap'+Hq')\frac{dp}{dn}+(Hp'+Bq')\frac{dq}{dn}=0.$$

Hence

$$\frac{\dfrac{dp}{dn}}{-u_2}=\frac{\dfrac{dq}{dn}}{u_1}=\lambda,$$

where the common value of the two fractions is given by the equation

$$\frac{1}{\lambda^2}=Au_2^2-2Hu_1u_2+Bu_1^2=V^2.$$

Also, we had

$$\theta_1p'+\theta_2q'=0,$$

so that

$$\frac{p'}{\theta_2}=\frac{q'}{-\theta_1}=\kappa',$$

where

$$\frac{1}{\kappa'^2}=A\theta_2^2-2H\theta_2\theta_1+B\theta_1^2.$$

Thus

$$V\frac{dp}{dn}=-u_2=-(Hp'+Bq')=-\kappa'(H\theta_2-B\theta_1),$$

$$V\frac{dq}{dn}=\quad u_1=\quad Ap'+Hq'\ =\ \kappa'(A\theta_2-H\theta_1).$$

We term the magnitude $\dfrac{d\theta}{dn}$, the limiting value of the ratio of the parametric variation $d\theta$ to the rudimentary normal dn, the *dilatation* of the curve, and we denote it by θ_n; thus

$$\begin{aligned}\theta_n&=\theta_1\frac{dp}{dn}+\theta_2\frac{dq}{dn}\\&=\frac{1}{V}\kappa'(A\theta_2^2-2H\theta_2\theta_1+B\theta_1^2)=\frac{1}{V\kappa'}.\end{aligned}$$

The equations become

$$\left.\begin{aligned}V\theta_n\frac{dp}{ds}&=\theta_2\\V\theta_n\frac{dq}{ds}&=-\theta_1\end{aligned}\right\},\quad\left.\begin{aligned}V^2\theta_n\frac{dp}{dn}&=-H\theta_2+B\theta_1\\V^2\theta_n\frac{dq}{dn}&=A\theta_2-H\theta_1\end{aligned}\right\},$$

$$V^2\theta_n^2=A\theta_2^2-2H\theta_2\theta_1+B\theta_1^2;$$

and we have, from these values, as also from the fact that the directions on the surface are at right angles,

$$\frac{dp}{ds}\frac{dq}{dn}-\frac{dq}{ds}\frac{dp}{dn}=\frac{1}{V}.$$

Also the equations (§ 149) for the second variations of p and q along the curve now become

$$\left.\begin{aligned}\frac{d^2p}{dt^2}+\Gamma_{11}\left(\frac{dp}{ds}\right)^2+2\Gamma_{12}\frac{dp}{ds}\frac{dq}{ds}+\Gamma_{22}\left(\frac{dq}{ds}\right)^2=\frac{1}{\gamma}\frac{dp}{dn}\\ \frac{d^2q}{dt^2}+\Delta_{11}\left(\frac{dq}{ds}\right)^2+2\Delta_{12}\frac{dp}{ds}\frac{dq}{ds}+\Delta_{22}\left(\frac{dq}{ds}\right)^2=\frac{1}{\gamma}\frac{dq}{dn}\end{aligned}\right\},$$

while the radius of flexure γ is given by each of the forms

$$-\frac{\theta_n}{\gamma}=\vartheta_{11}p'^2+2\vartheta_{12}p'q'+\vartheta_{22}q'^2,$$

$$-\frac{V^2\theta_n{}^3}{\gamma}=\vartheta_{11}\theta_2{}^2-2\vartheta_{12}\theta_2\theta_1+\vartheta_{22}\theta_1{}^2.$$

We note, in passing, that if the curve $\theta=0$ is a geodesic of the surface, the function θ must satisfy the relation

$$(\theta_{11}-\theta_1\Gamma_{11}-\theta_2\Delta_{11})\theta_2{}^2-2(\theta_{12}-\theta_1\Gamma_{12}-\theta_2\Delta_{12})\theta_2\theta_1+(\theta_{22}-\theta_1\Gamma_{22}-\theta_2\Delta_{22})\theta_1{}^2=0,$$

which accordingly is the partial differential equation of all geodesics on the surface. Also that, if $f(p, q)$ be any function of position alone on the surface (that is, a function independent of directions at a point), then

$$\frac{df}{ds}=\frac{1}{V\theta_n}\left(\theta_2\frac{\partial f}{\partial p}-\theta_1\frac{\partial f}{\partial q}\right),$$

$$\frac{df}{dn}=\frac{1}{V^2\theta_n}\left\{(-H\theta_2+B\theta_1)\frac{\partial f}{\partial p}+(A\theta_2-H\theta_1)\frac{\partial f}{\partial q}\right\}.$$

Thus

$$V\theta_n\frac{dV}{dn}=(\Gamma_{11}+\Delta_{12})(-H\theta_2+B\theta_1)+(\Gamma_{12}+\Delta_{22})(A\theta_2-H\theta_1),$$

$$V^2\theta_n\frac{dA}{dn}=2(A\Gamma_{11}+H\Delta_{11})(-H\theta_2+B\theta_1)+2(A\Gamma_{12}+H\Delta_{12})(A\theta_2-H\theta_1),$$

and similarly for variations of H, B, along the normal to the curve.

Proceeding from the equation

$$V^2\theta_n{}^2=A\theta_2{}^2-2H\theta_2\theta_1+B\theta_1{}^2,$$

and differentiating along the curve, we have

$$\begin{aligned}2V^2\theta_n\frac{d\theta_n}{ds}&+2V^2\theta_n{}^2\{(\Gamma_{11}+\Delta_{12})p'+(\Gamma_{12}+\Delta_{22})q'\}\\ &=2(A\theta_2-H\theta_1)(\theta_{12}p'+\theta_{22}q')+2(-H\theta_2+B\theta_1)(\theta_{11}p'+\theta_{12}q')\\ &\qquad+\theta_2{}^2\frac{dA}{ds}-2\theta_2\theta_1\frac{dH}{ds}+\theta_1{}^2\frac{dB}{ds}.\end{aligned}$$

The first line on the right-hand side

$$=2(A\theta_2-H\theta_1)(\vartheta_{12}p'+\vartheta_{22}q')+2(-H\theta_2+B\theta_1)(\vartheta_{11}p'+\vartheta_{12}q')$$
$$+2(A\theta_2-H\theta_1)\{(\theta_1\Gamma_{12}+\theta_2\Delta_{12})p'+(\theta_1\Gamma_{22}+\theta_2\Delta_{22})q'\}$$
$$+2(-H\theta_2+B\theta_1)\{(\theta_1\Gamma_{11}+\theta_2\Delta_{11})p'+(\theta_1\Gamma_{12}+\theta_2\Delta_{12})q'\}.$$

When this value is used, when the values of A', H', B', are inserted, and when reduction is effected, we have

$$V^2\theta_n\frac{d\theta_n}{ds}=(-H\theta_2+B\theta_1)(\vartheta_{11}p'+\vartheta_{12}q')+(A\theta_2-H\theta_1)(\vartheta_{12}p'+\vartheta_{22}q');$$

and therefore

$$\frac{d\theta_n}{ds}=\vartheta_{11}p'\frac{dp}{dn}+\vartheta_{12}\left(p'\frac{dq}{dn}+q'\frac{dp}{dn}\right)+\vartheta_{22}q'\frac{dq}{dn},$$

providing a covariant in the system of concomitants, together with its geometrical significance.

Proceeding from the same equation for $V^2\theta_n^2$, and differentiating along the normal to the curve, we have

$$2V^2\theta_n\frac{d^2\theta}{dn^2}+2\theta_n\{(\Gamma_{11}+\Delta_{12})(-H\theta_2+B\theta_1)+(\Gamma_{12}+\Delta_{22})(A\theta_2-H\theta_1)\}$$
$$=2(A\theta_2-H\theta_1)\left(\theta_{12}\frac{dp}{dn}+\theta_{22}\frac{dq}{dn}\right)+2(-H\theta_2+B\theta_1)\left(\theta_{11}\frac{dp}{dn}+\theta_{12}\frac{dq}{dn}\right)$$
$$+\theta_2^2\frac{dA}{dn}-2\theta_2\theta_1\frac{dH}{dn}+\theta_1^2\frac{dB}{dn}.$$

The first line on the right-hand side

$$=2V^2\theta_n\left\{\theta_{11}\left(\frac{dp}{dn}\right)^2+2\theta_{12}\frac{dp}{dn}\frac{dq}{dn}+\theta_{22}\left(\frac{dq}{dn}\right)^2\right\}$$
$$=2V^2\theta_n\left\{\vartheta_{11}\left(\frac{dp}{dn}\right)^2+2\vartheta_{12}\frac{dp}{dn}\frac{dq}{dn}+\vartheta_{22}\left(\frac{dq}{dn}\right)^2\right\}$$
$$+2V^2\theta_n\left\{(\theta_1\Gamma_{11}+\theta_2\Delta_{11})\left(\frac{dp}{dn}\right)^2+2(\theta_1\Gamma_{12}+\theta_2\Delta_{12})\frac{dp}{dn}\frac{dq}{dn}+(\theta_1\Gamma_{22}+\theta_2\Delta_{22})\left(\frac{dq}{dn}\right)^2\right\}.$$

When this value is used, when the values of the normal-derivatives of A, H, B, are inserted, and when reduction is effected, we find

$$\frac{d^2\theta}{dn^2}=\vartheta_{11}\left(\frac{dp}{dn}\right)^2+2\vartheta_{12}\frac{dp}{dn}\frac{dq}{dn}+\vartheta_{22}\left(\frac{dq}{dn}\right)^2.$$

We thus have three magnitudes,

$$-\frac{\theta_n}{\gamma},\quad\frac{d}{ds}\left(\frac{d\theta}{dn}\right),\quad\frac{d^2\theta}{dn^2},$$

which are linear in the quantities $\vartheta_{11}, \vartheta_{12}, \vartheta_{22}$; and we find

$$\left.\begin{aligned}
\frac{1}{V^2}\vartheta_{11} &= \left(\frac{dq}{ds}\right)^2 \frac{d^2\theta}{dn^2} - 2\frac{dq}{ds}\frac{dq}{dn}\frac{d\theta_n}{ds} + \left(\frac{dq}{dn}\right)^2\left(-\frac{\theta_n}{\gamma}\right)\\
\frac{1}{V^2}\vartheta_{12} &= -\frac{dq}{ds}\frac{dp}{ds}\frac{d^2\theta}{dn^2} + \left(\frac{dp}{ds}\frac{dq}{dn} + \frac{dq}{ds}\frac{dp}{dn}\right)\frac{d\theta_n}{ds} - \frac{dq}{dn}\frac{dp}{dn}\left(-\frac{\theta_n}{\gamma}\right)\\
\frac{1}{V^2}\vartheta_{22} &= \left(\frac{dp}{ds}\right)^2 \frac{d^2\theta}{dn^2} - 2\frac{dp}{ds}\frac{dp}{dn}\frac{d\theta_n}{ds} + \left(\frac{dp}{dn}\right)^2\left(-\frac{\theta_n}{\gamma}\right)
\end{aligned}\right\}.$$

Also, we have

$$\begin{aligned}
\frac{d^2\theta}{dn^2}\left(-\frac{\theta_n}{\gamma}\right) - \left(\frac{d\theta_n}{ds}\right)^2
&= (\vartheta_{11}p'^2 + 2\vartheta_{12}p'q' + \vartheta_{22}q'^2)\left\{\vartheta_{11}\left(\frac{dp}{dn}\right)^2 + 2\vartheta_{12}\frac{dp}{dn}\frac{dq}{dn} + \vartheta_{22}\left(\frac{dq}{dn}\right)^2\right\}\\
&\quad - \left\{\vartheta_{11}p'\frac{dp}{dn} + \vartheta_{12}\left(p'\frac{dq}{dn} + q'\frac{dp}{dn}\right) + \vartheta_{22}q'\frac{dq}{dn}\right\}^2\\
&= (\vartheta_{11}\vartheta_{22} - \vartheta_{12}^2)\left(p'\frac{dq}{dn} - q'\frac{dp}{dn}\right)^2,
\end{aligned}$$

and therefore

$$\vartheta_{11}\vartheta_{22} - \vartheta_{12}^2 = -V^2\left\{\frac{1}{\gamma}\theta_n\frac{d^2\theta}{dn^2} + \left(\frac{d\theta_n}{ds}\right)^2\right\},$$

providing another concomitant of the system, together with its geometrical significance. The result can also be deduced from the foregoing values of $\vartheta_{11}, \vartheta_{12}, \vartheta_{22}$.

Ex. 1. Verify the following relations:

(i) $\overline{A}p'\frac{dp}{dn} + \overline{H}\left(p'\frac{dq}{dn} + q'\frac{dp}{dn}\right) + \overline{B}q'\frac{dq}{dn} = -\frac{1}{\sigma}$;

(ii) $\overline{A}\left(\frac{dp}{dn}\right)^2 + 2\overline{H}\frac{dp}{dn}\frac{dq}{dn} + \overline{B}\left(\frac{dq}{dn}\right)^2 = \rho\left(\frac{1}{\sigma^2} + \frac{1}{\tau^2} + K\right)$;

(iii) $A\vartheta_{22} - 2H\vartheta_{12} + B\vartheta_{11} = V^2\left(\frac{d^2\theta}{dn^2} - \frac{\theta_n}{\gamma}\right)$;

(iv) $\frac{1}{V^2}(\overline{A}\vartheta_{22} - 2\overline{H}\vartheta_{12} + \overline{B}\vartheta_{11}) = \frac{1}{\rho}\frac{d^2\theta}{dn^2} + \frac{2}{\sigma}\frac{d\theta_n}{ds} - \theta_n\frac{\rho}{\gamma}\left(\frac{1}{\sigma^2} + \frac{1}{\tau^2} + K\right)$.

It should be noted that all the quantities, on the left-hand sides of these relations, are members of the system of concomitants belonging to the surface and the curve.

Ex. 2. Prove that, if $f(p, q)$ be any function of position on the surface,

$$V^2\left\{\left(\frac{df}{ds}\right)^2 + \left(\frac{df}{dn}\right)^2\right\} = B\left(\frac{\partial f}{\partial p}\right)^2 - 2H\frac{\partial f}{\partial p}\frac{\partial f}{\partial q} + A\left(\frac{\partial f}{\partial q}\right)^2.$$

Superficial variations normal to the curve.

155. Proceeding from the equation

$$V\theta_n \frac{dp}{ds} = \theta_2,$$

differentiating along the arc of the curve and (as usual) denoting the element of arc by dt, while ds is reserved for repeated differentiation along the geodesic tangent, we have

$$\begin{aligned} V\theta_n \frac{d^2p}{dt^2} + V\theta_n \frac{dp}{ds}\left\{(\Gamma_{11}+\Delta_{12})\frac{dp}{ds} + (\Gamma_{12}+\Delta_{22})\frac{dq}{ds}\right\} + V\frac{dp}{ds}\frac{d\theta_n}{ds} \\ = \theta_{12}\frac{dp}{ds} + \theta_{12}\frac{dq}{ds} \\ = \vartheta_{12}\frac{dp}{ds} + \vartheta_{22}\frac{dq}{ds} + (\theta_1\Gamma_{12}+\theta_2\Delta_{12})\frac{dp}{ds} + (\theta_1\Gamma_{22}+\theta_2\Delta_{22})\frac{dq}{ds}. \end{aligned}$$

When, in the term $V\frac{dp}{ds}\frac{d\theta_n}{ds}$, the value of $\frac{d\theta_n}{ds}$ is inserted and the whole quantity is transferred to the right-hand side of the equation, the coefficient of ϑ_{11} on that right-hand side

$$= -V\frac{dp}{dn}\left(\frac{dp}{ds}\right)^2;$$

the coefficient of ϑ_{12}

$$= \frac{dp}{ds} - V\frac{dp}{ds}\left(\frac{dq}{ds}\frac{dp}{dn} + \frac{dp}{ds}\frac{dq}{dn}\right) = -2V\frac{dp}{dn}\frac{dp}{ds}\frac{dq}{ds},$$

because of the relation

$$V\left(\frac{dp}{ds}\frac{dq}{dn} - \frac{dq}{ds}\frac{dp}{dn}\right) = 1;$$

the coefficient of ϑ_{22}

$$= \frac{dq}{ds} - V\frac{dp}{ds}\frac{dq}{ds}\frac{dq}{dn} = -V\frac{dp}{dn}\left(\frac{dq}{ds}\right)^2,$$

because of the same relation; and therefore the aggregate of the terms involving ϑ_{11}, ϑ_{12}, ϑ_{22},

$$= -V\frac{dp}{dn}(\vartheta_{11}p'^2 + 2\vartheta_{12}p'q' + \vartheta_{22}q'^2) = V\frac{\theta_n}{\gamma}\frac{dp}{dn}.$$

When the other terms are collected, the equation leads to the result

$$\frac{d^2p}{dt^2} + \Gamma_{11}\left(\frac{dp}{ds}\right)^2 + 2\Gamma_{12}\frac{dp}{ds}\frac{dq}{ds} + \Gamma_{22}\left(\frac{dq}{ds}\right)^2 = \frac{1}{\gamma}\frac{dp}{dn},$$

in accordance with the result already established. Similarly for the second curve-derivative of the parameter q.

Proceeding from the same equation, and differentiating along a superficial direction normal to the curve, we have

$$V\theta_n \frac{d}{dn}\left(\frac{dp}{ds}\right) + V\frac{d^2\theta}{dn^2}\frac{dp}{ds} + V\theta_n \frac{dp}{ds}\left\{(\Gamma_{11}+\Delta_{12})\frac{dp}{dn} + (\Gamma_{12}+\Delta_{22})\frac{dq}{dn}\right\}$$

$$= \theta_{12}\frac{dp}{dn} + \theta_{22}\frac{dq}{dn} = \vartheta_{12}\frac{dp}{dn} + \vartheta_{22}\frac{dq}{dn} + (\theta_1\Gamma_{12}+\theta_2\Delta_{12})\frac{dp}{dn} + (\theta_1\Gamma_{22}+\theta_2\Delta_{22})\frac{dq}{dn}.$$

We transpose the term $V\dfrac{d^2\theta}{dn^2}\dfrac{dp}{ds}$ to the right-hand side, at the same time substituting the value of $\dfrac{d^2\theta}{dn^2}$ from § 154; then, in the same way as in deducing the preceding value of $\dfrac{d^2p}{dt^2}$, the aggregate of terms involving ϑ_{11}, ϑ_{12}, ϑ_{22}, is found to be

$$= -V\frac{dp}{dn}\left\{\vartheta_{11}p'\frac{dp}{dn} + \vartheta_{12}\left(p'\frac{dq}{dn} + q'\frac{dp}{dn}\right) + \vartheta_{22}q'\frac{dq}{dn}\right\} = -V\frac{dp}{dn}\frac{d\theta_n}{ds}.$$

When the other terms are collected, the equation leads to the result

$$\frac{d}{dn}\left(\frac{dp}{ds}\right) + \Gamma_{11}\frac{dp}{ds}\frac{dp}{dn} + \Gamma_{12}\left(\frac{dp}{ds}\frac{dq}{dn} + \frac{dq}{ds}\frac{dp}{dn}\right) + \Gamma_{22}\frac{dq}{ds}\frac{dq}{dn} = -\frac{1}{\theta_n}\frac{d\theta_n}{ds}\frac{dp}{dn}.$$

Similarly

$$\frac{d}{dn}\left(\frac{dq}{ds}\right) + \Delta_{11}\frac{dp}{ds}\frac{dp}{dn} + \Delta_{12}\left(\frac{dp}{ds}\frac{dq}{dn} + \frac{dq}{ds}\frac{dp}{dn}\right) + \Delta_{22}\frac{dq}{ds}\frac{dq}{dn} = -\frac{1}{\theta_n}\frac{d\theta_n}{ds}\frac{dq}{dn}.$$

Next, we proceed similarly from the expressions for $\dfrac{dp}{dn}$, $\dfrac{dq}{dn}$. In the first place, we take the equivalent form

$$V\frac{dp}{dn} = -H\frac{dp}{ds} - B\frac{dq}{ds},$$

and differentiate along the curve; then

$$V\frac{d}{ds}\left(\frac{dp}{dn}\right) + V\frac{dp}{dn}\left\{(\Gamma_{11}+\Delta_{12})\frac{dp}{ds} + (\Gamma_{12}+\Delta_{22})\frac{dq}{ds}\right\}$$

$$= -H\frac{d^2p}{dt^2} - B\frac{d^2q}{dt^2} - \frac{dp}{ds}\frac{dH}{ds} - \frac{dq}{ds}\frac{dB}{ds}.$$

On the right-hand side, we substitute the values of $\dfrac{d^2p}{dt^2}$, $\dfrac{d^2q}{dt^2}$, verified above. From the terms involving $\dfrac{1}{\gamma}$, we have the aggregate

$$-\frac{1}{\gamma}\left(H\frac{dp}{dn} + B\frac{dq}{dn}\right) = -\frac{1}{\gamma}Vp'.$$

When the remaining terms are collected, the equation leads to the result

$$\frac{d}{ds}\left(\frac{dp}{dn}\right) + \Gamma_{11}\frac{dp}{ds}\frac{dp}{dn} + \Gamma_{12}\left(\frac{dp}{ds}\frac{dq}{dn} + \frac{dq}{ds}\frac{dp}{dn}\right) + \Gamma_{22}\frac{dq}{ds}\frac{dq}{dn} = -\frac{1}{\gamma}\frac{dp}{ds}.$$

Similarly, we find

$$\frac{d}{ds}\left(\frac{dq}{dn}\right)+\Delta_{11}\frac{dq}{ds}\frac{dq}{dn}+\Delta_{12}\left(\frac{dp}{ds}\frac{dq}{dn}+\frac{dq}{ds}\frac{dp}{dn}\right)+\Delta_{22}\frac{dq}{ds}\frac{dq}{dn}=-\frac{1}{\gamma}\frac{dq}{ds}.$$

It is to be noted that

$$\frac{d}{dn}\left(\frac{dp}{ds}\right)-\frac{d}{ds}\left(\frac{dp}{dn}\right)=\frac{1}{\gamma}\frac{dp}{ds}-\frac{1}{\theta_n}\frac{d\theta_n}{ds}\frac{dp}{dn},\quad \frac{d}{dn}\left(\frac{dq}{ds}\right)-\frac{d}{ds}\left(\frac{dq}{dn}\right)=\frac{1}{\gamma}\frac{dq}{ds}-\frac{1}{\theta_n}\frac{d\theta_n}{ds}\frac{dq}{dn}.$$

More generally, for any function $\phi(p, q)$, we have

$$\frac{d}{dn}\left(\frac{d\phi}{ds}\right)=\frac{d}{dn}\left(\phi_1\frac{dp}{ds}+\phi_2\frac{dq}{ds}\right),\quad \frac{d}{ds}\left(\frac{d\phi}{dn}\right)=\frac{d}{ds}\left(\phi_1\frac{dp}{dn}+\phi_2\frac{dq}{dn}\right),$$

and therefore

$$\frac{d}{dn}\left(\frac{d\phi}{ds}\right)-\frac{d}{ds}\left(\frac{d\phi}{dn}\right)=\phi_1\left\{\frac{d}{dn}\left(\frac{dp}{ds}\right)-\frac{d}{ds}\left(\frac{dp}{dn}\right)\right\}+\phi_2\left\{\frac{d}{dn}\left(\frac{dq}{ds}\right)-\frac{d}{ds}\left(\frac{dq}{dn}\right)\right\},$$

the terms in ϕ_{11}, ϕ_{12}, ϕ_{22}, cancelling: that is,

$$\begin{aligned}\frac{d}{dn}\left(\frac{d\phi}{ds}\right)-\frac{d}{ds}\left(\frac{d\phi}{dn}\right)&=\frac{1}{\gamma}\left(\phi_1\frac{dp}{ds}+\phi_2\frac{dq}{ds}\right)-\frac{1}{\theta_n}\frac{d\theta_n}{ds}\left(\phi_1\frac{dp}{dn}+\phi_2\frac{dq}{dn}\right)\\&=\frac{1}{\gamma}\frac{d\phi}{ds}-\frac{1}{\theta_n}\frac{d\theta_n}{ds}\frac{d\phi}{nd}.\end{aligned}$$

We thus verify the expectation that, for any function of position alone (even the simplest) on the surface, and *à fortiori* for a function which also involves direction, the operations $\frac{d^2}{dn\,ds}$ and $\frac{d^2}{ds\,dn}$ are not equivalent. An example of this property for a function of position is provided by θ, originating as it does with a suggestion (though not the fact) of direction; for $\frac{d\theta}{ds}$ is always zero, so that $\frac{d}{dn}\left(\frac{d\theta}{ds}\right)$ is always zero, while $\frac{d}{ds}\left(\frac{d\theta}{dn}\right)$, $=\frac{d\theta_n}{ds}$, is different from zero.

In the second place, we proceed from the same expression for $\frac{dp}{dn}$ in the form

$$V\frac{dp}{dn}=-H\frac{dp}{ds}-B\frac{dq}{ds},$$

and differentiate along a superficial direction normal to the curve, so that

$$\begin{aligned}V\frac{d^2p}{dn^2}+V\frac{dp}{dn}&\left\{(\Gamma_{11}+\Delta_{12})\frac{dp}{dn}+(\Gamma_{12}+\Delta_{22})\frac{dq}{dn}\right\}\\&=-H\frac{d}{dn}\left(\frac{dp}{ds}\right)-B\frac{d}{dn}\left(\frac{dq}{ds}\right)-\frac{dp}{ds}\frac{dH}{dn}-\frac{dq}{ds}\frac{dB}{dn}.\end{aligned}$$

We substitute for $\frac{d}{dn}\left(\frac{dp}{ds}\right)$ and $\frac{d}{dn}\left(\frac{dq}{ds}\right)$ their values already obtained; and after reductions, similar to those which precede, we find

$$\frac{d^2p}{dn^2}+\Gamma_{11}\left(\frac{dp}{dn}\right)^2+2\Gamma_{12}\frac{dp}{dn}\frac{dq}{dn}+\Gamma_{22}\left(\frac{dq}{dn}\right)^2=\frac{1}{\theta_n}\frac{d\theta_n}{ds}\frac{dp}{ds}.$$

Similarly

$$\frac{d^2q}{dn^2}+\Delta_{11}\left(\frac{dq}{dn}\right)^2+2\Delta_{12}\frac{dp}{dn}\frac{dq}{dn}+\Delta_{22}\left(\frac{dq}{dn}\right)^2=\frac{1}{\theta_n}\frac{d\theta_n}{ds}\frac{dq}{ds}.$$

156. In the expression for the flexure of the curve, second parametric derivatives of the quantity θ occur; and its third derivatives are required for the curve-derivative and the normal-derivative of that flexure. It is found convenient to introduce magnitudes ϑ_{ijk}, bearing to θ an analytical relation similar to that borne to y by the magnitudes η_{ijk} of § 100: we write

$$\left.\begin{aligned}
\vartheta_{111}={}&\frac{\partial^3\theta}{\partial p^3}-\frac{\partial\theta}{\partial p}\{\Gamma_{111}+3(\Gamma_{11}\Gamma_{11}+\Gamma_{12}\Delta_{11})\}-\frac{\partial\theta}{\partial q}\{\Delta_{111}+3(\Delta_{11}\Gamma_{11}+\Delta_{11}\Delta_{12})\}\\
&-3(\vartheta_{11}\Gamma_{11}+\vartheta_{12}\Delta_{11})\\
\vartheta_{112}={}&\frac{\partial^3\theta}{\partial p^2\,\partial q}-\frac{\partial\theta}{\partial p}\{\Gamma_{112}+2(\Gamma_{12}\Gamma_{11}+\Delta_{12}\Gamma_{12})+(\Gamma_{11}\Gamma_{12}+\Delta_{11}\Gamma_{22})\}\\
&-\frac{\partial\theta}{\partial q}\{\Delta_{112}+2(\Delta_{11}\Gamma_{12}+\Delta_{12}\Delta_{12})+(\Delta_{12}\Gamma_{11}+\Delta_{22}\Delta_{11})\}\\
&-2(\vartheta_{11}\Gamma_{12}+\vartheta_{12}\Delta_{12})-(\vartheta_{12}\Gamma_{11}+\vartheta_{22}\Delta_{11})\\
\vartheta_{122}={}&\frac{\partial^3\theta}{\partial p\,\partial q^2}-\frac{\partial\theta}{\partial p}\{\Gamma_{122}+(\Gamma_{11}\Gamma_{22}+\Gamma_{12}\Delta_{22})+2(\Gamma_{12}\Gamma_{12}+\Gamma_{22}\Delta_{12})\}\\
&-\frac{\partial\theta}{\partial q}\{\Delta_{122}+(\Delta_{11}\Gamma_{22}+\Delta_{12}\Delta_{22})+2(\Delta_{12}\Gamma_{12}+\Delta_{22}\Delta_{12})\}\\
&-(\vartheta_{11}\Gamma_{22}+\vartheta_{12}\Delta_{22})-2(\vartheta_{12}\Gamma_{12}+\vartheta_{22}\Delta_{12})\\
\vartheta_{222}={}&\frac{\partial^3\theta}{\partial p^3}-\frac{\partial\theta}{\partial p}\{\Gamma_{222}+3(\Gamma_{12}\Gamma_{22}+\Gamma_{22}\Delta_{22})\}-\frac{\partial\theta}{\partial q}\{\Delta_{222}+3(\Gamma_{22}\Delta_{12}+\Delta_{22}\Delta_{22})\}\\
&-3(\vartheta_{12}\Gamma_{22}+\vartheta_{22}\Delta_{22})
\end{aligned}\right\}.$$

We have

$$\vartheta_{11}=\theta_{11}-\theta_1\Gamma_{11}-\theta_2\Delta_{11},$$

and therefore

$$\frac{\partial\vartheta_{11}}{\partial p}=\frac{\partial^3\theta}{\partial p^3}-\frac{\partial^2\theta}{\partial p^2}\Gamma_{11}-\frac{\partial^2\theta}{\partial p\,\partial q}\Delta_{11}-\theta_1\frac{\partial\Gamma_{11}}{\partial p}-\theta_2\frac{\partial\Delta_{11}}{\partial p}.$$

When we insert the values of $\dfrac{\partial\Gamma_{11}}{\partial p}$ and $\dfrac{\partial\Delta_{11}}{\partial p}$ from § 97, substitute

$$\frac{\partial^2\theta}{\partial p^2}=\vartheta_{11}+\theta_1\Gamma_{11}+\theta_2\Delta_{11},\quad \frac{\partial^2\theta}{\partial p\,\partial q}=\vartheta_{12}+\theta_1\Gamma_{12}+\theta_2\Delta_{12},$$

and reduce, we find

$$\frac{\partial\vartheta_{11}}{\partial p}=\vartheta_{111}+2(\vartheta_{11}\Gamma_{11}+\vartheta_{12}\Delta_{11}).$$

Similarly for the remaining first parametric derivatives of ϑ_{11}, ϑ_{12}, ϑ_{22}; the complete set of values is

$$\left.\begin{aligned}
\frac{\partial\vartheta_{11}}{\partial p} &= \vartheta_{111} + 2(\vartheta_{11}\Gamma_{11} + \vartheta_{12}\Delta_{11}) \\
\frac{\partial\vartheta_{11}}{\partial q} &= \vartheta_{112} + 2(\vartheta_{11}\Gamma_{12} + \vartheta_{12}\Delta_{12}) - \tfrac{2}{3} KV^2\theta_n \frac{dq}{dn} \\
\frac{\partial\vartheta_{12}}{\partial p} &= \vartheta_{112} + \vartheta_{11}\Gamma_{12} + \vartheta_{12}\Delta_{12} + \vartheta_{12}\Gamma_{11} + \vartheta_{22}\Delta_{11} + \tfrac{1}{3} KV^2\theta_n \frac{dq}{dn} \\
\frac{\partial\vartheta_{12}}{\partial q} &= \vartheta_{122} + \vartheta_{11}\Gamma_{22} + \vartheta_{12}\Delta_{22} + \vartheta_{12}\Gamma_{12} + \vartheta_{22}\Delta_{12} + \tfrac{1}{3} KV^2\theta_n \frac{dp}{dn} \\
\frac{\partial\vartheta_{22}}{\partial p} &= \vartheta_{122} + 2(\vartheta_{12}\Gamma_{12} + \vartheta_{22}\Delta_{12}) - \tfrac{2}{3} KV^2\theta_n \frac{dp}{dn} \\
\frac{\partial\vartheta_{22}}{\partial q} &= \vartheta_{222} + 2(\vartheta_{12}\Gamma_{22} + \vartheta_{22}\Delta_{22})
\end{aligned}\right\}.$$

which are analogous to the formulæ in § 101, regard being paid to the relations defining $\frac{dp}{dn}$ and $\frac{dq}{dn}$ in terms of θ_1 and θ_2.

We have

$$-\frac{\theta_n}{\gamma} = \vartheta_{11}p'^2 + 2\vartheta_{12}p'q' + \vartheta_{22}q'^2;$$

and therefore, differentiating along the curve,

$$-\frac{d}{dt}\left(\frac{\theta_n}{\gamma}\right) = \frac{\partial\vartheta_{11}}{\partial p}p'^3 + \left(\frac{\partial\vartheta_{11}}{\partial q} + 2\frac{\partial\vartheta_{12}}{\partial p}\right)p'^2q' + \left(2\frac{\partial\vartheta_{12}}{\partial q} + \frac{\partial\vartheta_{22}}{\partial p}\right)p'q'^2 + \frac{\partial\vartheta_{22}}{\partial q}q'^3$$
$$+ 2(\vartheta_{11}p' + \vartheta_{12}q')\frac{d^2p}{dt^2} + 2(\vartheta_{12}p' + \vartheta_{22}q')\frac{d^2q}{dt^2}.$$

When substitution is effected for the first derivatives of ϑ_{11}, ϑ_{12}, ϑ_{22}, the terms involving K disappear from the aggregate; and the full set of terms involving the magnitudes ϑ_{ijk}

$$= (\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222} \between p', q')^3.$$

When substitution is made for $\frac{d^2p}{dt^2}$ and $\frac{d^2q}{dt^2}$, the set of terms involving $1/\gamma$

$$= \frac{2}{\gamma}\left\{(\vartheta_{11}p' + \vartheta_{12}q')\frac{dp}{dn} + (\vartheta_{12}p' + \vartheta_{22}q')\frac{dq}{dn}\right\}$$
$$= \frac{2}{\gamma}\frac{d\theta_n}{dt}.$$

It appears that all the remaining terms, now aggregated on the right-hand side, vanish; and therefore

$$-\frac{d}{dt}\left(\frac{\theta_n}{\gamma}\right) = (\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222} \between p', q')^3 + \frac{2}{\gamma}\frac{d\theta_n}{dt},$$

which can be modified into the slightly different forms

$$-\theta_n \frac{d}{dt}\left(\frac{1}{\gamma}\right)=(\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222}\between p', q')^3+\frac{3}{\gamma}\frac{d\theta_n}{dt},$$

$$-\frac{1}{\theta_n{}^2}\frac{d}{dt}\left(\frac{\theta_n{}^3}{\gamma}\right)=(\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222}\between p', q')^3,$$

$$-V^3\theta_n\frac{d}{dt}\left(\frac{\theta_n{}^3}{\gamma}\right)=(\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222}\between \theta_2, -\theta_1)^3.$$

Ex. It is easy to deduce the relation

$$\frac{d}{dt}\left(\frac{1}{\gamma}\right)-\frac{d}{ds}\left(\frac{1}{\gamma}\right)=-\frac{2}{\gamma\theta_n}\frac{d\theta_n}{dt}.$$

Next, differentiating the relation

$$\frac{d^2\theta}{dn^2}=\vartheta_{11}\left(\frac{dp}{dn}\right)^2+2\vartheta_{12}\frac{dp}{dn}\frac{dq}{dn}+\vartheta_{22}\left(\frac{dq}{dn}\right)^2$$

along the superficial direction normal to the curve, we find

$$\begin{aligned}\frac{d^3\theta}{dn^3}=\frac{\partial\vartheta_{11}}{\partial p}\left(\frac{dp}{dn}\right)^3&+\left(\frac{\partial\vartheta_{11}}{dq}+2\frac{\partial\vartheta_{12}}{\partial p}\right)\left(\frac{dp}{dn}\right)^2\frac{dq}{dn}\\ &+\left(2\frac{\vartheta_{12}}{\partial q}+\frac{\partial\vartheta_{22}}{\partial p}\right)\frac{dp}{dn}\left(\frac{dq}{dn}\right)^2+\frac{\partial\vartheta_{22}}{\partial q}\left(\frac{dq}{dn}\right)^3\\ &+2\left(\vartheta_{11}\frac{dp}{dn}+\vartheta_{12}\frac{dq}{dn}\right)\frac{d^2p}{dn^2}+2\left(\vartheta_{12}\frac{dp}{dn}+\vartheta_{22}\frac{dq}{dn}\right)\frac{d^2q}{dn^2}.\end{aligned}$$

When we substitute for the derivatives of ϑ_{11}, ϑ_{12}, ϑ_{22}, the aggregate of terms involving the magnitudes ϑ_{ijk}

$$=\left(\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222}\between\frac{dp}{dn}, \frac{dq}{dn}\right)^3;$$

and the aggregate of terms involving K is equal to zero. When we substitute for $\frac{d^2p}{dn^2}$ and $\frac{d^2q}{dn^2}$, the aggregate of terms involving the quantity $\frac{d\theta_n}{ds}$

$$\begin{aligned}&=\frac{2}{\theta_n}\frac{d\theta_n}{ds}\left\{\left(\vartheta_{11}\frac{dp}{dn}+\vartheta_{12}\frac{dq}{dn}\right)\frac{dp}{ds}+\left(\vartheta_{12}\frac{dp}{dn}+\vartheta_{22}\frac{dq}{dn}\right)\frac{dq}{ds}\right\}\\ &=\frac{2}{\theta_n}\left(\frac{d\theta_n}{ds}\right)^2.\end{aligned}$$

The aggregate of all the terms now remaining after both sets of both substitutions is found to be zero. Hence we have

$$\frac{d^3\theta}{dn^3}=\left(\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222}\between\frac{dp}{dn}, \frac{dq}{dn}\right)^3+\frac{2}{\theta_n}\left(\frac{d\theta_n}{ds}\right)^2.$$

Differentiating the same relation along the arc of the curve, we have

$$\frac{d}{dt}\left(\frac{d^2\theta}{dn^2}\right)=\left(\frac{dp}{dn}\right)^2\left(\frac{\partial\vartheta_{11}}{\partial p}p'+\frac{\partial\vartheta_{11}}{\partial q}q'\right)$$

$$+2\frac{dp}{dn}\frac{dq}{dn}\left(\frac{\partial\vartheta_{12}}{\partial p}p'+\frac{\partial\vartheta_{12}}{\partial q}q'\right)+\left(\frac{dq}{dn}\right)^2\left(\frac{\partial\vartheta_{22}}{\partial p}p'+\frac{\partial\vartheta_{22}}{\partial q}q'\right)$$

$$+2\left(\vartheta_{11}\frac{dp}{dn}+\vartheta_{12}\frac{dq}{dn}\right)\frac{d}{dt}\left(\frac{dp}{dn}\right)+2\left(\vartheta_{12}\frac{dp}{dn}+\vartheta_{22}\frac{dq}{dn}\right)\frac{d}{dt}\left(\frac{dq}{dn}\right).$$

When the values of the derivatives of ϑ_{11}, ϑ_{12}, ϑ_{22}, are substituted, the aggregate of the terms in the magnitudes ϑ_{ijk}

$$=\left(\vartheta_{111},\ \vartheta_{112},\ \vartheta_{122},\ \vartheta_{222}\between p',\ q'\between\frac{dp}{dn},\ \frac{dq}{dn}\right)^2,$$

while the aggregate of the terms involving K is zero. When we substitute for $\frac{d}{dt}\left(\frac{dp}{dn}\right)$ and $\frac{d}{dt}\left(\frac{dq}{dn}\right)$, there is an aggregate of terms in $\frac{1}{\gamma}$

$$=-\frac{1}{\gamma}\left\{2\left(\vartheta_{11}\frac{dp}{dn}+\vartheta_{12}\frac{dq}{dn}\right)\frac{dp}{ds}+2\left(\vartheta_{12}\frac{dp}{dn}+\vartheta_{22}\frac{dq}{dn}\right)\frac{dq}{ds}\right\}$$

$$=-\frac{2}{\gamma}\frac{d\theta_n}{ds}.$$

The aggregate of all the remaining terms, after both sets of substitutions have been made, is found to be zero. Hence

$$\frac{d}{dt}\left(\frac{d^2\theta}{dn^2}\right)=\left(\vartheta_{111},\ \vartheta_{112},\ \vartheta_{122},\ \vartheta_{222}\between p',\ q'\between\frac{dp}{dn},\ \frac{dq}{dn}\right)^2-\frac{2}{\gamma}\frac{d\theta_n}{ds}.$$

Proceeding similarly to differentiate the relation

$$\frac{d\theta_n}{ds}=\vartheta_{11}p'\frac{dp}{dn}+\vartheta_{12}\left(p'\frac{dq}{dn}+q'\frac{dp}{dn}\right)+\vartheta_{22}q'\frac{dq}{dn}$$

along a direction normal to the curve, and using the expressions for

$$\frac{dp'}{dn},\quad\frac{d^2p}{dn^2},\quad\frac{dq'}{dn},\quad\frac{d^2q}{dn^2},$$

we find

$$\frac{d}{dn}\left(\frac{d\theta_n}{ds}\right)=\left(\vartheta_{111},\ \vartheta_{112},\ \vartheta_{122},\ \vartheta_{222}\between p',\ q'\between\frac{dp}{dn},\ \frac{dq}{dn}\right)^2-\frac{1}{\gamma}\frac{d\theta_n}{ds}-\frac{1}{\theta_n}\frac{d\theta_n}{ds}\frac{d^2\theta}{dn^2}.$$

A comparison of the last two results leads to another example of the non-equivalence of the operators $\frac{d^2}{dt\,dn}$ and $\frac{d^2}{dn\,dt}$; for they give the relation

$$\left(\frac{d^2}{dt\,dn}-\frac{d^2}{dn\,dt}\right)\theta_n=\left(\frac{1}{\theta_n}\frac{d^2\theta}{dn^2}-\frac{1}{\gamma}\right)\frac{d\theta_n}{ds}.$$

When the relation

$$\frac{d\theta_n}{ds} = \vartheta_{11} p' \frac{dp}{dn} + \vartheta_{12}\left(p' \frac{dq}{dn} + q' \frac{dp}{dn}\right) + \vartheta_{22} q' \frac{dq}{dn}$$

is differentiated along the curve, and the values of the quantities

$$\frac{d^2p}{dt^2},\quad \frac{d}{ds}\left(\frac{dp}{dn}\right),\quad \frac{d^2q}{dt^2},\quad \frac{d}{ds}\left(\frac{dq}{dn}\right),$$

are used, we find, after some reduction,

$$\frac{d^2\theta_n}{dt^2} = \left(\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222}\,\middle)\!\middle(\frac{dp}{dn}, \frac{dq}{dn}\,\middle)\!\middle(p', q'\right)^2 + \frac{1}{\gamma}\frac{d^2\theta}{dn^2} + \left(\tfrac{1}{3}K + \frac{1}{\gamma^2}\right)\theta_n .$$

Finally, when we differentiate the equation for the flexure

$$-\frac{\theta_n}{\gamma} = \vartheta_{11} p'^2 + 2\vartheta_{12} p' q' + \vartheta_{22} q'^2$$

along the normal to the curve, we find

$$-\theta_n \frac{d}{dn}\left(\frac{1}{\gamma}\right) = \left(\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222}\,\middle)\!\middle(\frac{dp}{dn}, \frac{dq}{dn}\,\middle)\!\middle(p', q'\right)^2 + \frac{1}{\gamma}\frac{d^2\theta}{dn^2} - \tfrac{2}{3}K\theta_n - \frac{2}{\theta_n}\left(\frac{d\theta_n}{ds}\right)^2 .$$

The last two lead to the equation

$$\frac{d^2\theta_n}{dt^2} + \theta_n \frac{d}{dn}\left(\frac{1}{\gamma}\right) - \frac{2}{\theta_n}\left(\frac{d\theta_n}{ds}\right)^2 = \left(K + \frac{1}{\gamma^2}\right)\theta_n .$$

We thus have relations, giving a single value to each of the forms

$$(\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222})(p', q')^3,$$

$$\left(\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222}\,\middle)\!\middle(\frac{dp}{dn}, \frac{dq}{dn}\right)^3,$$

and two values to each of the forms

$$\left(\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222}\,\middle)\!\middle(\frac{dp}{dn}, \frac{dq}{dn}\,\middle)\!\middle(p', q'\right)^2,$$

$$\left(\vartheta_{111}, \vartheta_{112}, \vartheta_{122}, \vartheta_{222}\,\middle)\!\middle(p', q'\,\middle)\!\middle(\frac{dp}{dn}, \frac{dq}{dn}\right)^2.$$

Alternative derivation of the binormal of the curve and the spatial torsion.

157. The results, obtained in § 152 for the direction of the binormal of the curve and the magnitude of the torsion, can be obtained in a slightly different manner by means of the formula for the curve-variation of the flexure. We use the fundamental equation

$$\frac{Y_c}{\ } - \frac{Y}{\ } = \frac{l_3}{\ },$$

where

$$\frac{1}{\rho_c^2}=\frac{1}{\rho^2}+\frac{1}{\gamma^2}.$$

When the last relation is differentiated along the curve, we find

$$\begin{aligned}\frac{1}{\rho_c}\frac{d}{dt}\left(\frac{1}{\rho_c}\right)&=\frac{1}{\rho}\frac{d}{dt}\left(\frac{1}{\rho}\right)+\frac{1}{\gamma}\frac{d}{dt}\left(\frac{1}{\gamma}\right)\\&=\frac{1}{\rho}\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-\frac{2}{\gamma\sigma}\right\}+\frac{1}{\gamma}\frac{d}{dt}\left(\frac{1}{\gamma}\right),\end{aligned}$$

all the quantities on the right-hand side now being known; and we can take

$$\frac{d}{dt}\left(\frac{1}{\rho_c}\right)=\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-\frac{2}{\gamma\sigma}\right\}\cos\psi+\frac{d}{dt}\left(\frac{1}{\gamma}\right)\sin\psi,$$

where an organic angle ψ has been introduced such that

$$\frac{\cos\psi}{\rho_c}=\frac{1}{\rho},\quad\frac{\sin\psi}{\rho_c}=\frac{1}{\gamma}.$$

When the fundamental equation is differentiated along the curve, we have

$$\frac{dY_c}{dt}=\frac{\lambda_3}{\sigma_c}-\frac{y'}{\rho_c},$$

with the former notation; and we have established (§ 151) the equations

$$\frac{d}{dt}\left(\frac{Y}{\rho}\right)=Y\frac{d}{ds}\left(\frac{1}{\rho}\right)-2\frac{Y}{\gamma\sigma}+\frac{1}{\rho}\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)+\frac{2l_4}{\gamma\tau},$$

$$\frac{dl_3}{dt}=\frac{dl_3}{ds}-\frac{y'}{\gamma}=\frac{l_4}{\tau}-\frac{Y}{\sigma}-\frac{y'}{\gamma}.$$

Accordingly,

$$\begin{aligned}&\frac{1}{\rho_c}\left(\frac{\lambda_3}{\sigma_c}-\frac{y'}{\rho_c}\right)+Y_c\frac{d}{dt}\left(\frac{1}{\rho_c}\right)\\&\quad=\frac{1}{\gamma}\left(\frac{l_4}{\tau}-\frac{Y}{\sigma}-\frac{y'}{\gamma}\right)+l_3\frac{d}{dt}\left(\frac{1}{\gamma}\right)+Y\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-\frac{2}{\gamma\sigma}\right\}-\frac{y'}{\rho^2}+\frac{l_3}{\rho\sigma}+\frac{2l_4}{\gamma\tau}.\end{aligned}$$

The terms in y' on the two sides cancel. Also one form of the fundamental equation is

$$Y_c=Y\cos\psi+l_3\sin\psi;$$

and thus the equation for the magnitude of the torsion and the direction of the binormal of the curve becomes

$$\begin{aligned}\frac{\lambda_3}{\rho_c\sigma_c}&=3\frac{l_4}{\gamma\tau}+l_3\left\{\frac{d}{dt}\left(\frac{1}{\gamma}\right)+\frac{1}{\rho\sigma}-\frac{d}{dt}\left(\frac{1}{\rho_c}\right)\sin\psi\right\}\\&\quad+Y\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-\frac{3}{\gamma\sigma}-\frac{d}{dt}\left(\frac{1}{\rho_c}\right)\cos\psi\right\}.\end{aligned}$$

The coefficient of l_3 can be expressed in the form

$$\left\{\frac{d}{dt}\left(\frac{1}{\gamma}\right)+\frac{1}{\rho\sigma}\right\}\cos^2\psi-\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-\frac{3}{\gamma\sigma}\right\}\sin\psi\cos\psi;$$

and the coefficient of Y can be expressed in the form

$$-\left\{\frac{d}{dt}\left(\frac{1}{\gamma}\right)+\frac{1}{\rho\sigma}\right\}\sin\psi\cos\psi+\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-\frac{3}{\gamma\sigma}\right\}\sin^2\psi;$$

if therefore we write

$$W=\left\{\frac{d}{dt}\left(\frac{1}{\gamma}\right)+\frac{1}{\rho\sigma}\right\}\cos\psi-\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-\frac{3}{\gamma\sigma}\right\}\sin\psi,$$

the equation can be written

$$\frac{\lambda_3}{\rho_c\sigma_c}=3\frac{l_4}{\gamma\tau}+(l_3\cos\psi-Y\sin\psi)W.$$

Accordingly, we can take

$$\lambda_3=(l_3\cos\psi-Y\sin\psi)\cos\chi+l_4\sin\chi,$$

and we have

$$\left.\begin{aligned}\frac{\sin\chi}{\rho_c\sigma_c}&=\frac{3}{\gamma\tau}\\ \frac{\cos\chi}{\rho_c\sigma_c}&=W\end{aligned}\right\},$$

two equations for the determination of χ and σ_c. The former equation can be taken in the earlier form (§ 152)

$$\frac{\sin\chi}{\sigma_c}=\frac{3\sin\psi}{\tau}.$$

SECTION III: REGIONS

CHAPTER XIV

FREE REGIONS: PRELIMINARY

Parametric representation of a region.

158. We pass to the consideration of three-dimensional configurations in multiple space. When such a configuration is homaloidal, it is called a *flat*: otherwise, it is called a *region*.

A region can be defined by the analytical expressions of any contained point. These involve three parameters which will be denoted by p, q, r, (instead of x_1, x_2, x_3, as used for general amplitudes, though occasionally it will be convenient to revert to x_1, x_2, x_3). The plenary homaloidal space is of any number of dimensions, though necessarily for a region that number is not less than four. As always, we denote the typical space-coordinate by y. In our intrinsic geometry, the space-coordinates occur most frequently through their parametric derivatives; even so, these occur in combinations of a covariantive character. For the sake of brevity, we shall specify derivatives by numerical subscripts affixed to the typical coordinate y, at any rate for the first three orders: thus we take

$$y_1, y_2, y_3 = \frac{\partial y}{\partial p}, \frac{\partial y}{\partial q}, \frac{\partial y}{\partial r} = \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial y}{\partial x_3}, \quad y_{ij} = \frac{\partial^2 y}{\partial x_i \, \partial x_j},$$

where $x_1 = p$, $x_2 = q$, $x_3 = r$, and $i, j, = 1, 2, 3$; and, with the same range of values of i, j, k,

$$y_{ijk} = \frac{\partial^3 y}{\partial x_i \, \partial x_j \, \partial x_k}.$$

Thus

$$y_{111} = \frac{\partial^3 y}{\partial p^3}, \quad y_{123} = \frac{\partial^3 y}{\partial p \, \partial q \, \partial r},$$

and so on. With this significance *, each subscript 1 is associated with a dif-

* Later (§ 176), for explicit or implicit derivatives of order higher than the second, a different notation is used, according to the definition

$$\frac{\partial^{l+m+n} y}{\partial p^l \partial q^m \partial r^n} = y_{lmn};$$

for all positive integer values (zero included, but stated when it occurs) of l, m, n. Due notice of the change will be indicated when it occurs; meanwhile, the second notation is less suited for the cases $l + m + n = 1$, or 2.

ferentiation with respect to p, likewise for a subscript 2 and q, and for a subscript 3 and r.

The notation used to denote regional magnitudes is an extension of the notation used to denote magnitudes belonging to surfaces in free space. The magnitudes of the first order, usually called primary magnitudes, are denoted by A, B, C, F, G, H, according to the significance

$$A=\sum y_1^2, \quad B=\sum y_2^2, \quad C=\sum y_3^2,$$
$$F=\sum y_2 y_3, \quad G=\sum y_3 y_1, \quad H=\sum y_1 y_2,$$

the summation in each instance extending over all the point-variables, unspecified in number and typified by the single variable y. The arc-element in the ultimate plenary homaloidal space is given by the equation

$$ds^2=\sum dy^2;$$

when the arc-element belongs to the region, we have

$$\begin{aligned} ds^2 &= \sum (y_1 dp + y_2 dq + y_3 dr)^2 \\ &= A\,dp^2 + 2H\,dp\,dq + B\,dq^2 + 2G\,dp\,dr + 2F\,dq\,dr + C\,dr^2, \end{aligned}$$

a relation often written in the form

$$ds^2=\sum A\,dp^2,$$

where the summation-symbol now implies extension only over the homogeneous second-order combinations of the differentials of the three parameters. It is customary to write

$$\frac{dp}{ds}=p', \quad \frac{dq}{ds}=q', \quad \frac{dr}{ds}=r';$$

and thus there is a relation

$$\sum Ap'^2 = Ap'^2 + 2Hp'q' + Bq'^2 + 2Gp'r' + 2Fq'r' + Cr'^2 = 1,$$

often called the permanent arc-relation.

This quantity $\sum Ap'^2$ is a ternary quadratic. We denote its discriminant by Ω, so that

$$\Omega = \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = ABC + 2FGH - AF^2 - BG^2 - CH^2;$$

and we denote the first minors of Ω by a, b, c, f, g, h, so that

$$\left.\begin{aligned} a &= BC - F^2 = \frac{\partial\Omega}{\partial A}, \quad & f &= GH - AF = \tfrac{1}{2}\frac{\partial\Omega}{\partial F} \\ b &= CA - G^2 = \frac{\partial\Omega}{\partial B}, \quad & g &= HF - BG = \tfrac{1}{2}\frac{\partial\Omega}{\partial G} \\ c &= AB - H^2 = \frac{\partial\Omega}{\partial C}, \quad & h &= FG - CH = \tfrac{1}{2}\frac{\partial\Omega}{\partial H} \end{aligned}\right\}.$$

Types of variables for a region.

159. The variables dp, dq, dr, with the associated magnitudes p', q', r', are connected with a direction at a position in the region, and may be described as line-variables or direction-variables for the region. But we have to consider not merely lines and elements of length along lines, but also surfaces and elements of area on surfaces; and additional variables are required to represent orientations on surfaces. We have taken a length ds in connection with a set of line-variables dp, dq, dr. We take a different set of line-variables at the place p, q, r, in the form δp, δq, δr, with a corresponding length δs, so that

$$\delta s^2 = \sum A\,\delta p^2.$$

These two small lengths ds and δs, taken as adjacent sides of a rudimentary parallelogram, determine an element of area dS; and, if ω be the angle between the directions of ds and δs, we have

$$dS = ds\,.\,\delta s\,.\,\sin\omega.$$

Now, as usual, we have

$$\begin{aligned}\cos\omega &= \sum \frac{dy}{ds}\frac{\delta y}{\delta s}\\ &= \sum\left(y_1\frac{dp}{ds}+y_2\frac{dq}{ds}+y_3\frac{dr}{ds}\right)\left(y_1\frac{\delta p}{\delta s}+y_2\frac{\delta q}{\delta s}+y_3\frac{\delta r}{\delta s}\right):\end{aligned}$$

that is,

$$\begin{aligned}ds\,.\,\delta s\,.\,\cos\omega &= \sum (y_1\,dp+y_2\,dq+y_3\,dr)(y_1\,\delta p+y_2\,\delta q+y_3\,\delta r)\\ &= A\,dp\,\delta p + H(dp\,\delta q+dq\,\delta p)+B\,dq\,\delta q\\ &\qquad + G(dp\,\delta r+dr\,\delta p)+F(dq\,\delta r+dr\,\delta q)+C\,dr\,\delta r,\end{aligned}$$

which is often written

$$ds\,.\,\delta s\,.\,\cos\omega = \sum A\,dp\,\delta p.$$

Hence

$$\begin{aligned}ds^2\,.\,\delta s^2\,.\,\sin^2\omega &= ds^2\,.\,\delta s^2-(ds\,.\,\delta s\,.\,\cos\omega)^2\\ &= (\textstyle\sum A\,dp^2)(\sum A\,\delta p^2)-(\sum A\,dp\,\delta p)^2.\end{aligned}$$

When the expressions on the right-hand side are expanded, certain combinations of dp, dq, dr, and δp, δq, δr, occur in the forms

$$\xi = \begin{vmatrix} dq, & dr \\ \delta q, & \delta r\end{vmatrix}, \qquad \eta = \begin{vmatrix} dr, & dp \\ \delta r, & \delta p\end{vmatrix}, \qquad \zeta = \begin{vmatrix} dp, & dq \\ \delta p, & \delta q\end{vmatrix};$$

and we have

$$dS^2 = a\xi^2+2h\xi\eta+b\eta^2+2g\xi\zeta+2f\eta\zeta+c\zeta^2,$$

an expression for a general element of area within the region. These new quantities ξ, η, ζ, being bilinear combinations of line-variables, are called surface-variables, or orientation-variables.

There is no necessity, in the case of regions, to consider variables for the representation of volumes, in a manner analogous to the representation of surfaces

by variables ξ, η, ζ. In fact, if ds, δs, ∂s, be three conterminous arcs with the parametric elements dp, dq, dr; $\delta p, \delta q, \delta r$; $\partial p, \partial q, \partial r$; then the element of volume, being that of the parallelepiped which has ds, δs, ∂s, for conterminous edges, is

$$=\Omega^{\frac{1}{2}}\begin{vmatrix} \dfrac{dp}{ds}, & \dfrac{dq}{ds}, & \dfrac{dr}{ds} \\ \dfrac{\delta p}{\delta s}, & \dfrac{\delta q}{\delta s}, & \dfrac{\delta r}{\delta s} \\ \dfrac{\partial p}{\partial s}, & \dfrac{\partial q}{\partial s}, & \dfrac{\partial r}{\partial s} \end{vmatrix} ds\,\delta s\,\partial s = \Omega^{\frac{1}{2}}\begin{vmatrix} dp, & dq, & dr \\ \delta p, & \delta q, & \delta r \\ \partial p, & \partial q, & \partial r \end{vmatrix},$$

so that a regional variable, arising out of three non-complanar combinations of the line-variables, is a covariant in itself.

It thus appears that, for a region, there are two generically distinct classes of variables: the line-variables (or direction-variables), denoted for one set by dp, dq, dr: the surface-variables (or orientation-variables), denoted for one set by ξ, η, ζ.

Moreover, the surface-variables can be associated uniquely with a parametric equation $\theta(p, q, r)=0$, which represents a regional surface that may be individual or may be a member of a family. Let dS be an element of area on this surface, arising in connection with two linear directions dp, dq, dr; $\delta p, \delta q, \delta r$; on the surface. Then, because

$$\theta_1\,dp+\theta_2\,dq+\theta_3\,dr=0, \quad \theta_1\,\delta p+\theta_2\,\delta q+\theta_3\,\delta r=0,$$

we have

$$\frac{\xi}{\theta_1}=\frac{\eta}{\theta_2}=\frac{\zeta}{\theta_3}=\mu,$$

where μ temporarily denotes the common value of the fractions. Further, let ∂n be an element of length, measured in the region and normal to the surface; and let $\partial p, \partial q, \partial r$, be the parametric variations specifying ∂n. As ∂n is perpendicular to every direction in the surface, we have

$$(A\,dp+H\,dq+G\,dr)\,\partial p+(H\,dp+B\,dq+F\,dr)\,\partial q+(G\,dp+F\,dq+C\,dr)\,\partial r=0,$$
$$(A\,\delta p+H\,\delta q+G\,\delta r)\,\partial p+(H\,\delta p+B\,\delta q+F\,\delta r)\,\partial q+(G\,\delta p+F\,\delta q+C\,\delta r)\,\partial r=0.$$

Hence

$$\frac{\partial p}{(\sum H\,dp)(\sum G\,\delta p)-(\sum H\,\delta p)(\sum G\,dp)}$$
$$=\frac{\partial q}{(\sum G\,dp)(\sum A\,\delta p)-(\sum A\,dp)(\sum G\,\delta p)}$$
$$=\frac{\partial r}{(\sum A\,dp)(\sum H\,\delta p)-(\sum H\,dp)(\sum A\,\delta p)}:$$

that is,

$$\frac{\partial p}{a\xi+h\eta+g\zeta}=\frac{\partial q}{h\xi+b\eta+f\zeta}=\frac{\partial r}{g\xi+f\eta+c\zeta}=\lambda,$$

where λ temporarily denotes the common values of the fractions. Now

$$\begin{aligned}\partial n^2 &= \sum A\,\partial p^2\\ &= \lambda^2 \sum A\,(a\xi + h\eta + g\zeta)^2\\ &= \lambda^2 \Omega \sum a\xi^2 = \lambda^2\Omega\, dS^2,\end{aligned}$$

so that

$$\lambda = \frac{\partial n}{\Omega^{\frac{1}{2}} dS}.$$

Again, if we denote by θ_n the magnitude $\dfrac{d\theta}{dn}$ which may be called the *normal dilatation* of the surface, we have

$$\begin{aligned}\theta_n &= \theta_1 \frac{\partial p}{\partial n} + \theta_2 \frac{\partial q}{\partial n} + \theta_3 \frac{\partial r}{\partial n}\\ &= \frac{1}{\Omega^{\frac{1}{2}} dS} \{\theta_1(a\xi + h\eta + g\zeta) + \theta_2(h\xi + b\eta + f\zeta) + \theta_3(g\xi + f\eta + c\zeta)\}\\ &= \frac{\mu}{\Omega^{\frac{1}{2}} dS} \sum a\theta_1^2,\end{aligned}$$

from the relations connecting ξ, η, ζ, with θ_1, θ_2, θ_3. But, from the same relations,

$$dS^2 = \sum a\xi^2 = \mu^2 \sum a\theta_1^2\,;$$

and therefore

$$\Omega\theta_n^2 = \sum a\theta_1^2,$$

$$\mu = \frac{dS}{\Omega^{\frac{1}{2}}\theta_n}.$$

Consequently the relations between the surface-variables for the surface $\theta = 0$ and the derivatives of the parametric function θ are

$$\frac{\xi}{\theta_1} = \frac{\eta}{\theta_2} = \frac{\zeta}{\theta_3} = \frac{dS}{\Omega^{\frac{1}{2}}\theta_n},$$

dS denoting an element of the surface.

Moreover, there are two concomitants in variables of the system; and their geometrical significance is given by

$$\sum a\xi^2 = dS^2, \quad \sum a\theta_1^2 = \Omega\theta_n^2.$$

The regional direction-variables for the normal to the surface will be written $\dfrac{dp}{dn}, \dfrac{dq}{dn}, \dfrac{dr}{dn}$; and they are given by

$$\left.\begin{aligned}\Omega^{\frac{1}{2}} \frac{dp}{dn} dS &= a\xi + h\eta + g\zeta\\ \Omega^{\frac{1}{2}} \frac{dq}{dn} dS &= h\xi + b\eta + f\zeta\\ \Omega^{\frac{1}{2}} \frac{dr}{dn} dS &= g\xi + f\eta + c\zeta\end{aligned}\right\}, \qquad \left.\begin{aligned}\Omega\theta_n \frac{dp}{dn} &= a\theta_1 + h\theta_2 + g\theta_3\\ \Omega\theta_n \frac{dq}{dn} &= h\theta_1 + b\theta_2 + f\theta_3\\ \Omega\theta_n \frac{dr}{dn} &= g\theta_1 + f\theta_2 + c\theta_3\end{aligned}\right\},$$

as well as by the equivalent forms

$$\left.\begin{aligned}\xi\Omega^{\frac{1}{2}}&=\left(A\frac{dp}{dn}+H\frac{dq}{dn}+G\frac{dr}{dn}\right)dS\\ \eta\Omega^{\frac{1}{2}}&=\left(H\frac{dp}{dn}+B\frac{dq}{dn}+F\frac{dr}{dn}\right)dS\\ \zeta\Omega^{\frac{1}{2}}&=\left(G\frac{dp}{dn}+F\frac{dq}{dn}+C\frac{dr}{dn}\right)dS\end{aligned}\right\},\qquad \left.\begin{aligned}\theta_1&=\theta_n\left(A\frac{dp}{dn}+H\frac{dq}{dn}+G\frac{dr}{dn}\right)\\ \theta_2&=\theta_n\left(H\frac{dp}{dn}+B\frac{dq}{dn}+F\frac{dr}{dn}\right)\\ \theta_3&=\theta_n\left(G\frac{dp}{dn}+F\frac{dq}{dn}+C\frac{dr}{dn}\right)\end{aligned}\right\}.$$

The angle ω, between two directions dp, dq, dr, and δp, δq, δr, corresponding to elementary arcs ds, δs, in the region, was given by

$$ds\,.\,\delta s\,.\cos\omega=\sum A\,dp\,\delta p.$$

The angle η, between two orientations ξ, η, ζ, and ξ_0, η_0, ζ_0, corresponding to two elementary areas dS, dS_0, in the region, is given by

$$dS\,.\,dS_0\,.\cos\eta=\sum a\xi\xi_0,$$

where

$$dS=(\sum a\xi^2)^{\frac{1}{2}},\quad dS_0=(\sum a\xi_0{}^2)^{\frac{1}{2}}.$$

This result follows by using the customary definition of the inclination of two surfaces as the inclination of their unique positively-drawn regional normals at a point; thus, if the normals have the direction-variables

$$\frac{dp}{dn},\ \frac{dq}{dn},\ \frac{dr}{dn};\ \text{and}\ \frac{dp}{dn_0},\ \frac{dq}{dn_0},\ \frac{dr}{dn_0},$$

we have

$$\cos\eta=\sum A\frac{dp}{dn}\frac{dp}{dn_0}.$$

Hence

$$\begin{aligned}dS_0\,.\cos\eta&=\frac{dp}{dn}\left[\left(A\frac{dp}{dn_0}+H\frac{dq}{dn_0}+G\frac{dr}{dn_0}\right)dS_0\right]\\&\quad+\frac{dq}{dn}\left[\left(H\frac{dp}{dn_0}+B\frac{dq}{dn_0}+F\frac{dr}{dn_0}\right)dS_0\right]\\&\quad+\frac{dr}{dn}\left[\left(G\frac{dp}{dn_0}+F\frac{dq}{dn_0}+C\frac{dr}{dn_0}\right)dS_0\right]\\&=\Omega^{\frac{1}{2}}\left(\xi_0\frac{dp}{dn}+\eta_0\frac{dq}{dn}+\zeta_0\frac{dr}{dn}\right);\end{aligned}$$

and therefore

$$\begin{aligned}dS\,.\,dS_0\,.\cos\eta&=\xi_0\,\Omega^{\frac{1}{2}}\frac{dp}{dn}dS+\eta_0\,\Omega^{\frac{1}{2}}\frac{dq}{dn}dS+\zeta_0\,\Omega^{\frac{1}{2}}\frac{dr}{dn}dS\\&=\sum a\xi\xi_0.\end{aligned}$$

The result can also be established by considering the direction of intersection of

the surfaces, the perpendiculars to this direction in the respective surfaces, and still taking η as the inclination of these perpendiculars.

Ex. 1. Shew that the bisectors of two directions in the region lying in the orientation ξ, so that

$$\xi p' + \eta q' + \zeta r' = 0,$$

while their variables satisfy the equation

$$a_{11}p'^2 + 2a_{12}p'q' + a_{22}q'^2 + 2a_{13}p'r' + 2a_{23}q'r' + a_{33}r'^2 = 0,$$

have variables satisfying the equation

$$\begin{vmatrix} a_{11}p' + a_{12}q' + a_{13}r', & Ap' + Hq' + Gr', & \xi \\ a_{21}p' + a_{22}q' + a_{23}r', & Hp' + Bq' + Fr', & \eta \\ a_{31}p' + a_{32}q' + a_{33}r', & Gp' + Fq' + Cr', & \zeta \end{vmatrix} = 0.$$

Ex. 2. Two orientations in the region are composed of the directions 1, 2 ; 1, 3 ; respectively. Shew that

$$\sum a\xi_{12}\xi_{13} = \cos \widehat{23} - \cos \widehat{12} \cos \widehat{13},$$

where $\widehat{ab}$ denotes the angle between the directions p_a', q_a', r_a' : and p_b', q_b', r_b'.

Christoffel symbols for a region.

160. There are various sets of combinations of second derivatives of the space-coordinates of a point in a region.

The simplest of the sets defines magnitudes Γ, Δ, Θ, according to the relations

$$\left.\begin{aligned} \sum y_1 y_{ij} &= A\Gamma_{ij} + H\Delta_{ij} + G\Theta_{ij} \\ \sum y_2 y_{ij} &= H\Gamma_{ij} + B\Delta_{ij} + F\Theta_{ij} \\ \sum y_3 y_{ij} &= G\Gamma_{ij} + F\Delta_{ij} + C\Theta_{ij} \end{aligned}\right\},$$

for all the combinations $i, j, = 1, 2, 3$; the subscripts in Γ_{ij}, Δ_{ij}, Θ_{ij}, are merely symbolic, to connote association with the derivatives y_{ij}, where the subscripts denote differentiations. When the Christoffel symbols for any multiple space are used, there are the equivalences

$$\sum y_1 y_{ij} = [ij, 1], \quad \sum y_2 y_{ij} = [ij, 2], \quad \sum y_3 y_{ij} = [ij, 3],$$
$$\Gamma_{ij} = \{ij, 1\}, \qquad \Delta_{ij} = \{ij, 2\}, \qquad \Theta_{ij} = \{ij, 3\}.$$

The first derivatives of the primary magnitudes A, B, C, F, G, H, and the first derivatives of a, b, c, f, g, h, the minors of Ω, can be expressed in terms of Γ, Δ, Θ, the expressions being

$$\left.\begin{aligned} \tfrac{1}{2}A_i &= A\Gamma_{1i} + H\Delta_{1i} + G\Theta_{1i} \\ \tfrac{1}{2}B_i &= H\Gamma_{2i} + B\Delta_{2i} + F\Theta_{2i} \\ \tfrac{1}{2}C_i &= G\Gamma_{3i} + F\Delta_{3i} + C\Theta_{3i} \end{aligned}\right\},$$

$$\left.\begin{aligned} F_i&=(H\Gamma_{3i}+B\Delta_{3i}+F\Theta_{3i})+(G\Gamma_{2i}+F\Delta_{2i}+C\Theta_{2i})\\ G_i&=(G\Gamma_{1i}+F\Delta_{1i}+C\Theta_{1i})+(A\Gamma_{3i}+H\Delta_{3i}+G\Theta_{3i})\\ H_i&=(A\Gamma_{2i}+H\Delta_{2i}+G\Theta_{2i})+(H\Gamma_{1i}+B\Delta_{1i}+F\Theta_{1i})\end{aligned}\right\},$$

$$\tfrac{1}{2}\frac{\Omega_i}{\Omega}=\Gamma_{1i}+\Delta_{2i}+\Theta_{3i},$$

$$\left.\begin{aligned} \tfrac{1}{2}\left(a_i-a\frac{\Omega_i}{\Omega}\right)&=-(a\Gamma_{1i}+h\Gamma_{2i}+g\Gamma_{3i})\\ \tfrac{1}{2}\left(b_i-b\frac{\Omega_i}{\Omega}\right)&=-(h\Delta_{1i}+b\Delta_{2i}+f\Delta_{3i})\\ \tfrac{1}{2}\left(c_i-c\frac{\Omega_i}{\Omega}\right)&=-(g\Theta_{1i}+f\Theta_{2i}+c\Theta_{3i})\end{aligned}\right\},$$

$$\left.\begin{aligned} f_i-f\frac{\Omega_i}{\Omega}&=-(g\Delta_{1i}+f\Delta_{2i}+c\Delta_{3i})-(h\Theta_{1i}+b\Theta_{2i}+f\Theta_{3i})\\ g_i-g\frac{\Omega_i}{\Omega}&=-(a\Theta_{1i}+h\Theta_{2i}+g\Theta_{3i})-(g\Gamma_{1i}+f\Gamma_{2i}+c\Gamma_{3i})\\ h_i-h\frac{\Omega_i}{\Omega}&=-(h\Gamma_{1i}+b\Gamma_{2i}+f\Gamma_{3i})-(a\Delta_{1i}+h\Delta_{2i}+g\Delta_{3i})\end{aligned}\right\},$$

for all the values $i=1, 2, 3$.

These results can be more compendiously expressed by taking

$$E_{ij}=\sum y_i y_j,$$

using e_{ij} to represent the minor of E_{ij} in Ω, and using t_1, t_2, t_3, to denote p, q, r, respectively; then

$$\frac{\partial E_{ij}}{\partial t_k}=(E_{1j}\Gamma_{ki}+E_{2j}\Delta_{ki}+E_{3j}\Theta_{ki})+(E_{1i}\Gamma_{kj}+E_{2i}\Delta_{kj}+E_{3i}\Theta_{kj}),$$

$$-\Omega\frac{\partial}{\partial t_k}\left(\frac{e_{ij}}{\Omega}\right)=e_{1j}\{k1, i\}+e_{2j}\{k2, i\}+e_{3j}\{k3, i\}+e_{1i}\{k1, j\}+e_{2i}\{k2, j\}+e_{3i}\{k3, j\},$$

for all the combinations i, j, k, $=1, 2, 3$.

All the formulæ are, in fact, particularised instances of the formulæ, as given in § 12, relating to the corresponding magnitudes of a configuration of n dimensions existing in a multiple plenary space. They can, of course, be established at once, without recourse to the general formulæ.

Regional geodesics.

161. To obtain the intrinsic equations of geodesics in the region, we minimise the integral

$$\int\left\{\sum A\left(\frac{dp}{dt}\right)^2\right\}^{\frac{1}{2}}dt.$$

The three critical equations are, first,

$$\frac{d}{dt}\left(A\frac{dp}{ds}+H\frac{dq}{ds}+G\frac{dr}{ds}\right)=\tfrac{1}{2}\left\{\sum A_1\left(\frac{dp}{ds}\right)^2\right\}\frac{ds}{dt}:$$

that is,

$$\frac{d}{ds}\left(A\frac{dp}{ds}+H\frac{dq}{ds}+G\frac{dr}{ds}\right)=\tfrac{1}{2}\sum A_1\left(\frac{dp}{ds}\right)^2,$$

which easily reduces to

$$\begin{aligned}Ap''+Hq''+Gr''=&-\tfrac{1}{2}A_1p'^2-A_2p'q'-A_3p'r'\\&+(\tfrac{1}{2}B_1-H_2)q'^2+(F_1-H_3-G_2)q'r'+(\tfrac{1}{2}C_1-G_3)r'^2.\end{aligned}$$

In accordance with previous notations, we write

$$\sum\Phi_{11}p'^2=\Phi_{11}p'^2+2\Phi_{12}p'q'+\Phi_{22}q'^2+2\Phi_{13}p'r'+2\Phi_{23}q'r'+\Phi_{33}r'^2,$$

for $\Phi=\Gamma, \Delta, \Theta$, successively; and now the right-hand side can be expressed in the form

$$-A\sum\Gamma_{11}p'^2-H\sum\Delta_{11}p'^2-G\sum\Theta_{11}p'^2,$$

so that the first critical equation becomes

$$A(p''+\sum\Gamma_{11}p'^2)+H(q''+\sum\Delta_{11}p'^2)+G(r''+\sum\Theta_{11}p'^2)=0.$$

The other two critical equations similarly become

$$\begin{aligned}H(p''+\sum\Gamma_{11}p'^2)+B(q''+\sum\Delta_{11}p'^2)+F(r''+\sum\Theta_{11}p'^2)=0,\\ G(p''+\sum\Gamma_{11}p'^2)+F(q''+\sum\Delta_{11}p'^2)+C(r''+\sum\Theta_{11}p'^2)=0.\end{aligned}$$

As the determinant Ω of the coefficients on the left-hand side does not vanish, these equations are equivalent to the set

$$p''+\sum\Gamma_{11}p'^2=0,\quad q''+\sum\Delta_{11}p'^2=0,\quad r''+\sum\Theta_{11}p'^2=0,$$

which accordingly are the intrinsic equations of a geodesic in the region.

The three equations provide only two independent equations when account is taken of the permanent relation

$$\sum Ap'^2=1;$$

for differentiation of this relation along any arc in the region yields the condition

$$(Ap'+Hq'+Gr')p''+(Hp'+Bq'+Fr')q''+(Gp'+Fq'+Cr')r''+\tfrac{1}{2}\sum\frac{dA}{ds}p'^2=0.$$

Now

$$\begin{aligned}\frac{dA}{ds}&=A_1p'+A_2q'+A_3r'\\&=2A(\Gamma_{11}p'+\Gamma_{12}q'+\Gamma_{13}r')+2H(\Delta_{11}p'+\Delta_{12}q'+\Delta_{13}r')\\&\qquad+2G(\Theta_{11}p'+\Theta_{12}q'+\Theta_{13}r'),\\ \frac{dH}{ds}&=H_1p'+H_2q'+H_3r'\\&=A(\Gamma_{12}p'+\Gamma_{22}q'+\Gamma_{23}r')+H(\Delta_{12}p'+\Delta_{22}q'+\Delta_{23}r')+G(\Theta_{12}p'+\Theta_{22}q'+\Theta_{23}r')\\&\qquad+H(\Gamma_{11}p'+\Gamma_{12}q'+\Gamma_{13}r')+B(\Delta_{11}p'+\Delta_{12}q'+\Delta_{13}r')\\&\qquad\qquad+F(\Theta_{11}p'+\Theta_{12}q'+\Theta_{13}r'),\end{aligned}$$

and similarly for the others ; hence

$$\begin{aligned}\tfrac{1}{2}\sum\frac{dA}{ds}p'^2 &= \tfrac{1}{2}\frac{dA}{ds}p'^2+\frac{dH}{ds}p'q'+\frac{dG}{ds}p'r'+\tfrac{1}{2}\frac{dB}{ds}q'^2+\frac{dF}{ds}q'r'+\tfrac{1}{2}\frac{dC}{ds}r'^2\\ &= (Ap'+Hq'+Gr')(\textstyle\sum\Gamma_{11}p'^2)\\ &\qquad +(Hp'+Bq'+Fr')(\textstyle\sum\Delta_{11}q'^2)\\ &\qquad\qquad +(Gp'+Fq'+Cr')(\textstyle\sum\Theta_{11}q'^2).\end{aligned}$$

Thus the differentiated permanent relation, affecting all arcs in the region, leads to the necessary condition

$$\begin{aligned}&(Ap'+Hq'+Gr')(p''+\textstyle\sum\Gamma_{11}p'^2)\\ &\qquad +(Hp'+Bq'+Fr')(q''+\textstyle\sum\Delta_{11}q'^2)\\ &\qquad\qquad +(Gp'+Fq'+Cr')(r''+\textstyle\sum\Theta_{11}q'^2)=0.\end{aligned}$$

It follows, from this universal condition, that the three critical equations for regional geodesics are effectively equivalent to only two independent equations, under the permanent arc-relation which applies to all arcs in the region.

Let the direction-cosines, in the plenary homaloidal space of the region, of the prime normal of a regional geodesic be denoted by $Y_1, Y_2, \ldots, Y_N$; a typical direction-cosine, corresponding to the typical point-variable y, will be denoted by Y. When ρ is used to denote the radius of circular curvature of the geodesic, then

$$Y=\rho y'',$$

so that

$$\begin{aligned}\frac{Y}{\rho} &= y''=y_1p''+y_2q''+y_3r''\\ &\qquad +y_{11}p'^2+2y_{12}p'q'+2y_{13}p'r'+y_{22}q'^2+2y_{23}q'r'+y_{33}r'^2.\end{aligned}$$

On the right-hand side, we substitute for p'', q'', r'', their values in terms of p', q', r', which arise from the property of the geodesic; and we introduce symbols η_{ij}, for all the combinations $i, j, =1, 2, 3$, according to the definition

$$\eta_{ij}=y_{ij}-y_1\Gamma_{ij}-y_2\Delta_{ij}-y_3\Theta_{ij}.$$

Then the typical direction-cosine of the prime normal of the geodesic is given by the equation

$$\frac{Y}{\rho}=\eta_{11}p'^2+2\eta_{12}p'q'+2\eta_{13}p'r'+\eta_{22}q'^2+2\eta_{23}q'r'+\eta_{33}r'^2,$$

which often will be taken in the form

$$\frac{Y}{\rho}=\sum\eta_{11}p'^2.$$

There are N such equations, corresponding to the N dimensions of the homaloidal plenary space. There is also the general relation $\sum Y^2=1$, the unspecified axes

in the plenary space being assumed an orthogonal set. Thus the equations are adequate to determine, both the circular curvature of the regional geodesic and the direction of its prime normal.

We note, in passing, one property of the symbols η_{ij}. We have

$$\sum y_1\eta_{ij}=0, \quad \sum y_2\eta_{ij}=0, \quad \sum y_3\eta_{ij}=0,$$

verified by substituting the value of η_{ij} and noting the original definition of the magnitudes Γ, Δ, Θ.

Ex. A region of constant positive sphericity $1/\kappa$ has its arc-element represented by the equation

$$ds^2=\frac{1}{D^2}(dp^2+dq^2+dr^2),$$

where

$$D=1+\frac{1}{4\kappa}(p^2+q^2+r^2).$$

Obtain the general integral equations of its geodesics * in the form

$$\frac{2}{D}=1+\cos a\cos t,$$

$$p=2\kappa^{\frac{1}{2}}(D-1)^{\frac{1}{2}}\cos\mu\cos(u+\beta),$$
$$q=2\kappa^{\frac{1}{2}}(D-1)^{\frac{1}{2}}\{\cos\lambda\sin(u+\beta)-\sin\lambda\sin\mu\sin(u+\beta)\},$$
$$r=2\kappa^{\frac{1}{2}}(D-1)^{\frac{1}{2}}\{\sin\lambda\sin(u+\beta)+\cos\lambda\sin\mu\cos(u+\beta),$$

where $t=s\kappa^{-\frac{1}{2}}$, and

$$\tan t=\tan u\sin a,$$

the quantities a, β, λ, μ, being arbitrary constants, and an arbitrary constant being absorbed into the variable s.

Obtain the corresponding equations for a region of constant negative sphericity.

Derivatives of the quantities Γ, Δ, Θ.

162. We shall require the first derivatives of the magnitudes Γ, Δ, Θ. In the first place, certain linear relations between these derivatives are to be satisfied, arising out of identities of the type

$$\frac{\partial E_3}{\partial q}=\frac{\partial E_2}{\partial r}, \quad \frac{\partial E_1}{\partial r}=\frac{\partial E_3}{\partial p}, \quad \frac{\partial E_2}{\partial p}=\frac{\partial E_1}{\partial q},$$

where E denotes any one of the six primary magnitudes A, B, C, F, G, H; and in the second place, the actual values of these derivatives are used in connection with third parametric derivatives of the spatial coordinates.

To formulate the relations, we use the notation

$$\begin{aligned}(*\between ij\between kl)= \; & A\Gamma_{ij}\Gamma_{kl}+F(\Delta_{ij}\Theta_{kl}+\Theta_{ij}\Delta_{kl})\\ &+B\Delta_{ij}\Delta_{kl}+G(\Theta_{ij}\Gamma_{kl}+\Gamma_{ij}\Theta_{kl})\\ &+C\Theta_{ij}\Theta_{kl}+H(\Gamma_{ij}\Delta_{kl}+\Delta_{ij}\Gamma_{kl}),\end{aligned}$$

* The result is the special case of *Ex.* 1 in § 17 when $n=3$.

for all the combinations $i, j, k, l, =1, 2, 3$, independently of one another. To obtain the necessary relations, we first construct certain combinations of the second parametric derivatives of the spatial variables. For example, we have

$$\sum y_2 y_{11} = H_1 - \tfrac{1}{2}A_2, \quad \sum y_2 y_{12} = \tfrac{1}{2}B_1,$$

and therefore

$$\begin{aligned} -\tfrac{1}{2}(A_{22} - 2H_{12} + B_{11}) &= \sum y_2 y_{112} + \sum y_{11} y_{22} - \sum y_2 y_{112} - \sum y_{12}^2 \\ &= \sum (y_{11} y_{22} - y_{12}^2). \end{aligned}$$

Now we had quantities η_{ij} such that

$$y_{ij} = \eta_{ij} + y_1 \Gamma_{ij} + y_2 \Delta_{ij} + y_3 \Theta_{ij},$$

while

$$\sum y_k \eta_{ij} = 0,$$

for all the admissible values of i, j, k; and therefore

$$\sum y_{ij} y_{kl} = \sum \eta_{ij} \eta_{kl} + (* \between ij \between kl),$$

for all combinations. Consequently

$$-\tfrac{1}{2}(A_{22} - 2H_{12} + B_{11}) = \sum \eta_{11}\eta_{22} - \sum \eta_{12}^2 + (* \between 11 \between 22) - (* \between 12 \between 12).$$

There are six such relations for a region, the other five being obtained by a similar process: in full tale, they are

$$\left.\begin{aligned} (23, 23) = k_{11} &= \sum \eta_{22}\eta_{33} - \sum \eta_{23}^2 \\ &= -\tfrac{1}{2}(B_{33} - 2F_{23} + C_{22}) - (* \between 22 \between 33) + (* \between 23 \between 23) \\ (31, 31) = k_{22} &= \sum \eta_{33}\eta_{11} - \sum \eta_{31}^2 \\ &= -\tfrac{1}{2}(C_{11} - 2G_{31} + A_{33}) - (* \between 33 \between 11) + (* \between 31 \between 31) \\ (12, 12) = k_{33} &= \sum \eta_{11}\eta_{22} - \sum \eta_{12}^2 \\ &= -\tfrac{1}{2}(A_{22} - 2H_{12} + B_{11}) - (* \between 11 \between 22) + (* \between 12 \between 12) \\ (12, 31) = k_{23} &= \sum \eta_{12}\eta_{13} - \sum \eta_{11}\eta_{23} \\ &= \tfrac{1}{2}(A_{23} - G_{12} - H_{13} + F_{11}) - (* \between 12 \between 13) + (* \between 11 \between 23) \\ (23, 12) = k_{31} &= \sum \eta_{23}\eta_{21} - \sum \eta_{22}\eta_{31} \\ &= \tfrac{1}{2}(B_{31} - H_{23} - F_{21} + G_{22}) - (* \between 23 \between 21) + (* \between 22 \between 31) \\ (31, 23) = k_{12} &= \sum \eta_{31}\eta_{32} - \sum \eta_{33}\eta_{12} \\ &= \tfrac{1}{2}(C_{12} - F_{31} - G_{32} + H_{33}) - (* \between 31 \between 32) + (* \between 33 \between 12) \end{aligned}\right\}.$$

It will be noted that these quantities k_{ij}, in value, are the six non-vanishing four-index Riemann symbols of a region, as in § 16.

Ex. Verify, as in the example in § 90, that all the six quantities k_{ij} vanish when the region is a flat.

In the next place, we consider certain combinations of third-order parametric derivatives of the space-coordinates of any position in the region. On the analogy of the definition of the Christoffel symbols $[ij, k]$, we define symbols R_{ijk}, S_{ijk}, T_{ijk}, according to the scheme

$$\left.\begin{aligned}\sum y_1\eta_{ijk} &= AR_{ijk}+HS_{ijk}+GT_{ijk}\\ \sum y_2 y_{ijk} &= HR_{ijk}+BS_{ijk}+FT_{ijk}\\ \sum y_3 y_{ijk} &= GR_{ijk}+FS_{ijk}+CT_{ijk}\end{aligned}\right\},$$

for all combinations $i, j, k, =1, 2, 3$, in all possible combinations, ten in number.

There was a relation

$$\sum y_1 y_{ij} = A\Gamma_{ij}+H\Delta_{ij}+G\Theta_{ij},$$

holding for the six possible combinations ij; hence, differentiating with respect to x_k (where $x_1=p$, $x_2=q$, $x_3=r$), we have

$$\begin{aligned}&\sum y_1 y_{ijk}+\sum y_{1k}y_{ij}\\ &= A\frac{\partial\Gamma_{ij}}{\partial x_k}+2\Gamma_{ij}(A\Gamma_{1k}+H\Delta_{1k}+G\Theta_{1k})\\ &\quad+H\frac{\partial\Delta_{ij}}{\partial x_k}+\Delta_{ij}(A\Gamma_{2k}+H\Delta_{2k}+G\Theta_{2k}+H\Gamma_{1k}+B\Delta_{1k}+F\Theta_{1k})\\ &\quad+G\frac{\partial\Theta_{ij}}{\partial x_k}+\Theta_{ij}(A\Gamma_{3k}+H\Delta_{3k}+G\Theta_{3k}+G\Gamma_{1k}+F\Delta_{1k}+C\Theta_{1k}).\end{aligned}$$

When we substitute for $\sum y_{1k}y_{ij}$ its value obtained on p. 462, this equation becomes

$$\begin{aligned}\sum y_1 y_{ijk}+\sum \eta_{1k}\eta_{ij} = \; & A\left(\frac{\partial\Gamma_{ij}}{\partial x_k}+\Gamma_{ij}\Gamma_{1k}+\Delta_{ij}\Gamma_{2k}+\Theta_{ij}\Gamma_{3k}\right)\\ &+H\left(\frac{\partial\Delta_{ij}}{\partial x_k}+\Gamma_{ij}\Delta_{1k}+\Delta_{ij}\Delta_{2k}+\Theta_{ij}\Delta_{3k}\right)\\ &+G\left(\frac{\partial\Theta_{ij}}{\partial x_k}+\Gamma_{ij}\Theta_{1k}+\Delta_{ij}\Theta_{2k}+\Theta_{ij}\Theta_{3k}\right).\end{aligned}$$

Proceeding from the relations

$$\sum y_2 y_{ij} = H\Gamma_{ij}+B\Delta_{ij}+G\Theta_{ij}, \quad \sum y_3 y_{ij} = G\Gamma_{ij}+F\Delta_{ij}+C\Theta_{ij},$$

differentiating with respect to x_k, and re-arranging, we obtain expressions for

$$\sum y_2 y_{ijk}+\sum \eta_{2k}\eta_{ij}, \quad \sum y_3 y_{ijk}+\sum \eta_{3k}\eta_{ij},$$

similar to the preceding expression, with coefficients H, B, F, in the first, and coefficients G, F, C, in the second.

In the three equations, let the values of the quantities

$$\sum y_l y_{ijk} \quad (l=1, 2, 3)$$

be substituted; there then are three equations, linear in the magnitudes R_{ijk}, S_{ijk}, T_{ijk}; and when they are resolved, they give the results

$$R_{ijk}=\frac{\partial\Gamma_{ij}}{\partial x_k}+\Gamma_{ij}\Gamma_{1k}+\Delta_{ij}\Gamma_{2k}+\Theta_{ij}\Gamma_{3k}-\frac{1}{\Omega}(a\sum\eta_{ij}\eta_{1k}+h\sum\eta_{ij}\eta_{2k}+g\sum\eta_{ij}\eta_{3k}),$$

$$S_{ijk}=\frac{\partial\Delta_{ij}}{\partial x_k}+\Gamma_{ij}\Delta_{1k}+\Delta_{ij}\Delta_{2k}+\Theta_{ij}\Delta_{3k}-\frac{1}{\Omega}(h\sum\eta_{ij}\eta_{1k}+b\sum\eta_{ij}\eta_{2k}+f\sum\eta_{ij}\eta_{3k}),$$

$$T_{ijk}=\frac{\partial\Theta_{ij}}{\partial x_k}+\Gamma_{ij}\Theta_{1k}+\Delta_{ij}\Theta_{2k}+\Theta_{ij}\Theta_{3k}-\frac{1}{\Omega}(g\sum\eta_{ij}\eta_{1k}+f\sum\eta_{ij}\eta_{2k}+c\sum\eta_{ij}\eta_{3k}).$$

Now in the definition of these symbols R_{ijk}, S_{ijk}, T_{ijk}, re-arrangement of the numbers in the combination ijk leaves the values unaltered. The same property therefore must appertain to the values obtained for those symbols respectively; and therefore there must be relations

$$\begin{aligned}
&\frac{\partial\Gamma_{ij}}{\partial x_k}+\Gamma_{ij}\Gamma_{1k}+\Delta_{ij}\Gamma_{2k}+\Theta_{ij}\Gamma_{3k}-\frac{1}{\Omega}(a\sum\eta_{ij}\eta_{1k}+h\sum\eta_{ij}\eta_{2k}+g\sum\eta_{ij}\eta_{3k})\\
&\quad=\frac{\partial\Gamma_{jk}}{\partial x_i}+\Gamma_{jk}\Gamma_{1i}+\Delta_{jk}\Gamma_{2i}+\Theta_{jk}\Gamma_{3i}-\frac{1}{\Omega}(a\sum\eta_{jk}\eta_{1i}+h\sum\eta_{jk}\eta_{2i}+g\sum\eta_{jk}\eta_{3i})\\
&\quad=\frac{\partial\Gamma_{ki}}{\partial x_j}+\Gamma_{ki}\Gamma_{1j}+\Delta_{ki}\Gamma_{2j}+\Theta_{ki}\Gamma_{3j}-\frac{1}{\Omega}(a\sum\eta_{ki}\eta_{1j}+h\sum\eta_{ki}\eta_{2j}+g\sum\eta_{ki}\eta_{3j}),
\end{aligned}$$

arising out of the value of R_{ijk}: relations

$$\begin{aligned}
&\frac{\partial\Delta_{ij}}{\partial x_k}+\Gamma_{ij}\Delta_{1k}+\Delta_{ij}\Delta_{2k}+\Theta_{ij}\Delta_{3k}-\frac{1}{\Omega}(h\sum\eta_{ij}\eta_{1k}+b\sum\eta_{ij}\eta_{2k}+f\sum\eta_{ij}\eta_{3k})\\
&\quad=\frac{\partial\Delta_{jk}}{\partial x_i}+\Gamma_{jk}\Delta_{1i}+\Delta_{jk}\Delta_{2i}+\Theta_{jk}\Delta_{3i}-\frac{1}{\Omega}(h\sum\eta_{jk}\eta_{1i}+b\sum\eta_{jk}\eta_{2i}+f\sum\eta_{jk}\eta_{3i})\\
&\quad=\frac{\partial\Delta_{ki}}{\partial x_j}+\Gamma_{ki}\Delta_{1j}+\Delta_{ki}\Delta_{2j}+\Theta_{ki}\Delta_{3j}-\frac{1}{\Omega}(h\sum\eta_{ki}\eta_{1j}+b\sum\eta_{ki}\eta_{2j}+f\sum\eta_{ki}\eta_{3j}),
\end{aligned}$$

arising out of the value S_{ijk}: and relations

$$\begin{aligned}
&\frac{\partial\Theta_{ij}}{\partial x_k}+\Gamma_{ij}\Theta_{1k}+\Delta_{ij}\Theta_{2k}+\Theta_{ij}\Theta_{3k}-\frac{1}{\Omega}(g\sum\eta_{ij}\eta_{1k}+f\sum\eta_{ij}\eta_{2k}+c\sum\eta_{ij}\eta_{3k})\\
&\quad=\frac{\partial\Theta_{jk}}{\partial x_i}+\Gamma_{jk}\Theta_{1i}+\Delta_{jk}\Theta_{2i}+\Theta_{jk}\Theta_{3i}-\frac{1}{\Omega}(g\sum\eta_{jk}\eta_{1i}+f\sum\eta_{jk}\eta_{2i}+c\sum\eta_{jk}\eta_{3i})\\
&\quad=\frac{\partial\Theta_{ki}}{\partial x_j}+\Gamma_{ki}\Theta_{1j}+\Delta_{ki}\Theta_{2j}+\Theta_{ki}\Theta_{3j}-\frac{1}{\Omega}(g\sum\eta_{ki}\eta_{1j}+f\sum\eta_{ki}\eta_{2j}+c\sum\eta_{ki}\eta_{3j}),
\end{aligned}$$

arising out of the value T_{ijk}.

It is easy to see that there are eight significant relations affecting derivatives of the quantities Γ, eight affecting those of Δ, and eight affecting those of Θ. As there are six quantities Γ, and as there are three possible derivatives of each, it follows that the eighteen first derivatives of the quantities Γ provide only ten

independent quantities, which is in fact the number of magnitudes R_{ijk}. Similarly for the magnitudes S_{ijk}, T_{ijk}.

Moreover, these relations can be arranged so as to be expressible in terms of the Riemann four-index symbols instead of the quantities $\sum \eta_{ij}\eta_{kl}$, and therefore in terms of the magnitudes k_{ij} which (as already pointed out) are the six Riemann symbols that do not vanish for a region. In fact, we have

$$\Omega\left\{\left(\frac{\partial \Phi_{ij}}{\partial x_k}+\Phi_{1k}\Gamma_{ij}+\Phi_{2k}\Delta_{ij}+\Phi_{3k}\Theta_{ij}\right)-\left(\frac{\partial \Phi_{jk}}{\partial x_i}+\Phi_{1i}\Gamma_{jk}+\Phi_{2i}\Delta_{jk}+\Phi_{3i}\Theta_{jk}\right)\right\}$$
$$=\alpha(j1, ik)+\beta(j2, ik)+\gamma(j3, ik),$$

where
$$\alpha,\ \beta,\ \gamma=a,\ h,\ g, \text{ when } \Phi=\Gamma,$$
$$=h,\ b,\ f, \text{ when } \Phi=\Delta,$$
$$=g,\ f,\ c, \text{ when } \Phi=\Theta.$$

The properties of these four-index symbols are given in § 16; in particular, the quantity denoted by a symbol (ij, kl) vanishes if $i=j$ or if $k=l$. And all these results can be compared with the corresponding results given (in § 22) for a general amplitude, of which they are the special instances when the amplitude is a region.

Values of p''', q''', r''', along a regional geodesic.

163. To obtain the values of p''', q''', r''', along a regional geodesic, we differentiate its characteristic equations. Thus the equation

$$-p''=\sum \Gamma_{11}p'^2,$$

differentiated along that geodesic, gives

$$\begin{aligned}-p'''=\frac{d\Gamma_{11}}{ds}p'^2&+2\frac{d\Gamma_{12}}{ds}p'q'+\frac{d\Gamma_{22}}{ds}q'^2+2\frac{d\Gamma_{13}}{ds}p'r'+2\frac{d\Gamma_{23}}{ds}q'r'+\frac{d\Gamma_{33}}{ds}r'^2\\&+2(\Gamma_{11}p'+\Gamma_{12}q'+\Gamma_{13}r')p''\\&+2(\Gamma_{12}p'+\Gamma_{13}q'+\Gamma_{23}r')q''\\&+2(\Gamma_{13}p'+\Gamma_{23}q'+\Gamma_{33}r')r''.\end{aligned}$$

When substitution is made for the arc-derivatives of Γ_{ij}, and for p'', q'', r'', the equation can be taken in the form

$$\begin{aligned}-p'''=\Gamma_{111}p'^3+3\Gamma_{112}p'^2q'&+3\Gamma_{122}p'q'^2+\Gamma_{222}q'^3=\sum \Gamma_{111}p'^3.\\+3\Gamma_{113}p'^2r'&+6\Gamma_{123}p'q'r'+3\Gamma_{223}q'^2r'\\&+3\Gamma_{133}p'r'^2+3\Gamma_{233}q'r'^2\\&+\Gamma_{333}r'^3\end{aligned}$$

To express these coefficients Γ_{ijk}, we use an abbreviated symbol with the definition

$$\Phi_{1\mu}\Gamma_{ij}+\Phi_{2\mu}\Delta_{ij}+\Phi_{3\mu}\Theta_{ij}=\Phi_{\mu}(ij),$$

for $\mu=1, 2, 3$, and $\Phi=\Gamma, \Delta, \Theta$. Then

$$\Gamma_{111}=\frac{\partial \Gamma_{11}}{\partial p}-2\Gamma_1(11),$$

$$3\Gamma_{112}=\frac{\partial \Gamma_{11}}{\partial q}+2\frac{\partial \Gamma_{12}}{\partial p}-4\Gamma_1(12)-2\Gamma_2(11),$$

and so on. Now the parametric derivatives of Γ, Δ, Θ, have already received one form of expression, by means of the quantities R_{ijk}, S_{ijk}, T_{ijk}, in § 162. When these are used, we find two kinds of relations: first, those connecting the magnitudes Γ_{ijk} and R_{ijk}, similar to the relations in § 23 connecting

$$\begin{bmatrix} ijk \\ \mu \end{bmatrix} \text{ and } \{ijk, \mu\}:$$

next, those expressing the derivatives of Γ_{ij} individually in terms of the quantities Γ_{ijk}. Thus we have

$$R_{112}=\frac{\partial \Gamma_{11}}{\partial q}+\Gamma_2(11)-\frac{1}{\Omega}\{a(\sum \eta_{11}\eta_{12})+h(\sum \eta_{11}\eta_{22})+g(\sum \eta_{11}\eta_{23})\},$$

$$R_{112}=\frac{\partial \Gamma_{12}}{\partial p}+\Gamma_1(12)-\frac{1}{\Omega}\{a(\sum \eta_{12}\eta_{11})+h(\sum \eta_{12}\eta_{12})+g(\sum \eta_{12}\eta_{13})\};$$

when these are combined with the foregoing expression for Γ_{112}, we find

$$\begin{aligned} R_{112}-\Gamma_{112}=2\Gamma_1(12)&+\Gamma_2(11) \\ &-\frac{1}{3\Omega}[3a(\sum \eta_{11}\eta_{12})+h\{(\sum \eta_{11}\eta_{22})+2(\sum \eta_{12}\eta_{12})\} \\ &\qquad +g\{(\sum \eta_{11}\eta_{23})+2(\sum \eta_{12}\eta_{13})\}], \end{aligned}$$

$$\frac{\partial \Gamma_{11}}{\partial q}=\Gamma_{112}+2\Gamma_1(12)+\frac{2}{3\Omega}(hk_{33}-gk_{23}),$$

$$\frac{\partial \Gamma_{12}}{\partial p}=\Gamma_{112}+\Gamma_1(12)+\Gamma_2(11)+\frac{1}{3\Omega}(-hk_{33}+gk_{23}),$$

and so for all the other coefficients. The full tale of values is

$$\begin{aligned} \frac{\partial \Gamma_{ij}}{\partial x_k}=\Gamma_{ijk}&+\Gamma_i(jk)+\Gamma_j(ik) \\ &-\frac{1}{3\Omega}[a\{(1i, jk)+(1j, ik)\}+h\{(2i, jk)+(2j, ik)\}+g\{(3i, jk)+(3j, ik)\}], \end{aligned}$$

$$\begin{aligned} R_{ijk}=\Gamma_{ijk}&+\Gamma_i(jk)+\Gamma_j(ki)+\Gamma_k(ij) \\ &-\frac{1}{3\Omega}[a\{\sum(\eta_{1i}\eta_{jk}+\eta_{1j}\eta_{ki}+\eta_{1k}\eta_{ij})\} \\ &\qquad +h\{\sum(\eta_{2i}\eta_{jk}+\eta_{2j}\eta_{ki}+\eta_{2k}\eta_{ij})\} \\ &\qquad\qquad +g\{\sum(\eta_{3i}\eta_{jk}+\eta_{3j}\eta_{ki}+\eta_{3k}\eta_{ij})\}], \end{aligned}$$

with the conventions $x_1=p$, $x_2=q$, $x_3=r$.

Similarly for q''', r''': we have, in all

$$-p'''=\sum \Gamma_{111}p'^3, \quad -q'''=\sum \Delta_{111}p'^3, \quad -r'''=\sum \Theta_{111}p'^3.$$

The relations between the quantities S_{ijk}, the parametric derivatives of Δ_{ij}, and the magnitudes Δ_{ijk}; and the relations between the quantities T_{ijk}, the parametric derivatives of Θ_{ij}, and the magnitudes Θ_{ijk}; respectively are

$$\begin{aligned}\frac{\partial\Delta_{ij}}{\partial x_k}&=\Delta_{ijk}+\Delta_i(jk)+\Delta_j(ik)\\&\quad-\frac{1}{3\Omega}[h\{(1i, jk)+(1j, ik)\}+b\{(2i, jk)+(2j, ik)\}+f\{(3i, jk)+(3j, ik)\}],\end{aligned}$$

$$\begin{aligned}S_{ijk}&=\Delta_{ijk}+\Delta_i(jk)+\Delta_j(ki)+\Delta_k(ij)\\&\quad-\frac{1}{3\Omega}[h\{\textstyle\sum(\eta_{1i}\eta_{jk}+\eta_{1j}\eta_{ki}+\eta_{1k}\eta_{ij})\}\\&\qquad+b\{\textstyle\sum(\eta_{2i}\eta_{jk}+\eta_{2j}\eta_{ki}+\eta_{2k}\eta_{ij})\}\\&\qquad\quad+f\{\textstyle\sum(\eta_{3i}\eta_{jk}+\eta_{3j}\eta_{ki}+\eta_{3k}\eta_{ij})\}];\end{aligned}$$

$$\begin{aligned}\frac{\partial\Theta_{ij}}{\partial x_k}&=\Theta_{ijk}+\Theta_i(jk)+\Theta_j(ik)\\&\quad-\frac{1}{3\Omega}[g\{(1i, jk)+(1j, ik)\}+f\{(2i, jk)+(2j, ik)\}+c\{(3i, jk)+(3j, ik)\}],\end{aligned}$$

$$\begin{aligned}T_{ijk}&=\Theta_{ijk}+\Theta_i(jk)+\Theta_j(ki)+\Theta_k(ij)\\&\quad-\frac{1}{3\Omega}[g\{\textstyle\sum(\eta_{1i}\eta_{jk}+\eta_{1j}\eta_{ki}+\eta_{1k}\eta_{ij})\}\\&\qquad+f\{\textstyle\sum(\eta_{2i}\eta_{jk}+\eta_{2j}\eta_{ki}+\eta_{2k}\eta_{ij})\}\\&\qquad\quad+c\{\textstyle\sum(\eta_{3i}\eta_{jk}+\eta_{3j}\eta_{ki}+\eta_{3k}\eta_{ij})\}].\end{aligned}$$

The relations between the derivatives of Γ_{ij} and the magnitudes R_{ijk}, as given on p. 464, are verifiable by the foregoing values; and likewise for the derivatives of Δ_{ij} and the magnitudes S_{ijk}, and for the derivatives of Θ_{ij} and the magnitudes T_{ijk}.

Geodesic polar coordinates.

164. The intrinsic equations of a geodesic in a region are

$$p''+\textstyle\sum\Gamma_{11}p'^2=0,\quad q''+\sum\Delta_{11}p'^2=0,\quad r''+\sum\Theta_{11}p'^2=0.$$

We proceed to obtain the form of the expression for the regional arc, when one of the parametric curves is a regional geodesic.

Let the parametric curve

$$p=\text{variable},\quad q=\text{constant},\quad r=\text{constant},$$

be a geodesic of the region; then the equations

$$p''+\Gamma_{11}p'^2=0,\quad \Delta_{11}p'^2=0,\quad \Theta_{11}p'^2=0,$$

must be satisfied, and therefore

$$\Delta_{11}=0,\quad \Theta_{11}=0.$$

Hence, from the general results in § 160, we have

$$\tfrac{1}{2}A_1=A\Gamma_{11},\quad H_1-\tfrac{1}{2}A_2=H\Gamma_{11},\quad G_1-\tfrac{1}{2}A_3=G\Gamma_{11},$$

so that

$$\frac{\partial}{\partial p}\left(\frac{H}{A^{\frac{1}{2}}}\right)=\frac{\partial}{\partial q}(A^{\frac{1}{2}}),\quad \frac{\partial}{\partial q}\left(\frac{G}{A^{\frac{1}{2}}}\right)=\frac{\partial}{\partial r}(A^{\frac{1}{2}}).$$

There therefore exists some function l, of the parameters of the region, such that

$$A^{\frac{1}{2}}=\frac{\partial l}{\partial p},\quad \frac{H}{A^{\frac{1}{2}}}=\frac{\partial l}{\partial q},\quad \frac{G}{A^{\frac{1}{2}}}=\frac{\partial l}{\partial r}.$$

Now, as the arc ds of the region is given by $ds^2=\sum A\,dp^2$, the substitution of these values leads to the modified form

$$\begin{aligned} ds^2 &= \left(\frac{\partial l}{\partial p}\right)^2 dp^2+2\frac{\partial l}{\partial p}\frac{\partial l}{\partial q}\partial p\,dq+2\frac{\partial l}{\partial p}\frac{\partial l}{\partial r}dp\,dr+B\,dq^2+2F\,dq\,dr+C\,dr^2 \\ &= dl^2+B_0\,dq^2+2F_0\,dq\,dr+C_0\,dr^2, \end{aligned}$$

where

$$B_0=B-\left(\frac{\partial l}{\partial q}\right)^2,$$

$$F_0=F-\frac{\partial l}{\partial q}\frac{\partial l}{\partial r},$$

$$C_0=C-\left(\frac{\partial l}{\partial r}\right)^2.$$

In this modified expression, the quantity l manifestly is the length of the geodesic arc measured along the selected parametric geodesic curve.

The surfaces in the region, for which the element of arc is given by the expression

$$dt^2=B_0dq^2+2F_0dq\,dr+C_0dr^2,$$

are also given by the equation $l=a$ parametric constant; and therefore any direction on such a surface, with $\frac{dp}{dt}, \frac{dq}{dt}, \frac{dr}{dt}$, as direction-variables, is such that

$$\frac{\partial l}{\partial p}\frac{dp}{dt}+\frac{\partial l}{\partial q}\frac{dq}{dt}+\frac{\partial l}{\partial r}\frac{dr}{dt}=0,$$

that is,

$$A\frac{dp}{dt}+H\frac{dq}{dt}+G\frac{dr}{dt}=0.$$

The direction-variables of the parametric geodesic are

$$\frac{dp}{ds}=\frac{1}{A^{\frac{1}{2}}},\quad \frac{dq}{ds}=0,\quad \frac{dr}{ds}=0.$$

Hence we have

$$\sum A\frac{dp}{dt}\frac{dp}{ds}=0,$$

and therefore every direction in the surface is at right angles to the parametric geodesic. Consequently the regional surfaces, given by $l=$ a parametric constant, are orthogonal to the parametric geodesic.

On the analogy of polar coordinates in homaloidal triple space, the representation of a regional arc in the form

$$dl^2+(P, Q, R \between d\phi, d\chi)^2,$$

where l, ϕ, χ, are three independent functions of p, q, r, is called a *geodesic polar* representation; occasionally, the new parameters l, ϕ, χ, are called geodesic polar coordinates.

The partial differential equation, satisfied by the surfaces orthogonal to a polar geodesic and referred to the initial representation $ds^2=\sum A\,dp^2$ of the regional arc, can be constructed as follows. Equating the equivalent expressions for the regional arc, we have

$$\sum A\,dp^2=dl^2+(P, Q, R \between d\phi, d\chi)^2,$$

so that

$$\begin{aligned} A-l_1{}^2&=(P, Q, R \between \phi_1, \chi_1)^2, & F-l_2l_3&=(P, Q, R \between \phi_2, \chi_2 \between \phi_3, \chi_3),\\ B-l_2{}^2&=(P, Q, R \between \phi_2, \chi_2)^2, & G-l_3l_1&=(P, Q, R \between \phi_3, \chi_3 \between \phi_1, \chi_1),\\ C-l_3{}^2&=(P, Q, R \between \phi_3, \chi_3)^2, & H-l_1l_2&=(P, Q, R \between \phi_1, \chi_1 \between \phi_2, \chi_2). \end{aligned}$$

Hence

$$\begin{vmatrix} A-l_1{}^2, & H-l_1l_2, & G-l_1l_3 \\ H-l_1l_2, & B-l_2{}^2, & F-l_2l_3 \\ G-l_1l_3, & F-l_2l_3, & C-l_3{}^2 \end{vmatrix}=0,$$

for the substitution of the values of the various constituents of the determinant, as given in the preceding six relations, makes the determinant vanish identically. When this equation in the derivatives of l is expanded, we find the equivalent form

$$\sum a l_1{}^2=\Omega,$$

the partial differential equation required.

This result is in accordance with the equation

$$\sum a\epsilon_1{}^2=\Omega\epsilon_n{}^2$$

for the normal dilatation (§ 159) of any parametric surface $\epsilon(p, q, r)=$ constant; for, in the case of surfaces orthogonal to polar geodesics, we have

$$\epsilon=l, \quad dn=dl,$$

and therefore $\epsilon_n=1$.

Ex. 1. Prove that, when an integral of the partial differential equation of the surfaces orthogonal to polar geodesics is known, the quantities ϕ and χ are two independent integrals of the partial differential equation

$$\{a-(Bl_3^2-2Fl_2l_3+Cl_2^2)\}^{\frac{1}{2}}\frac{\partial\theta}{\partial p}$$
$$+\{b-(Cl_1^2+2Gl_1l_3+Al_3^2)\}^{\frac{1}{2}}\frac{\partial\theta}{\partial q}$$
$$+\{c-(Al_2^2-2Hl_1l_2+Bl_1^2)\}^{\frac{1}{2}}\frac{\partial\theta}{\partial r}=0.$$

Ex. 2. In connection with the four-index symbols, and their expressions (§ 162) in terms of the derivatives of the primary magnitudes of the region, verify that the following results hold when the expression for the regional arc is

$$ds^2=dp^2+(B,\ F,\ C,\between dq,\ dr)^2.$$

The values of Γ, Δ, Θ, are given by

$$\Gamma_{11}=0,\quad \Delta_{11}=0,\quad \Theta_{11}=0;$$
$$\Gamma_{12}=0,\quad \Gamma_{13}=0,\quad \Gamma_{22}=-\tfrac{1}{2}B_1,\quad \Gamma_{23}=-\tfrac{1}{2}F_1,\quad \Gamma_{33}=-\tfrac{1}{2}C_1;$$
$$\left.\begin{array}{l}B\Delta_{12}+F\Theta_{12}=\tfrac{1}{2}B_1\\ F\Delta_{12}+C\Theta_{12}=\tfrac{1}{2}F_1\end{array}\right\};\quad \left.\begin{array}{l}B\Delta_{13}+F\Theta_{13}=\tfrac{1}{2}F_1\\ F\Delta_{13}+C\Theta_{13}=\tfrac{1}{2}C_1\end{array}\right\};$$
$$\left.\begin{array}{l}B\Delta_{22}+F\Theta_{22}=\tfrac{1}{2}B_2\\ F\Delta_{22}+C\Theta_{22}=F_2-\tfrac{1}{2}B_3\end{array}\right\};\quad \left.\begin{array}{l}B\Delta_{23}+F\Theta_{23}=\tfrac{1}{2}B_3\\ F\Delta_{23}+C\Theta_{23}=\tfrac{1}{2}C_2\end{array}\right\};\quad \left.\begin{array}{l}B\Delta_{33}+F\Theta_{33}=F_3-\tfrac{1}{2}C_2\\ F\Delta_{33}+C\Theta_{33}=\tfrac{1}{2}C_3\end{array}\right\}.$$

The values of the four-index symbols are

$$\begin{aligned}k_{11}=&-\tfrac{1}{2}(B_{33}-2F_{23}+C_{22})\\ &-\tfrac{1}{4}B_1C_1-\tfrac{1}{4}\frac{1}{BC-F^2}[C(2B_2F_3-B_2C_2)\\ &\qquad\qquad -2F\{B_2C_3+(2F_2-B_3)(2F_3-C_2)\}+B(2F_2C_3-B_3C_3)]\\ &+\tfrac{1}{4}F_1^2+\tfrac{1}{4}\frac{1}{BC-F^2}[CB_3^2-2FC_2B_3+BC_2^2],\end{aligned}$$

$$\begin{aligned}k_{12}=&\ \tfrac{1}{2}(C_{12}-F_{13})-(B,\ F,\ C\between\Delta_{13},\ \Theta_{13}\between\Delta_{23},\ \Theta_{23})+(B,\ F,\ C\between\Delta_{12},\ \Theta_{12}\between\Delta_{33},\ \Theta_{33}),\\ k_{13}=&\ \tfrac{1}{2}(B_{13}-F_{12})-(B,\ F,\ C\between\Delta_{23},\ \Theta_{23}\between\Delta_{12},\ \Theta_{12})+(B,\ F,\ C\between\Delta_{13},\ \Theta_{13}\between\Delta_{22},\ \Theta_{22}).\\ k_{22}=&-\tfrac{1}{2}C_{11}+(B,\ F,\ C\between\Delta_{13},\ \Theta_{13})^2,\\ k_{23}=&\ \tfrac{1}{2}F_{11}-(B,\ F,\ C\between\Delta_{12},\ \Theta_{12}\between\Delta_{13},\ \Theta_{13}),\\ k_{33}=&-\tfrac{1}{2}B_{11}+(B,\ F,\ C\between\Delta_{12},\ \Theta_{12})^2.\end{aligned}$$

Ex. 3. Prove that the Riemann measure K (the sphericity, in §§ 65, 217) of the region in the orientation of a surface orthogonal to the geodesic p = variable, when the expression of the regional arc is

$$ds^2=dp^2+(B,\ F,\ C\between dq,\ dr)^2$$

is given by

$$K=\frac{k_{11}}{BC-F^2}.$$

Ex. 4. Prove that, if a region could have two families of geodesics orthogonal to one another, its elements of arc would be expressible in the form

$$ds^2 = dp^2 + dq^2 + Cdr^2;$$

and obtain the four-index symbols in the form

$$\begin{aligned} k_{11} &= -\tfrac{1}{2}C_{22} + \tfrac{1}{4}\frac{C_2^{\,2}}{C}, & k_{13} &= 0. \\ -k_{12} &= -\tfrac{1}{2}C_{12} + \tfrac{1}{4}\frac{C_1 C_2}{C}, & k_{23} &= 0, \\ k_{22} &= -\tfrac{1}{2}C_{11} + \tfrac{1}{4}\frac{C_1^{\,2}}{C}, & k_{33} &= 0. \end{aligned}$$

Hence (or otherwise than by shewing that the region would then contain a parametric family of surfaces with zero sphericity) prove that the region is of limited generality as being developable into a flat.

Minimal regions in free space.

165. The general equations of minimal regions in a plenary homaloidal space of n dimensions (where $n > 3$) are obtained by the customary methods of the calculus of variations. We use y as a typical space-coordinate of any point; and, in the region, we have

$$ds^2 = \sum A\, dp^2.$$

The element of volume of the region is

$$\Omega^{\frac{1}{2}}\, dp\, dq\, dr,$$

where Ω is the determinant of the primary coefficients in ds^2; and accordingly, we have to minimise the triple integral

$$\iiint \Omega^{\frac{1}{2}}\, dp\, dq\, dr,$$

the independent variables being the parameters p, q, r, and the dependent variables being the space-coordinates. The critical equation *

$$-\frac{\partial \Omega^{\frac{1}{2}}}{\partial y} + \frac{d}{dp}\left(\frac{\partial \Omega^{\frac{1}{2}}}{\partial y_1}\right) + \frac{d}{dq}\left(\frac{\partial \Omega^{\frac{1}{2}}}{\partial y_2}\right) + \frac{d}{dr}\left(\frac{\partial \Omega^{\frac{1}{2}}}{\partial y_3}\right) = 0,$$

holds for each of the dependent variables typified by y.

Now $\Omega^{\frac{1}{2}}$ is composed of magnitudes constituted from derivatives of these dependent variables and it does not contain any one of such variables explicitly; hence

$$\frac{\partial \Omega^{\frac{1}{2}}}{\partial y} = 0.$$

* See my *Calculus of Variations*, § 606, and chapter xii generally.

Again, as

$$A=\sum y_1^2\,,\quad B=\sum y_2^2\,,\quad C=\sum y_3^2\,,$$
$$F=\sum y_2y_3,\quad G=\sum y_3y_1,\quad H=\sum y_1y_2,$$

we have

$$\frac{\partial\Omega^{\frac{1}{2}}}{\partial y_1}=\frac{1}{2\Omega^{\frac{1}{2}}}\frac{\partial\Omega}{\partial y_1}=\frac{1}{\Omega^{\frac{1}{2}}}(ay_1+hy_2+gy_3),$$
$$\frac{\partial\Omega^{\frac{1}{2}}}{\partial y_2}=\frac{1}{\Omega^{\frac{1}{2}}}(hy_1+by_2+fy_3),$$
$$\frac{\partial\Omega^{\frac{1}{2}}}{\partial y_3}=\frac{1}{\Omega^{\frac{1}{2}}}(gy_1+fy_2+cy_3).$$

Thus the typical critical equation becomes

$$\frac{1}{\Omega^{\frac{1}{2}}}(ay_{11}+2hy_{12}+2gy_{13}+by_{22}+2fy_{23}+cy_{33})$$
$$+\sum\left[y_1\left\{\frac{d}{dp}\left(\frac{a}{\Omega^{\frac{1}{2}}}\right)+\frac{d}{dq}\left(\frac{h}{\Omega^{\frac{1}{2}}}\right)+\frac{d}{dr}\left(\frac{g}{\Omega^{\frac{1}{2}}}\right)\right\}\right]=0.$$

Now

$$\frac{d}{dp}\left(\frac{a}{\Omega^{\frac{1}{2}}}\right)=\frac{1}{\Omega^{\frac{1}{2}}}\left(a_1-\tfrac{1}{2}a\frac{\Omega_1}{\Omega}\right)$$
$$=\frac{1}{\Omega^{\frac{1}{2}}}\left\{\tfrac{1}{2}a\frac{\Omega_1}{\Omega}-2(a\Gamma_{11}+h\Gamma_{12}+g\Gamma_{13})\right\},$$

by the formulæ in § 160, with similar expressions for the other like quantities. When these are substituted, and the typical equation is re-arranged, it becomes

$$a\eta_{11}+2h\eta_{12}+2g\eta_{13}+b\eta_{22}+2f\eta_{23}+c\eta_{33}=0.$$

There is one such equation for each of the space-variables; the complete set are therefore the point-equations of the minimal regions in a plenary homaloidal space.

Ex. 1. When the plenary homaloidal space is quadruple, so that there are four such equations, and when the properties of regions in quadruple space* are used, the four equations lead to the central relation

$$a\bar{A}+2h\bar{H}+2g\bar{G}+b\bar{B}+2f\bar{F}+c\bar{C}=0.$$

Thus the sum of the three principal circular curvatures of the geodesics belonging to a minimal region in quadruple space is zero.

Ex. 2. Shew that the (primary) sub-amplitude

$$a^2(y_1^2+\ldots+y_m^2)=f(ay_{m+1}+c),$$

in a plenary homaloidal space of $m+1$ dimensions, is minimal if the function $f(u)$ is such that

$$f'^2(u)=\{f(u)\}^m-\frac{8}{m-1}f(u),$$

the quantities a and c denoting constants which affect only scale and position.

* *G.F.D.*, vol. ii, § 416.

It has been proved that nul-lines satisfy the equations of minimal lines (geodesics) in an amplitude (§ 18) and that nul-surfaces satisfy the equations of minimal surfaces in an amplitude (§ 75). The same property, as between nul-regions and minimal regions, holds for any amplitude; it can be verified at once when the amplitude is the plenary homaloidal space of the regions.

The typical critical equation, as Ω does not involve the variables y explicitly, has the form

$$\frac{d}{dp}\left(\frac{\partial \Omega^{\frac{1}{2}}}{\partial y_1}\right)+\frac{d}{dq}\left(\frac{\partial \Omega^{\frac{1}{2}}}{\partial y_2}\right)+\frac{d}{dr}\left(\frac{\partial \Omega^{\frac{1}{2}}}{\partial y_3}\right)=0.$$

We have

$$\frac{\partial \Omega^{\frac{1}{2}}}{\partial y_1}=\frac{1}{\Omega^{\frac{1}{2}}}Y_1,\quad \frac{\partial \Omega^{\frac{1}{2}}}{\partial y_2}=\frac{1}{\Omega^{\frac{1}{2}}}Y_2,\quad \frac{\partial \Omega^{\frac{1}{2}}}{\partial y_3}=\frac{1}{\Omega^{\frac{1}{2}}}Y_3,$$

where

$$Y_1=ay_1+hy_2+gy_3,\quad Y_2=hy_1+by_2+fy_3,\quad Y_3=gy_1+fy_2+cy_3.$$

Hence

$$\frac{d}{dp}\left(\frac{\partial \Omega^{\frac{1}{2}}}{\partial y_1}\right)=-\frac{1}{2\Omega^{\frac{3}{2}}}\frac{d\Omega}{dp}Y_1+\frac{1}{\Omega^{\frac{1}{2}}}\frac{dY_1}{dp},$$

$$\frac{d}{dq}\left(\frac{\partial \Omega^{\frac{1}{2}}}{\partial y_2}\right)=-\frac{1}{2\Omega^{\frac{3}{2}}}\frac{d\Omega}{dq}Y_2+\frac{1}{\Omega^{\frac{1}{2}}}\frac{dY_2}{dq},$$

$$\frac{d}{dr}\left(\frac{\partial \Omega^{\frac{1}{2}}}{\partial y_3}\right)=-\frac{1}{2\Omega^{\frac{3}{2}}}\frac{d\Omega}{dr}Y_3+\frac{1}{\Omega^{\frac{1}{2}}}\frac{dY_3}{dr},$$

and thus the typical equation becomes

$$Y_1\frac{d\Omega}{dp}+Y_2\frac{d\Omega}{dq}+Y_3\frac{d\Omega}{dr}=2\Omega\left(\frac{dY_1}{dp}+\frac{dY_2}{dq}+\frac{dY_3}{dr}\right),$$

holding for each of the n variables y; and n is greater than 3.

These equations, which are characteristic of minimal regions in free space, manifestly are satisfied by a permanent equation

$$\Omega=0,$$

that is, by the equation of nul-regions.

The tangent flat of a region.

166. The equations of any line, tangential to the region, are

$$\left\|\frac{\bar{y}-y}{y'}\right\|=\lambda,$$

where λ is the parameter of a current point on the line, so that any point on the line is represented by the typical equation

$$\bar{y}-y=\lambda y'=y_1\lambda p'+y_2\lambda q'+y_3\lambda r'.$$

We obtain different points on the same line by varying values of λ: we obtain different lines by the various sets of values of p', q', r', which are subject to the single permanent arc-relation of a quadratic order in these variables. Hence the coordinates of every point on every line, through the initial point typified by y, satisfy the equations

$$\left\| \bar{y}-y, \quad y_1, \quad y_2, \quad y_3 \right\| = 0,$$

which, in a plenary space of N dimensions, constitute a set of $N-3$ linearly independent equations. They therefore represent a flat in that plenary space. Every straight line, in the flat and passing through the initial point typified by y, touches the region; consequently, the flat thus constituted is the tangent flat of the region at the initial point.

Two inferences can be drawn at once. In the first place, the space-coordinates of any point in the flat can be represented by the typical equations

$$\bar{y}-y = \alpha y_1 + \beta y_2 + \gamma y_3,$$

where α, β, γ, are parameters. In the second place, let any direction in the flat have spatial direction-cosines l_1, l_2, ... l_N, typified by l; then we have the typical equation

$$l = \epsilon y_1 + \eta y_2 + \omega y_3,$$

where ϵ, η, ω, are parameters; but, as $\sum l^2 = 1$, the three direction-parameters are subject to the relation

$$A\epsilon^2 + 2H\epsilon\eta + 2G\epsilon\omega + B\eta^2 + 2F\eta\omega + C\omega^2 = 1.$$

Also, the conditions that a space-direction, denoted by l_1, l_2, ..., l_N, shall lie in the tangent flat of the region are

$$\left\| l_i, \quad \frac{\partial y_i}{\partial p}, \quad \frac{\partial y_i}{\partial q}, \quad \frac{\partial y_i}{\partial r} \right\| = 0.$$

Moreover, there are orientation-variables of different types in the plenary space. It suffices, in connection with a region, to consider merely the surface-variables, because the region itself provides a single volumetric orientation at a point in the plenary space. As the parametric curves at the origin provide, by their tangents, a set of leading lines of the flat, so the orientations at the origin, composed from pairs of these parametric directions, provide a set of leading surface-orientations of the flat. For this purpose, we take

$$\omega_P^{(ij)}, \quad \omega_Q^{(ij)}, \quad \omega_R^{(ij)} = \left\| \begin{matrix} \dfrac{\partial y_i}{\partial p}, & \dfrac{\partial y_i}{\partial q}, & \dfrac{\partial y_i}{\partial r} \\ \dfrac{\partial y_j}{\partial p}, & \dfrac{\partial y_j}{\partial q}, & \dfrac{\partial y_j}{\partial r} \end{matrix} \right\|.$$

Let any surface-orientation in the plenary space be contained in the flat. Any two lines in that orientation, taken as guiding lines, must lie in the flat; let their spatial direction-cosines be $l_1, l_2, \ldots, l_N$, and $m_1, m_2, \ldots, m_N$, so that we must have

$$l_i = \frac{\partial y_i}{\partial p}\alpha_1 + \frac{\partial y_i}{\partial q}\beta_1 + \frac{\partial y_i}{\partial r}\gamma_1, \quad m_j = \frac{\partial y_j}{\partial p}\alpha_2 + \frac{\partial y_j}{\partial q}\beta_2 + \frac{\partial y_j}{\partial r}\gamma_2,$$

for all values $1, \ldots, N$, of i and j, the quantities α, β, γ, being parametric magnitudes for the respective directions. The surface-orientation in the plenary space has variables denoted by $\omega^{(ij)}$, such that

$$\begin{aligned}\omega^{(ij)} &= l_i m_j - l_j m_i \\ &= \begin{vmatrix} \frac{\partial y_i}{\partial q}, & \frac{\partial y_i}{\partial r} \\ \frac{\partial y_j}{\partial q}, & \frac{\partial y_j}{\partial r} \end{vmatrix} \begin{vmatrix} \beta_1, & \gamma_1 \\ \beta_2, & \gamma_2 \end{vmatrix} + \begin{vmatrix} \frac{\partial y_i}{\partial r}, & \frac{\partial y_i}{\partial p} \\ \frac{\partial y_j}{\partial r}, & \frac{\partial y_j}{\partial p} \end{vmatrix} \begin{vmatrix} \gamma_1, & \alpha_1 \\ \gamma_2, & \alpha_2 \end{vmatrix} + \begin{vmatrix} \frac{\partial y_i}{\partial p}, & \frac{\partial y_i}{\partial q} \\ \frac{\partial y_j}{\partial p}, & \frac{\partial y_j}{\partial q} \end{vmatrix} \begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix} \\ &= \theta\omega_P^{(ij)} + \phi\omega_Q^{(ij)} + \psi\omega_R^{(ij)},\end{aligned}$$

where θ, ϕ, ψ, now are parametric magnitudes for the spatial surface-orientation represented as lying in the flat. If then the quantity ω be taken as typical of all the variables $\omega^{(ij)}$ specifying an orientation, we have

$$\omega = \theta\omega_P + \phi\omega_Q + \psi\omega_R;$$

and the conditions that a surface-orientation in the plenary space, with orientation variables $\omega^{(ij)}$, shall lie in the tangent flat of the region are

$$\| \omega^{(ij)}, \quad \omega_P^{(ij)}, \quad \omega_Q^{(ij)}, \quad \omega_R^{(ij)} \| = 0.$$

We may assume, initially, that the selected leading lines in the surface orientation in the plenary space are at right angles: then

$$\begin{aligned}\sum_i \sum_j \omega_{ij}^2 &= \sum_i \sum_j (l_i m_j - l_j m_i)^2 \\ &= \Big(\sum_i l_i^2\Big)\Big(\sum_j m_j^2\Big) - \Big(\sum_i l_i m_i\Big)^2 = 1.\end{aligned}$$

Accordingly the parameters θ, ϕ, ψ, in the foregoing representation must be such that

$$\sum_i \sum_j \{\theta\omega_P^{(ij)} + \phi\omega_Q^{(ij)} + \psi\omega_R^{(ij)}\}^2 = 1.$$

Now

$$\begin{aligned}\sum_i \sum_j \{\omega_P^{(ij)}\}^2 &= \sum_i \sum_j \left(\frac{\partial y_i}{\partial q}\frac{\partial y_j}{\partial r} - \frac{\partial y_j}{\partial q}\frac{\partial y_i}{\partial r}\right)^2 \\ &= \left\{\sum_i \left(\frac{\partial y_i}{\partial q}\right)^2\right\}\left\{\sum_j \left(\frac{\partial y_j}{\partial r}\right)^2\right\} - \left\{\sum_i \sum_j \frac{\partial y_i}{\partial r}\frac{\partial y_j}{\partial q}\right\}^2 \\ &= BC - F^2 = a,\end{aligned}$$

and so for the other like combinations: hence the relation, to be satisfied by the parameters θ, ϕ, ψ, is

$$a\theta^2 + 2h\theta\phi + 2g\theta\psi + b\phi^2 + 2f\phi\psi + c\psi^2 = 1.$$

167. Now consider the perpendicular drawn, upon the tangent flat of the region at O, from a point Q near O. We may proceed as in § 20 and shew that the direction of this perpendicular coincides, in the limit of a decreasing distance QO, with the prime normal at O of the geodesic OQ; or, as was the method adopted in § 94 when a surface was under consideration, we may take Q to be a point near O along a geodesic OQ in the region. Adopting the latter method, we denote the small geodesic arc OQ by t; the coordinates of Q, denoted by $\eta_1, \eta_2, \ldots$, will be represented by a typical variable η; and the foot of the perpendicular from Q on the tangent flat will have its coordinates denoted by $\bar{y}_1, \bar{y}_2, \ldots$, so that there exist parameters θ, ϕ, ψ, such that

$$\bar{y}_m - y_m = \theta \frac{\partial y_m}{\partial p} + \phi \frac{\partial y_m}{\partial q} + \psi \frac{\partial y_m}{\partial r},$$

for all the values $1, 2, \ldots, N$, of m. If Π be the length of the perpendicular, and if $X_1, X_2, \ldots$ denote its direction-cosines, with X as the typical direction-cosine, we have

$$X_m \Pi = \eta_m - \bar{y}_m.$$

In order that the line may be a perpendicular, the quantity $\sum(\eta_m - \bar{y}_m)^2$ must be a minimum along all the lengths from Q to the flat, and therefore

$$\sum \left\{\eta_m - \left(y_m + \theta \frac{\partial y_m}{\partial p} + \phi \frac{\partial y_m}{\partial q} + \psi \frac{\partial y_m}{\partial r}\right)\right\}^2$$

must be a minimum for all values of the independent parameters θ, ϕ, ψ. The critical equations are

$$\sum \left[\frac{\partial y_m}{\partial p}\left\{\eta_m - \left(y_m + \theta \frac{\partial y_m}{\partial p} + \phi \frac{\partial y_m}{\partial q} + \psi \frac{\partial y_m}{\partial r}\right)\right\}\right] = 0,$$

$$\sum \left[\frac{\partial y_m}{\partial q}\left\{\eta_m - \left(y_m + \theta \frac{\partial y_m}{\partial p} + \phi \frac{\partial y_m}{\partial q} + \psi \frac{\partial y_m}{\partial r}\right)\right\}\right] = 0,$$

$$\sum \left[\frac{\partial y_m}{\partial r}\left\{\eta_m - \left(y_m + \theta \frac{\partial y_m}{\partial p} + \phi \frac{\partial y_m}{\partial q} + \psi \frac{\partial y_m}{\partial r}\right)\right\}\right] = 0;$$

and the further conditions, necessary and sufficient to secure a minimum, are easily seen to be satisfied.

In the first place, these three equations may be written

$$\left(\sum X_m \frac{\partial y_m}{\partial p}\right)\Pi = 0, \quad \left(\sum X_m \frac{\partial y_m}{\partial q}\right)\Pi = 0, \quad \left(\sum X_m \frac{\partial y_m}{\partial r}\right)\Pi = 0;$$

hence, as Π is not zero, we have

$$\sum X_m \frac{\partial y_m}{\partial p} = 0, \quad \sum X_m \frac{\partial y_m}{\partial q} = 0, \quad \sum X_m \frac{\partial y_m}{\partial r} = 0:$$

that is, the perpendicular in question is at right angles to every direction in the flat, as is to be expected geometrically.

Next, the equations may be taken in the form

$$A\theta + H\phi + G\psi = \sum\left\{\frac{\partial y_m}{\partial p}(\eta_m - y_m)\right\},$$

$$H\theta + B\phi + F\psi = \sum\left\{\frac{\partial y_m}{\partial q}(\eta_m - y_m)\right\},$$

$$G\theta + F\phi + C\psi = \sum\left\{\frac{\partial y_m}{\partial r}(\eta_m - y_m)\right\}.$$

Now

$$\eta_m - y_m = t y_m' + \tfrac{1}{2}t^2 y_m'' + \tfrac{1}{6}t^3 y_m''' + \dots .$$

We have

$$\sum \frac{\partial y_m}{\partial p} y_m' = \sum \frac{\partial y_m}{\partial p}\left(\frac{\partial y_m}{\partial p}p' + \frac{\partial y_m}{\partial q}q' + \frac{\partial y_m}{\partial r}r'\right) = Ap' + Hq' + Gr' ;$$

and similarly

$$\sum \frac{\partial y_m}{\partial q} y_m' = Hp' + Bq' + Fr', \quad \sum \frac{\partial y_m}{\partial r} y_m' = Gp' + Fq' + Cr'.$$

Also, because

$$y_m'' = \frac{1}{\rho} Y_m = \sum \eta_{ij} p_i' p_j',$$

we have

$$\sum \frac{\partial y_m}{\partial p} y_m'' = \frac{1}{\rho}\sum_i \sum_j \left\{p_i' p_j' \left(\sum \frac{\partial y_m}{\partial p}\eta_{ij}\right)\right\} = 0 ;$$

and similarly

$$\sum \frac{\partial y_m}{\partial q} y_m'' = 0, \quad \sum \frac{\partial y_m}{\partial r} y_m'' = 0.$$

Hence the foregoing equations become

$$A(\theta - tp') + H(\phi - tq') + G(\psi - tr') = \tfrac{1}{6}t^3 \sum y_m''' \frac{\partial y_m}{\partial p} + \dots ,$$

$$H(\theta - tp') + B(\phi - tq') + F(\psi - tr') = \tfrac{1}{6}t^3 \sum y_m''' \frac{\partial y_m}{\partial q} + \dots ,$$

$$G(\theta - tp') + F(\phi - tq') + C(\psi - tr') = \tfrac{1}{6}t^3 \sum y_m''' \frac{\partial y_m}{dr} + \dots :$$

and therefore, up to the second power of t inclusive (t being a small quantity),

$$\theta = tp', \quad \phi = tq', \quad \psi = tr'.$$

Thus

$$X_m \Pi = \eta_m - \left(y_m + \theta\frac{\partial y_m}{\partial p} + \phi\frac{\partial y_m}{\partial q} + \psi\frac{\partial y_m}{\partial r}\right)$$

$$= y_m + t y_m' + \tfrac{1}{2}t^2 y_m'' - \left\{y_m + t\left(p'\frac{\partial y_m}{\partial p} + q'\frac{\partial y_m}{\partial q} + r'\frac{\partial y_m}{\partial r}\right)\right\},$$

accurately up to the second order inclusive: and so, up to that order,

$$X_m \Pi = \tfrac{1}{2} t^2 y_m'' = Y_m \frac{t^2}{2\rho},$$

for all the values of m. Hence, as t tends to zero, we have

$$\Pi \doteq \frac{t^2}{2\rho}, \quad X_m = Y_m,$$

also for all values of m: that is, the limiting position of the perpendicular from Q on the tangent flat, as Q tends to coincide with O, is the prime normal of the geodesic. The relation $2\rho\Pi = t^2$ implies a customary expression for the curvature of the regional geodesic.

We note also that the prime normal of every regional geodesic through O is orthogonal to the tangent flat of the region, while of course the tangent of that geodesic lies in the tangent flat.

Expressions for the circular curvature of a geodesic: secondary magnitudes.

168. An expression for the square of the circular curvature is obtained by squaring the foregoing typical equation and adding the results for all the space-dimensions; and the expression involves the sums of the squares and the sums of the bilinear products of the magnitudes η_{ij}. To represent these sums, the following special notation * is adopted:

$$\left.\begin{array}{lll} \sum \eta_{11}^2 = \kappa_{400}, & \sum \eta_{11}\eta_{12} = \kappa_{310}, & \sum \eta_{11}\eta_{13} = \kappa_{301} \\ \sum \eta_{22}^2 = \kappa_{040}, & \sum \eta_{12}\eta_{22} = \kappa_{130}, & \sum \eta_{22}\eta_{23} = \kappa_{031} \\ \sum \eta_{33}^2 = \kappa_{004}, & \sum \eta_{13}\eta_{33} = \kappa_{103}, & \sum \eta_{23}\eta_{33} = \kappa_{013} \end{array}\right\};$$

$$\left.\begin{array}{r} \sum \eta_{23}^2 = \kappa_{022} - \tfrac{1}{3}k_{11} \\ \sum \eta_{22}\eta_{33} = \kappa_{022} + \tfrac{2}{3}k_{11} \end{array}\right\}, \qquad \left.\begin{array}{r} \sum \eta_{11}\eta_{23} = \kappa_{211} - \tfrac{2}{3}k_{23} \\ \sum \eta_{12}\eta_{13} = \kappa_{211} + \tfrac{1}{3}k_{23} \end{array}\right\},$$

$$\left.\begin{array}{r} \sum \eta_{31}^2 = \kappa_{202} - \tfrac{1}{3}k_{22} \\ \sum \eta_{11}\eta_{33} = \kappa_{202} + \tfrac{2}{3}k_{22} \end{array}\right\}, \qquad \left.\begin{array}{r} \sum \eta_{22}\eta_{13} = \kappa_{122} - \tfrac{2}{3}k_{13} \\ \sum \eta_{12}\eta_{23} = \kappa_{122} + \tfrac{1}{3}k_{13} \end{array}\right\},$$

$$\left.\begin{array}{r} \sum \eta_{12}^2 = \kappa_{220} - \tfrac{1}{3}k_{33} \\ \sum \eta_{11}\eta_{22} = \kappa_{220} + \tfrac{2}{3}k_{33} \end{array}\right\}, \qquad \left.\begin{array}{r} \sum \eta_{33}\eta_{12} = \kappa_{112} - \tfrac{2}{3}k_{12} \\ \sum \eta_{13}\eta_{23} = \kappa_{112} + \tfrac{1}{3}k_{12} \end{array}\right\}.$$

* Later (§ 254), it is found convenient to adopt a different notation. With the double-suffix combinations in the quantities $\eta_{11}, \eta_{12}, \eta_{22}, \eta_{13}, \eta_{23}, \eta_{33}$, we associate the integers 1, 2, 3, 4, 5, 6, respectively in the ordered succession; and then we use symbols s_{ij} to denote the twenty-one magnitudes of the type $\sum \eta_{ij}\eta_{kl}$, according to the definitions such as

$$\begin{array}{llll} s_{11} = \sum \eta_{11}^2, & s_{12} = \sum \eta_{11}\eta_{12}, & s_{13} = \sum \eta_{11}\eta_{22}, & s_{14} = \sum \eta_{11}\eta_{13}, \\ s_{15} = \sum \eta_{11}\eta_{23}, & s_{16} = \sum \eta_{11}\eta_{33}, & s_{22} = \sum \eta_{12}^2, & s_{23} = \sum \eta_{12}\eta_{22}, \\ s_{24} = \sum \eta_{12}\eta_{13}, & s_{25} = \sum \eta_{12}\eta_{23}, & s_{26} = \sum \eta_{12}\eta_{33}, & s_{33} = \sum \eta_{22}^2. \end{array}$$

The symbols k_{ij}, for $i, j, =1, 2, 3$, which have appeared in § 162, will arise in connection with the expression of the Riemann sphericity of the region in any surface-orientation. The desired expression giving the magnitude of circular curvature of a regional geodesic is

$$\frac{1}{\rho^2}=\sum_\lambda \sum_\mu \sum_\nu \kappa_{\lambda\mu\nu} p'^\lambda q'^\mu r'^\nu = Q,$$

for numerical values of $\lambda, \mu, \nu, =0, 1, 2, 3, 4$, such that $\lambda+\mu+\nu=4$, the summation being taken for λ, μ, ν, independently of one another; and the term containing the specific coefficient $\kappa_{\lambda\mu\nu}$ has a numerical coefficient

$$\frac{4!}{\lambda!\,\mu!\,\nu!}.$$

Thus the circular curvature of the regional geodesic is given by a homogeneous ternary quartic form Q in the direction-variables.

Another form can be given to the expression of the circular curvature of a regional geodesic, by means of the secondary magnitudes $\bar{A}, \bar{B}, \bar{C}, \bar{F}, \bar{G}, \bar{H}$, of the region, defined by the equations

$$\bar{A}=\sum Y\eta_{11}, \quad \bar{B}=\sum Y\eta_{22}, \quad \bar{C}=\sum Y\eta_{33},$$
$$\bar{F}=\sum Y\eta_{23}, \quad \bar{G}=\sum Y\eta_{31}, \quad \bar{H}=\sum Y\eta_{12};$$

and we shall use also an alternative double-suffix notation, under the definition

$$\bar{A}_{ij}=\sum Y\eta_{ij}.$$

(Owing to the properties

$$\sum Yy_1=0, \quad \sum Yy_2=0, \quad \sum Yy_3=0,$$

any magnitude in the preceding definitions can be changed according to the general relation

$$\sum Y\eta_{ij}=\sum Yy_{ij},$$

where the suffix-notation for the typical variable y now connotes differentiation with respect to the parameters p, q, r, of the region.) With these definitions, and using the relation

$$\frac{Y}{\rho}=\sum_i \sum_j \eta_{ij} p_i' p_j', \quad (i, j, =1, 2, 3)$$

we have

$$\frac{1}{\rho}=\sum Y.\frac{Y}{\rho}=\bar{A}p'^2+2\bar{H}p'q'+\bar{B}q'^2+2\bar{G}p'r'+2\bar{F}q'r'+\bar{C}r'^2=\sum \bar{A}p'^2,$$

apparently a ternary quadratic expression in p', q', r'.

When the plenary space of the region is of only four dimensions, the typical direction-cosine Y of the prime normal of a geodesic is independent of p', q', r'; and then the magnitudes $\bar{A}, \bar{B}, \bar{C}, \bar{F}, \bar{G}, \bar{H}$, are functions of position only.

But when the plenary space of the region is of more than four dimensions, the typical direction-cosine Y of the prime normal of a geodesic is dependent on p', q', r', and varies from one geodesic to another; in fact, we have

$$Y=\frac{\sum \eta_{11}p'^2}{Q^{\frac{1}{2}}},$$

where Q is the foregoing ternary quartic in p', q', r'. Hence the secondary magnitudes, while certainly being functions of position in the region, implicitly involve also the direction-variables p', q', r', of the geodesic. We shall assume that usually the plenary space of a region is of at least five dimensions: any exception to the assumption will be noted definitely as it occurs.

More explicit expressions for the magnitudes $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{F}$, $\bar{G}$, $\bar{H}$, can be obtained by substituting the explicit value of Y in the equations of definition and by using the foregoing symbols for the different magnitudes $\sum \eta_{ij}\eta_{kl}$, for $i, j, k, l, =1, 2, 3,$ with the results

$$\begin{aligned}
\frac{\bar{A}}{\rho}&=\kappa_{400}p'^2+2\kappa_{310}p'q'+\kappa_{220}q'^2+2\kappa_{301}p'r'+2\kappa_{211}q'r'+\kappa_{202}r'^2\\
&\qquad+\tfrac{2}{3}(k_{33}q'^2-2k_{23}q'r'+k_{22}r'^2),\\
\frac{\bar{H}}{\rho}&=\kappa_{310}p'^2+2\kappa_{220}p'q'+\kappa_{130}q'^2+2\kappa_{211}p'r'+2\kappa_{122}q'r'+\kappa_{112}r'^2\\
&\qquad-\tfrac{2}{3}(k_{33}p'q'-k_{23}p'r'-k_{13}q'r'+k_{12}r'^2),\\
\frac{\bar{G}}{\rho}&=\kappa_{301}p'^2+2\kappa_{211}p'q'+\kappa_{122}q'^2+2\kappa_{202}p'r'+2\kappa_{112}q'r'+\kappa_{103}r'^2\\
&\qquad-\tfrac{2}{3}(k_{22}p'r'-k_{23}p'q'-k_{12}q'r'+k_{13}q'^2),\\
\frac{\bar{B}}{\rho}&=\kappa_{220}p'^2+2\kappa_{130}p'q'+\kappa_{040}q'^2+2\kappa_{121}p'r'+2\kappa_{031}q'r'+\kappa_{022}r'^2\\
&\qquad+\tfrac{2}{3}(k_{11}r'^2-2k_{13}p'r'+k_{33}p'^2),\\
\frac{\bar{F}}{\rho}&=\kappa_{211}p'^2+2\kappa_{122}p'q'+\kappa_{031}q'^2+2\kappa_{112}p'r'+2\kappa_{022}q'r'+\kappa_{013}r'^2\\
&\qquad-\tfrac{2}{3}(k_{11}q'r'-k_{13}p'q'-k_{12}p'r'+k_{23}p'^2),\\
\frac{\bar{C}}{\rho}&=\kappa_{202}p'^2+2\kappa_{112}p'q'+\kappa_{022}q'^2+2\kappa_{103}p'r'+2\kappa_{013}q'r'+\kappa_{004}r'^2\\
&\qquad+\tfrac{2}{3}(k_{22}p'^2-2k_{12}p'q'+k_{11}q'^2).
\end{aligned}$$

It will be noted that, in these six expressions, the first lines are equal to

$$\frac{1}{12}\frac{\partial^2 Q}{\partial p'^2},\quad \frac{1}{12}\frac{\partial^2 Q}{\partial p'\partial q'},\quad \frac{1}{12}\frac{\partial^2 Q}{\partial p'\partial r'},\quad \frac{1}{12}\frac{\partial^2 Q}{\partial q'^2},\quad \frac{1}{12}\frac{\partial^2 Q}{\partial q'\partial r'},\quad \frac{1}{12}\frac{\partial^2 Q}{\partial r'^2},$$

respectively. And it is easy to verify that, when these values of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{F}$, $\bar{G}$, $\bar{H}$, are substituted in the equation

$$\frac{1}{\rho}=\sum \bar{A}p'^2,$$

we return to the former equation

$$\frac{1}{\rho^2}=Q.$$

Later, it will be found convenient to use symbols u_1, u_2, u_3, and v_1, v_2, v_3, according to the definitions

$$\left.\begin{aligned} u_1 &= Ap' + Hq' + Gr' \\ u_2 &= Hp' + Bq' + Fr' \\ u_3 &= Gp' + Fq' + Cr' \end{aligned}\right\}, \qquad \left.\begin{aligned} v_1 &= \bar{A}p' + \bar{H}q' + \bar{G}r' \\ v_2 &= \bar{H}p' + \bar{B}q' + \bar{F}r' \\ v_3 &= \bar{G}p' + \bar{F}q' + \bar{C}r' \end{aligned}\right\},$$

so that we have

$$u_1 p' + u_2 q' + u_3 r' = 1,$$

$$v_1 p' + v_2 q' + v_3 r' = \frac{1}{\rho}.$$

By direct substitution of the values of the secondary magnitudes, we find

$$\begin{aligned} \frac{1}{\rho}(\bar{A}p' + \bar{H}q' + \bar{G}r') &= \tfrac{1}{12}\left(p'\frac{\partial^2 Q}{\partial p'^2} + q'\frac{\partial^2 Q}{\partial p'\,\partial q'} + r'\frac{\partial^2 Q}{\partial p'\,\partial r'}\right) \\ &= \tfrac{1}{4}\frac{\partial Q}{\partial p'} = \frac{1}{2\rho}\frac{\partial}{\partial p'}\left(\frac{1}{\rho}\right), \\ \frac{1}{\rho}(\bar{H}p' + \bar{B}q' + \bar{F}r') &= \tfrac{1}{12}\left(p'\frac{\partial^2 Q}{\partial p'\,\partial q'} + q'\frac{\partial^2 Q}{\partial q'^2} + r'\frac{\partial^2 Q}{\partial q'\,\partial r'}\right) \\ &= \tfrac{1}{4}\frac{\partial Q}{\partial q'} = \frac{1}{2\rho}\frac{\partial}{\partial q'}\left(\frac{1}{\rho}\right), \\ \frac{1}{\rho}(\bar{G}p' + \bar{F}q' + \bar{C}r') &= \tfrac{1}{12}\left(p'\frac{\partial^2 Q}{\partial p'\,\partial r'} + q'\frac{\partial^2 Q}{\partial q'\,\partial r'} + r'\frac{\partial^2 Q}{\partial r'^2}\right) \\ &= \tfrac{1}{4}\frac{\partial Q}{\partial r'} = \frac{1}{2\rho}\frac{\partial}{\partial r'}\left(\frac{1}{\rho}\right), \end{aligned}$$

for the aggregate of terms, involving the coefficients k_{ij} in the three selected combinations, disappears in each instance; hence we have the relations

$$\frac{\partial}{\partial p'}\left(\frac{1}{\rho}\right) = 2v_1, \quad \frac{\partial}{\partial q'}\left(\frac{1}{\rho}\right) = 2v_2, \quad \frac{\partial}{\partial r'}\left(\frac{1}{\rho}\right) = 2v_3,$$

and their equivalents

$$\frac{v_1}{\rho} = \frac{1}{4}\frac{\partial}{\partial p'}\left(\frac{1}{\rho^2}\right), \quad \frac{v_2}{\rho} = \frac{1}{4}\frac{\partial}{\partial q'}\left(\frac{1}{\rho^2}\right), \quad \frac{v_3}{\rho} = \frac{1}{4}\frac{\partial}{\partial r'}\left(\frac{1}{\rho^2}\right).$$

The binormal and the trinormal of a geodesic lie in the tangent flat.

169. Consider the (quadruple) block represented by the equations

$$\left\| \; \bar{y} - y, \quad \frac{\partial y}{\partial p}, \quad \frac{\partial y}{\partial q}, \quad \frac{\partial y}{\partial r}, \quad Y \; \right\| = 0.$$

Manifestly it contains the tangent flat of the region, and therefore the tangent to the regional geodesic in any contained direction p', q', r'. Also, it contains the

prime normal of that regional geodesic; and therefore it contains the osculating plane of the geodesic.

The tangent at a consecutive point $y+y'\delta$ of the geodesic, where δ is a small arc-distance, is represented by equations of the type

$$\frac{\bar{y}-(y+y'\delta)}{y'+y''\delta}=\theta,$$

where θ is a parameter of length along the tangent line: that is, by equations

$$\begin{aligned}\bar{y}-y&=(\delta+\theta)y'+\frac{\theta\delta}{\rho}Y\\&=(\delta+\theta)\left\{p'\frac{\partial y}{\partial p}+q'\frac{\partial y}{\partial q}+r'\frac{\partial y}{\partial r}\right\}+\frac{\theta\delta}{\rho}Y:\end{aligned}$$

and therefore this tangent at a consecutive point lies within the block. Hence the block, known to contain the osculating plane of the regional geodesic at the initial point, contains also the tangent at a consecutive point: that is, the block contains the osculating flat of the geodesic.

Now the osculating flat of any curve contains the binormal of the curve, as well as the tangent and the prime normal; hence the binormal of the regional geodesic lies within the foregoing block, so that its typical direction-cosine l_3 can be represented by equations

$$l_3=\lambda\frac{\partial y}{\partial p}+\mu\frac{\partial y}{\partial q}+\nu\frac{\partial y}{\partial r}+\omega Y,$$

where λ, μ, ν, ω, are parameters. The binormal is at right angles to the prime normal, and therefore

$$\sum l_3 Y=0;$$

also we have

$$\sum Y\frac{\partial y}{\partial p}=0,\quad \sum Y\frac{\partial y}{\partial q}=0,\quad \sum Y\frac{\partial y}{\partial r}=0;$$

consequently

$$\omega=0,$$

and the direction of the binormal is given by

$$l_3=\lambda\frac{\partial y}{\partial p}+\mu\frac{\partial y}{\partial q}+\nu\frac{\partial y}{\partial r}.$$

Thus the binormal of any regional geodesic lies in the tangent flat of the region.

Moreover, we have

$$Y'=\frac{l_3}{\sigma}-\frac{y'}{\rho},\quad Y=\rho y'',$$

so that

$$y''' = \frac{l_3}{\rho\sigma} - \frac{y'}{\rho^2} - \frac{\rho'}{\rho^2} Y$$

$$= \left(\frac{\lambda}{\rho\sigma} - \frac{p'}{\rho^2}\right)\frac{\partial y}{\partial p} + \left(\frac{\mu}{\rho\sigma} - \frac{q'}{\rho^2}\right)\frac{\partial y}{\partial q} + \left(\frac{\nu}{\rho\sigma} - \frac{r'}{\rho^2}\right)\frac{\partial y}{\partial r} - \frac{\rho'}{\rho^2} Y.$$

Now consider a third tangent at a further neighbouring point; or, what is analytically the equivalent, the terms of the second order in δ_0, the small arc-distance of that neighbouring point from O. The equations of this third tangent are of the form and type

$$\frac{\bar{y} - (y + y'\,\delta_0 + \frac{1}{2}y''\,\delta_0^2)}{y' + y''\,\delta_0 + \frac{1}{2}y'''\,\delta_0^2} = \phi,$$

the second power of δ_0 being retained; the quantity ϕ is a parameter of length along the tangent line. These equations are typified by

$$\begin{aligned}
\bar{y} - y = {} & (\delta_0 + \phi)y' + (\tfrac{1}{2}\delta_0^2 + \phi\delta_0)y'' + \tfrac{1}{2}\phi\delta_0^2 y''' \\
= {} & \frac{\partial y}{\partial p}\left\{(\delta_0 + \phi)p' + \tfrac{1}{2}\phi\delta_0^2\left(\frac{\lambda}{\rho\sigma} - \frac{p'}{\rho^2}\right)\right\} \\
& + \frac{\partial y}{\partial q}\left\{(\delta_0 + \phi)q' + \tfrac{1}{2}\phi\delta_0^2\left(\frac{\mu}{\rho\sigma} - \frac{q'}{\rho^2}\right)\right\} \\
& + \frac{\partial y}{\partial r}\left\{(\delta_0 + \phi)r' + \tfrac{1}{2}\phi\delta_0^2\left(\frac{\nu}{\rho\sigma} - \frac{r'}{\rho^2}\right)\right\} \\
& + Y\left\{(\tfrac{1}{2}\delta_0^2 + \phi\delta_0)\frac{1}{\rho} - \tfrac{1}{2}\phi\delta_0^2\frac{\rho'}{\rho^2}\right\},
\end{aligned}$$

accurately up to the second order of small quantities inclusive; and they shew that the line, thus represented, lies within the block.

Hence the block, known already to contain the osculating flat of the geodesic (and therefore the tangent, a consecutive tangent, and the prime normal of the geodesic) contains a further consecutive tangent, that is, one not contained in the osculating flat; and therefore it is the osculating block of the regional geodesic. The osculating block of any curve contains the trinormal of the curve; and therefore the particular block under consideration contains the trinormal of the regional geodesic. Accordingly, if l_4 denote the typical direction-cosine of the trinormal of the geodesic which thus lies in the block, there must be relations typified by the relation

$$l_4 = \alpha\frac{\partial y}{\partial p} + \beta\frac{\partial y}{\partial q} + \gamma\frac{\partial y}{\partial r} + \eta Y,$$

where α, β, γ, η, are parameters for the line. The trinormal is at right angles to the prime normal, so that

$$\sum l_4 Y = 0;$$

and, as ever,

$$\sum Y \frac{\partial y}{\partial p}=0, \quad \sum Y \frac{\partial y}{\partial q}=0, \quad \sum Y \frac{\partial y}{\partial r}=0:$$

hence

$$\eta=0,$$

and the typical direction-cosine of the trinormal is given by

$$l_4=\alpha \frac{\partial y}{\partial p}+\beta \frac{\partial y}{\partial q}+\gamma \frac{\partial y}{\partial r}.$$

Thus the trinormal of any regional geodesic lies in the tangent flat of the region.

It thus appears that the tangent flat of the region contains the tangent, the binormal, and the trinormal, of a regional geodesic. These three directions are at right angles to one another; and no direction, at right angles to each of the three, can lie in the flat which is only three-dimensional. Also, the block

$$\left\| \bar{y}-y, \quad \frac{\partial y}{\partial p}, \quad \frac{\partial y}{\partial q}, \quad \frac{\partial y}{\partial r}, \quad Y \right\|=0$$

has been proved to be the osculating block of the regional geodesic; it contains the prime normal of the geodesic, as well as the three principal lines of the geodesic which lie in the tangent flat. These four directions are at right angles to one another; and no direction, at right angles to each of the four, can lie in the block which is only four-dimensional.

It follows that, for the set of three directions, constituted by the tangents to the parametric curves of the region, and with direction-cosines typically represented by

$$A^{-\frac{1}{2}} \frac{\partial y}{\partial p}, \quad B^{-\frac{1}{2}} \frac{\partial y}{\partial q}, \quad C^{-\frac{1}{2}} \frac{\partial y}{\partial r},$$

there can be substituted the set of three directions, constituted by the tangent, the binormal, and the trinormal, of any regional geodesic. The substitution provides a set of leading lines particular to the geodesic which is specially useful in examining the properties of any individual (though quite general) geodesic; the earlier set of leading lines, constituted by the directions of the parametric curves at O, is the more useful in considering the characteristic properties of the region without special regard to the relations enforced upon any geodesic which it contains.

Ex. 1. After these results, it is clear that the equations

$$\| \bar{y}-y, \quad y_1, \quad y_2, \quad y_3, \quad Y \|=0$$

represent the osculating block of the geodesic. We have seen that the three directions, typified by y_1, y_2, y_3, and lying in the tangent flat of the region, as leading lines of that flat, are topographically equivalent to the three directions typified by y', l_3, l_4: that is, to the tangent, the binormal, and the trinormal of the geodesic.

It would therefore be an immediate inference that two such blocks, at contiguous points on the regional geodesic in the direction p', q', r', should intersect in the osculating flat of the geodesic. The preceding $N-4$ equations (where N denotes the number of dimensions of the plenary homaloidal space) are equivalent to the $N-4$ equations

$$\sum (\bar{y}-y)\,l_\mu = 0,$$

for $\mu = 5, 6, \ldots, N$. To obtain the intersection of this homaloid by the similar homaloid at a consecutive point along the geodesic, we join with these equations the further set of relations

$$\frac{d}{ds}\left\{\sum (\bar{y}-y)\,l_\mu\right\} = 0.$$

When $\mu = 5$, this relation is

$$\sum \left\{(\bar{y}-y)\left(\frac{l_6}{\rho_5} - \frac{l_4}{\rho_4}\right)\right\} - \sum y' l_5 = 0:$$

that is, in virtue of the remaining equations, the relation becomes

$$\sum (\bar{y}-y)\,l_4 = 0.$$

When $\mu = N$, the relation is

$$\sum (\bar{y}-y)\left(-\frac{l_{N-1}}{\rho_{N-1}}\right) - \sum y' l_N = 0:$$

that is, the relation becomes

$$\sum (\bar{y}-y)\,l_{N-1} = 0,$$

an equation already contained in the intersection-set.

When $5 < \mu < N$, the relation is

$$\sum \left\{(\bar{y}-y)\left(\frac{l_{\mu+1}}{\rho_\mu} - \frac{l_{\mu-1}}{\rho_{\mu-1}}\right)\right\} - \sum y' l_\mu = 0,$$

which, because of the relation $\sum y' l_\mu = 0$, is satisfied by the equations of the intersection-set.

Hence the aggregate of equations for the intersection of the two blocks is

$$\sum (\bar{y}-y)\,l_\mu = 0,$$

for $\mu = 4, 5, \ldots, N$: that is,

$$\|\ \bar{y}-y,\quad y',\quad Y,\quad l_3\ \| = 0,$$

which are the equations of the osculating flat of the geodesic. The inference is thus verified.

Ex. 2. A further inference will be verified, because the mode of verification employs a method which, in extended form, will be useful in the discussion (§ 260) of the orthogonal centre of a region in sextuple plenary space. We consider the intersection of two consecutive osculating flats of the same regional geodesic; it should emerge as the osculating plane. Any point in the foregoing osculating flat is represented by equations

$$\bar{y} - y = Py' + Ql_3 + RY;$$

any point, in the osculating flat of the geodesic at a point at a small arc-distance D along the regional geodesic, is represented by the equations

$$\begin{aligned}\bar{y} - y - y'D &= \bar{P}(y' + y''D) + \bar{Q}(l_3 + l'_3 D) + \bar{R}(Y + Y'D) \\ &= \bar{P}y' + \bar{Q}l_3 + \bar{R}Y \\ &\qquad + D\left\{\bar{P}\frac{Y}{\rho} + \bar{Q}\left(\frac{l_4}{\tau} - \frac{Y}{\sigma}\right) + \bar{R}\left(\frac{l_3}{\sigma} - \frac{y'}{\rho}\right)\right\}.\end{aligned}$$

Where (if at all) the two flats intersect, the two sets of values thus provided for the typical magnitude $\bar{y}$ must be the same; and therefore we have

$$\bar{P} - \bar{R}\frac{D}{\rho} + D = P,$$

$$\bar{P}\frac{D}{\rho} - \bar{Q}\frac{D}{\sigma} + \bar{R} = R,$$

$$\bar{Q} + \bar{R}\frac{D}{\sigma} = Q,$$

$$\bar{Q}D = 0.$$

Hence $\bar{Q} = 0$, from the last equation. In the limit as D tends to zero, the third equation requires the relation $Q = 0$; the second equation requires the relation $\bar{R} = R$; and the first equation requires the relation $\bar{P} = P$. Thus the required intersection is given by the typical relation

$$\bar{y} - y = Py' + RY,$$

where P and R are parametric for all the equations of the set. Hence the two consecutive flats of the same regional geodesic, considered in connection with the expression of their equations in terms of regional magnitudes, intersect in the osculating plane of the geodesic

$$\| \; \bar{y} - y, \quad y' \quad Y \; \| = 0:$$

a result which, if interpreted in connection with the geodesic alone, is merely a verification of a known property.

In the application mentioned, relating to the orthogonal centre of a region in sextuple space, it proves necessary to include terms involving squares of the small arc-distance D.

170. The analytical possibilities, required to ensure the property that the tangent flat contains the tangent, the binormal, and the trinormal, of any geodesic, are seen to suffice for the demands.

In the aggregate of the equations of the flat, there are $3N$ magnitudes in all, being

$$\frac{\partial y_i}{\partial p}, \quad \frac{\partial y_i}{\partial q}, \quad \frac{\partial y_i}{\partial r},$$

for $i = 1, \ldots, N$, where N denotes the dimensionality of the plenary space. Also, for any direction, there are three direction-variables p', q', r'. Hence, in connection with directions associated both with a regional geodesic and with the tangent flat, there are $3N + 3$ disposable quantities.

Now for the spatial direction-cosines of the tangent, being the magnitudes typified by y', there are N distinct cosines; thus N of the disposable quantities will be required.

For the spatial direction-cosines of the binormal, being the magnitudes typified by l_3, there are N distinct cosines; thus N of the disposable quantities will be required. But there is also one limitation, in the form of the condition

$$\sum y' l_3 = 0;$$

consequently, in effect, one more demand is made on the disposable quantities, and so their number must satisfy $N+1$ demands in all, for the determination of the binormal by means of directions.

Similarly, for the trinormal the direction-cosines of which are typified by l_4, there are N distinct cosines: so that N of the disposable quantities will be required. There now are also two limitations, in the form of the conditions

$$\sum y' l_4 = 0, \quad \sum l_3 l_4 = 0,$$

leading to two other requirements; thus, in determining the direction of the trinormal by means of the available magnitudes belonging to the tangent flat, there are $N+2$ demands in all.

Hence the total number of demands, thus made for expressing the direction-cosines of the tangent, the binormal, and the trinormal, of a regional geodesic, is $N+(N+1)+(N+2)$, $=3N+3$, in all, being the same as the number of available magnitudes. Each of the N, $+N$, $+N$, relations for the actual expressions of y', l_3, l_4, is linear in the magnitudes $\dfrac{\partial y}{\partial p}$, $\dfrac{\partial y}{\partial q}$, $\dfrac{\partial y}{\partial r}$; and so, because

$$\sum Y \frac{\partial y}{\partial p} = 0, \quad \sum Y \frac{\partial y}{\partial q} = 0, \quad \sum Y \frac{\partial y}{\partial r} = 0,$$

the conditions

$$\sum Y y' = 0, \quad \sum Y l_3 = 0, \quad \sum Y l_4 = 0,$$

are satisfied, being the analytical conditions that all the three directions thus obtained are at right angles to the prime normal of the regional geodesic.

Accordingly, the respective typical direction-cosines of the three lines are expressible in the respective forms

$$y' = p' \frac{\partial y}{\partial p} + q' \frac{\partial y}{\partial q} + r' \frac{\partial y}{\partial r},$$

$$l_3 = l \frac{\partial y}{\partial p} + m \frac{\partial y}{\partial q} + n \frac{\partial y}{\partial r},$$

$$l_4 = \bar{\alpha} \frac{\partial y}{\partial p} + \bar{\beta} \frac{\partial y}{\partial q} + \bar{\gamma} \frac{\partial y}{\partial r},$$

where the parameters l, m, n, of the binormal, and the parameters $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, of the trinormal, have to be determined.

CHAPTER XV

Torsion, Tilt, Coil, of Regional Geodesics

Generalities on ternariants.

171. At this stage, it is convenient to state some purely algebraical lemmas, relating to the concomitants of ternary forms.

When the parameters p, q, r, of a region are changed to another set p_0, q_0, r_0, by three independent relations of the form

$$f_i(p, q, r, p_0, q_0, r_0)=0, \quad (i=1, 2, 3),$$

the direction-variables p', q', r', are transformed to homogeneous linear combinations of p_0', q_0', r_0', the coefficients in these linear transformations being independent of directions. Accordingly, p', q', r', can be taken as variables for ternary forms; variables ξ, η, ζ, composed of two cogredient sets of such variables according to the law

$$\xi,\ \eta,\ \zeta = \begin{vmatrix} p_1', & q_1', & r_1' \\ p_2', & q_2', & r_2' \end{vmatrix},$$

are contragredient to p', q', r', as also are the operators

$$\frac{\partial}{\partial p'},\quad \frac{\partial}{\partial q'},\quad \frac{\partial}{\partial r'};$$

and a magnitude

$$\begin{vmatrix} p_1', & q_1', & r_1' \\ p_2', & q_2', & r_2' \\ p_3', & q_3', & r_3' \end{vmatrix},$$

composed of three cogredient sets, is a covariant independent of any configuration.

In connection with a region, there are homogeneous ternary forms. Thus there is the quadratic form

$$U=\sum Ap'^2,$$

which, in value, is equal to unity; for analytical purposes, it is convenient to retain the symbol U, and (as on p. 481) we shall write

$$u_1,\ u_2,\ u_3=\tfrac{1}{2}\frac{\partial U}{\partial p'},\quad \tfrac{1}{2}\frac{\partial U}{\partial q'},\quad \tfrac{1}{2}\frac{\partial U}{\partial r'}.$$

Again, there is the quadratic form

$$V=\sum \bar{A}p'^2,$$

which, in value, is the circular curvature of a regional geodesic in the direction p', q', r', through the point O: the coefficients $\bar{A}$ implicitly involve the variables p', q', r', but we are concerned (for the moment only) with explicit algebraical expressions. We shall write *

$$v_1,\ v_2,\ v_3 = \tfrac{1}{2}\frac{\partial V}{\partial p'},\quad \tfrac{1}{2}\frac{\partial V}{\partial q'},\quad \tfrac{1}{2}\frac{\partial V}{\partial r'},$$

where the partial derivatives with respect to p', q', r', relate solely to the explicit occurrences of p', q', r', in the quantity V, though (as at p. 71) the results are still correct if account is taken of the implicit occurrence of p', q', r', in the coefficients in V.

The set of variables u_1, u_2, u_3, is contragredient to the set p', q', r', as also is the set v_1, v_2, v_3; the set of variables

$$\left\| \begin{matrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{matrix} \right\|$$

is cogredient with p', q', r', as also are the two sets of operators

$$\frac{\partial}{\partial u_1},\quad \frac{\partial}{\partial u_2},\quad \frac{\partial}{\partial u_3};\quad \frac{\partial}{\partial v_1},\quad \frac{\partial}{\partial v_2},\quad \frac{\partial}{\partial v_3};$$

when these operate upon covariantive functions of u_1, u_2, u_3; v_1, v_2, v_3. Similar remarks apply to any ternary form, such as the quartic Q of § 168, and to its first-order partial derivatives with respect to p', q', r'.

The discriminant Ω of the form U is

$$\Omega = \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix},$$

and a, b, c, f, g, h, denote its first minors. Similarly, there is the formal discriminant $\bar{\Omega}$ of the form V, given by

$$\bar{\Omega} = \begin{vmatrix} \bar{A}, & \bar{H}, & \bar{G} \\ \bar{H}, & \bar{B}, & \bar{F} \\ \bar{G}, & \bar{F}, & \bar{C} \end{vmatrix},$$

while $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{f}$, $\bar{g}$, $\bar{h}$, are used to denote its first minors. Both of these quantities Ω, $\bar{\Omega}$, are invariants of the system for U and V, as also are the intermediate invariants

$$A\bar{a} + 2H\bar{h} + B\bar{b} + 2G\bar{g} + 2F\bar{f} + C\bar{c} = \bar{\omega},$$
$$\bar{A}a + 2\bar{H}h + \bar{B}b + 2\bar{G}g + 2\bar{F}f + \bar{C}c = \omega,$$

such that the discriminant of $\lambda U + \mu V$ is

$$\lambda^3\Omega + \lambda^2\mu\omega + \lambda\mu^2\bar{\omega} + \mu^3\bar{\Omega}.$$

* The notation u_1, u_2, u_3; v_1, v_2, v_3 is used in conformity with the notation of § 31.

There are concomitants

$$\sum a u_1^2, \quad \sum a u_1 v_1, \quad \sum a v_1^2, \quad \sum \bar{a} u_1^2, \quad \sum \bar{a} u_1 v_1, \quad \sum \bar{a} v_1^2,$$

belonging to the system of U and V.

In the same way, a set of surface-variables ξ, η, ζ, is contragredient to the set of variables p', q', r', the quantity

$$p'\xi + q'\eta + r'\zeta$$

being covariantive by itself. We already have had the concomitant $\sum a\xi^2$: other instances will occur later.

When such forms, whether invariants, covariants, contravariants, or mixed concomitants, occur in connection with a region, it usually is required to assign a geometrical significance to an invariantive expression. The following results will be found useful:

$$\left.\begin{array}{l} a u_1 + h u_2 + g u_3 = \Omega p' \\ h u_1 + b u_2 + f u_3 = \Omega q' \\ g u_1 + f u_2 + c u_3 = \Omega r' \end{array}\right\}, \qquad \left.\begin{array}{l} \bar{a} v_1 + \bar{h} v_2 + \bar{g} v_3 = \bar{\Omega} p' \\ \bar{h} v_1 + \bar{b} v_2 + \bar{f} v_3 = \bar{\Omega} q' \\ \bar{g} v_1 + \bar{f} v_2 + \bar{c} v_3 = \bar{\Omega} r' \end{array}\right\},$$

$$\sum a u_1^2 = \Omega, \qquad \sum \bar{a} v_1^2 = \frac{\bar{\Omega}}{\rho},$$

$$\sum a u_1 v_1 = \frac{\Omega}{\rho}, \qquad \sum \bar{a} u_1 v_1 = \bar{\Omega}.$$

These are, of course, only individual concomitants; all such concomitants can be expressed algebraically in terms of a selected set, which contain only a limited number of members.

Binormal of a geodesic: the torsion.

172. To determine the parameters l, m, n, in the expression for the typical direction-cosine l_3 of the binormal in the form

$$l_3 = l y_1 + m y_2 + n y_3,$$

we begin with the typical Frenet formula

$$Y' = \frac{l_3}{\sigma} - \frac{y'}{\rho},$$

so that the typical equation

$$\frac{1}{\sigma}(l y_1 + m y_2 + n y_3) = \frac{y'}{\rho} + Y'$$

holds for each of the variables corresponding to the space-dimensions. The values of l, m, n, must be obtained. Now

$$\sum Y y_1 = 0, \quad \sum Y y_2 = 0, \quad \sum Y y_3 = 0;$$

and therefore, differentiating each of these equations along the geodesic, we have

$$\sum y_1 Y' = -\sum Y y_1' = -\sum Y(y_{11}p' + y_{12}q' + y_{13}r') = -(\bar{A}p' + \bar{H}q' + \bar{G}r') = -v_1,$$
$$\sum y_2 Y' = -\sum Y y_2' = -\sum Y(y_{21}p' + y_{22}q' + y_{23}r') = -(\bar{H}p' + \bar{B}q + \bar{F}r') = -v_2,$$
$$\sum y_3 Y' = -\sum Y y_3' = -\sum Y(y_{31}p' + y_{32}q' + y_{33}r') = -(\bar{G}p' + \bar{F}q' + \bar{C}r') = -v_3.$$

Also we have

$$\sum y_1 y' = Ap' + Hq' + Gr' = u_1,$$
$$\sum y_2 y' = Hp' + Bq' + Fr' = u_2,$$
$$\sum y_3 y' = Gp' + Fq' + Cr' = u_3.$$

Hence, multiplying the equation in l, m, n, by y_1 and adding: then by y_2, and adding: lastly by y_3, and adding: we have, successively,

$$\frac{1}{\sigma}(Al + Hm + Gn) = \frac{1}{\rho}u_1 - v_1,$$
$$\frac{1}{\sigma}(Hl + Bm + Fn) = \frac{1}{\rho}u_2 - v_2,$$
$$\frac{1}{\sigma}(Gl + Fm + Cn) = \frac{1}{\rho}u_3 - v_3.$$

We write

$$\left.\begin{aligned} \Omega\bar{v}_1 &= av_1 + hv_2 + gv_3 \\ \Omega\bar{v}_2 &= hv_1 + bv_2 + fv_3 \\ \Omega\bar{v}_3 &= gv_1 + fv_2 + cv_3 \end{aligned}\right\}:$$

then the foregoing equations, resolved for l, m, n, give

$$\left.\begin{aligned} \frac{l}{\sigma} &= \frac{p'}{\rho} - \bar{v}_1 \\ \frac{m}{\sigma} &= \frac{q'}{\rho} - \bar{v}_2 \\ \frac{n}{\sigma} &= \frac{r'}{\rho} - \bar{v}_3 \end{aligned}\right\},$$

so that the direction-cosines of the binormal are known.

An expression for the torsion of the regional geodesic can be obtained. We have

$$\begin{aligned} 1 = \sum l_3^2 &= \sum (ly_1 + my_2 + ny_3)^2 \\ &= Al^2 + 2Hlm + Bm^2 + 2Gln + 2Fmn + Cn^2. \end{aligned}$$

Let the preceding values of l, m, n, be substituted. There are three sets of terms. In one of them, we have

$$\frac{\sigma^2}{\rho^2}\sum Ap'^2, \quad = \frac{\sigma^2}{\rho^2}.$$

In the second set, we have

$$-2\frac{\sigma^2}{\rho}\{p'(A\bar{v}_1+H\bar{v}_2+G\bar{v}_3)+q'(H\bar{v}_1+B\bar{v}_2+F\bar{v}_3)+r'(G\bar{v}_1+F\bar{v}_2+C\bar{v}_3)\};$$

but

$$A\bar{v}_1+H\bar{v}_2+G\bar{v}_3=v_1,\quad H\bar{v}_1+B\bar{v}_2+F\bar{v}_3=v_2,\quad G\bar{v}_1+F\bar{v}_2+C\bar{v}_3=v_3,$$

and

$$v_1p'+v_2q'+v_3r'=\frac{1}{\rho},$$

so that the second set of terms

$$=-2\frac{\sigma^2}{\rho^2}.$$

The third set of terms

$$=\sigma^2\sum A\bar{v}_1^{\,2}$$
$$=\sigma^2\{\bar{v}_1(A\bar{v}_1+H\bar{v}_2+G\bar{v}_3)+\bar{v}_2(H\bar{v}_1+B\bar{v}_2+G\bar{v}_3)+\bar{v}_3(G\bar{v}_1+F\bar{v}_2+C\bar{v}_3)\}$$
$$=\sigma^2(v_1\bar{v}_1+v_2\bar{v}_2+v_3\bar{v}_3)$$
$$=\frac{\sigma^2}{\Omega}\sum av_1^{\,2}.$$

Hence we have

$$\Omega\left(\frac{1}{\sigma^2}+\frac{1}{\rho^2}\right)=\sum av_1^{\,2},$$

a relation giving a value for the torsion, the circular curvature being known.

An expression for the torsion alone can be deduced at once. We have

$$\begin{aligned}(\textstyle\sum au_1^{\,2})(\sum av_1^{\,2})-(\sum au_1v_1)^2&=\textstyle\sum(bc-f^2)(u_2v_3-u_3v_2)^2\\&=\Omega\textstyle\sum A(u_2v_3-u_3v_2)^2;\end{aligned}$$

while

$$\sum au_1^{\,2}=\Omega,\quad \sum au_1v_1=\frac{\Omega}{\rho},\quad \sum av_1^{\,2}=\Omega\left(\frac{1}{\sigma^2}+\frac{1}{\rho^2}\right);$$

and therefore

$$\frac{\Omega}{\sigma^2}=\sum A(u_2v_3-u_3v_2)^2,$$

a result that will appear later, in connection with the direction-cosines of the trinormal.

The result can be written

$$\frac{\Omega}{\sigma^2}=\begin{vmatrix} A, & H, & G, & u_1, & v_1\\ H, & B, & F, & u_2, & v_2\\ G, & F, & C, & u_3, & v_3\\ u_1, & u_2, & u_3, & 0, & 0\\ v_1, & v_2, & v_3, & 0, & 0\end{vmatrix},$$

in accordance with the form of the result for the torsion of any amplitudinal geodesic (§ 35).

The condition $\sum l_3 y' = 0$, which is organic to the frame of the geodesic, must be satisfied by the analytical expression obtained for l_3. Now

$$\sum y' y_1 = u_1, \quad \sum y' y_2 = u_2, \quad \sum y' y_3 = u_3 ;$$

and therefore

$$\begin{aligned}\sum l_3 y' &= u_1 l + u_2 m + u_3 n \\ &= \sigma \left\{ \frac{1}{\rho} (u_1 p' + u_2 q' + u_3 r') - (u_1 \bar{v}_1 + u_2 \bar{v}_2 + u_3 \bar{v}_3) \right\}.\end{aligned}$$

But

$$u_1 p' + u_2 q' + u_3 r' = 1,$$

and

$$\begin{aligned}&u_1 \bar{v}_1 + u_2 \bar{v}_2 + u_3 \bar{v}_3 \\ &\qquad = \frac{1}{\Omega} \{u_1 (a v_1 + h v_2 + g v_3) + u_2 (h v_1 + b v_2 + f v_3) + u_3 (g v_1 + f v_2 + c v_3)\} ;\end{aligned}$$

also

$$a u_1 + h u_2 + g u_3 = \Omega p', \quad h u_1 + b u_2 + f u_3 = \Omega q', \quad g u_1 + f u_2 + c u_3 = \Omega r',$$

so that

$$u_1 \bar{v}_1 + u_2 \bar{v}_2 + u_3 \bar{v}_3 = v_1 p' + v_2 q' + v_3 r' = \frac{1}{\rho}.$$

Hence the analytic expressions give

$$\sum y' l_3 = 0,$$

as required.

Moreover, an expression can be inferred for Y' in terms of regional magnitudes. When the values of l, m, n, are substituted in the formula

$$l_3 = l y_1 + m y_2 + n y_3$$

for the typical direction-cosine of the binormal, we have

$$\begin{aligned}\frac{l_3}{\sigma} &= \frac{1}{\rho} (y_1 p' + y_2 q' + y_3 r') - (y_1 \bar{v}_1 + y_2 \bar{v}_2 + y_3 \bar{v}_3) \\ &= \frac{y'}{\rho} - (y_1 \bar{v}_1 + y_2 \bar{v}_2 + y_3 \bar{v}_3).\end{aligned}$$

The Frenet equation, which affects the direction of the binormal, is

$$\frac{l_3}{\sigma} = \frac{y'}{\rho} + Y' ;$$

consequently

$$Y' = -(y_1 \bar{v}_1 + y_2 \bar{v}_2 + y_3 \bar{v}_3),$$

the expression indicated.

Also, because

$$Y = y'' \rho,$$

differentiation along the geodesic gives

$$\begin{aligned}y''' &= -\frac{\rho'}{\rho} y'' + \frac{1}{\rho} Y' \\ &= -\frac{\rho'}{\rho^2} Y - \frac{1}{\rho} (y_1 \bar{v}_1 + y_2 \bar{v}_2 + y_3 \bar{v}_3).\end{aligned}$$

A more generally useful expression for y''' will be obtained immediately; meanwhile, we have a result

$$\sum Yy''' = -\frac{\rho'}{\rho^2} = \frac{d}{ds}\left(\frac{1}{\rho}\right),$$

one form of the value of the arc-rate of change of the circular curvature of the regional geodesic.

Also, it will be useful to have the relations

$$\begin{aligned}\sum y' y_{ij} &= \Gamma_{ij} u_1 + \Delta_{ij} u_2 + \Theta_{ij} u_3,\\ \sum Y' y_{ij} &= -(\Gamma_{ij} v_1 + \Delta_{ij} v_2 + \Theta_{ij} v_3);\end{aligned}$$

the verifications are simple.

It is convenient, in connection with these linear combinations in the magnitudes Γ_{ij}, Δ_{ij}, Θ_{ij}, to introduce some abbreviating symbols*, defined by the relations

$$\left.\begin{aligned}\Gamma_{11}p' + \Gamma_{12}q' + \Gamma_{13}r' &= a\\ \Gamma_{21}p' + \Gamma_{22}p' + \Gamma_{23}r' &= \beta\\ \Gamma_{31}p' + \Gamma_{32}p' + \Gamma_{33}r' &= \gamma\end{aligned}\right\},\quad \left.\begin{aligned}\Delta_{11}p' + \Delta_{12}q' + \Delta_{13}r' &= \xi\\ \Delta_{21}p' + \Delta_{22}q' + \Delta_{23}r' &= \eta\\ \Delta_{31}p' + \Delta_{32}q' + \Delta_{33}r' &= \zeta\end{aligned}\right\},\quad \left.\begin{aligned}\Theta_{11}p' + \Theta_{12}q' + \Theta_{13}r' &= \phi\\ \Theta_{21}p' + \Theta_{22}q' + \Theta_{23}r' &= \chi\\ \Theta_{31}p' + \Theta_{32}q' + \Theta_{33}r' &= \psi\end{aligned}\right\},$$

satisfying the equations

$$\left.\begin{aligned}-p'' &= ap' + \beta q' + \gamma r'\\ -q'' &= \xi p' + \eta q' + \zeta r'\\ -r'' &= \phi p' + \chi q' + \psi r'\end{aligned}\right\}.$$

In particular, we have (as in § 161)

$$\begin{aligned}\frac{dA}{ds} &= 2(Aa + H\xi + G\phi),\\ \frac{dB}{ds} &= 2(H\beta + B\eta + F\chi),\\ \frac{dC}{ds} &= 2(G\gamma + F\zeta + C\psi),\\ \frac{dF}{ds} &= (H\gamma + B\zeta + F\psi) + (G\beta + F\eta + C\chi),\\ \frac{dG}{ds} &= (Ga + F\xi + C\phi) + (A\gamma + H\zeta + G\psi),\\ \frac{dH}{ds} &= (A\beta + H\eta + G\chi) + (Ha + B\xi + F\phi).\end{aligned}$$

Ex. 1. Obtain values for the quantities $\sum y_i y_j'$, for all the combinations $i, j, = 1, 2, 3$, taken independently of one another.

* They have a particularised significance which is in agreement with the general symbols $g_{\mu i}^{(p)}$ of p. 148.

Ex. 2. Verify the formulæ:

$$\left.\begin{aligned}\textstyle\sum y'y_1' &= u_1\alpha + u_2\xi + u_3\phi\\ \textstyle\sum y'y_2' &= u_1\beta + u_2\eta + u_3\chi\\ \textstyle\sum y'y_3' &= u_1\gamma + u_2\zeta + u_3\psi\end{aligned}\right\},\qquad \left.\begin{aligned}-\textstyle\sum Y'y_1' &= v_1\alpha + v_2\xi + v_3\phi\\ -\textstyle\sum Y'y_2' &= v_1\beta + v_2\eta + v_3\chi\\ -\textstyle\sum Y'y_3' &= v_1\gamma + v_2\zeta + v_3\psi\end{aligned}\right\};$$

and deduce the values of $\sum l_3 y_i'$, for $i=1, 2, 3$.

Value of y''' along a geodesic.

173. The value of y'' along a regional geodesic is given by

$$y'' = \textstyle\sum \eta_{11} p'^2;$$

differentiation along the geodesic leads to the relation

$$\begin{aligned}y''' = \frac{d\eta_{11}}{ds} p'^2 + 2\frac{d\eta_{12}}{ds} p'q' + \frac{d\eta_{22}}{ds} q'^2 + 2\frac{d\eta_{13}}{ds} p'r' + 2\frac{d\eta_{23}}{ds} q'r' + \frac{d\eta_{33}}{ds} r'^2\\ + 2(\eta_{11}p' + \eta_{12}q' + \eta_{13}r')p'' + 2(\eta_{12}p' + \eta_{22}q' + \eta_{23}r')q''\\ + 2(\eta_{13}p' + \eta_{23}q' + \eta_{33}r')r''.\end{aligned}$$

In connection with various symmetrical sums, we use a symbol defined by the equation

$$\eta_{1k}\Gamma_{ij} + \eta_{2k}\Delta_{ij} + \eta_{3k}\Theta_{ij} = \eta_k(ij),$$

similar to the symbol (p. 465)

$$\Phi_{1\mu}\Gamma_{ij} + \Phi_{2k}\Delta_{ij} + \Phi_{3k}\Theta_{ij} = \Phi_\mu(ij),$$

for $\Phi = \Gamma, \Delta, \Theta$. When substitution is made for the arc-derivatives of the quantities η_{ij} and for p'', q'', r'', we have a homogeneous ternary form for y''', analogous to the forms (§ 163) for p''', q''', r'''; we take

$$\begin{aligned}y''' = \eta_{111}p'^3 + 3\eta_{112}p'^2q' + 3\eta_{122}p'q'^2 + \eta_{222}q'^3\\ + 3\eta_{113}p'^2r' + 6\eta_{123}p'q'r' + 3\eta_{223}q'^2r'\\ + 3\eta_{133}p'r'^2 + 3\eta_{233}q'r'^2\\ + \eta_{333}r'^3\\ = \textstyle\sum \eta_{111}p'^3,\end{aligned}$$

where

$$\eta_{111} = \frac{\partial\eta_{11}}{\partial p} - 2\eta_1(11),$$

$$3\eta_{112} = \frac{\partial\eta_{11}}{\partial q} + 2\frac{\partial\eta_{12}}{\partial p} - 4\eta_1(12) - 2\eta_2(11),$$

and so on, the formulæ being similar to the formulæ (§ 163) for Γ_{ijk} with the change of Γ_{ij} into η_{ij}, for all values of i, j.

We have

$$\eta_{ij} = y_{ij} - y_1\Gamma_{ij} - y_2\Delta_{ij} - y_3\Theta_{ij};$$

and therefore, with the conventions $x_1=p$, $x_2=q$, $x_3=r$,

$$\frac{\partial \eta_{ij}}{\partial x_k}=y_{ijk}-y_{1k}\Gamma_{ij}-y_{2k}\Delta_{ij}-y_{3k}\Theta_{ij}-y_1\frac{\partial \Gamma_{ij}}{\partial x_k}-y_2\frac{\partial \Delta_{ij}}{\partial x_k}-y_3\frac{\partial \Theta_{ij}}{\partial x_k}.$$

But

$$\begin{aligned} y_{1k}\Gamma_{ij}+y_{2k}\Delta_{ij}+y_{3k}\Theta_{ij} &= (\eta_{1k}+y_1\Gamma_{1k}+y_2\Delta_{1k}+y_3\Theta_{1k})\Gamma_{ij} \\ &\quad +(\eta_{2k}+y_1\Gamma_{2k}+y_2\Delta_{2k}+y_3\Theta_{2k})\Delta_{ij} \\ &\quad +(\eta_{3k}+y_1\Gamma_{3k}+y_2\Delta_{3k}+y_3\Theta_{3k})\Theta_{ij} \\ &= \eta_k(ij)+y_1\Gamma_k(ij)+y_2\Delta_k(ij)+y_3\Theta k(ij)\,; \end{aligned}$$

and from the values of the derivatives of Γ_{ij}, Δ_{ij}, Θ_{ij}, in § 162,

$$\begin{aligned} & y_1\frac{\partial \Gamma_{ij}}{\partial x_k}+y_2\frac{\partial \Delta_{ij}}{\partial x_k}+y_3\frac{\partial \Theta_{ij}}{\partial x_k} \\ &\quad = y_1[\Gamma_{ijk}+\Gamma_i(jk)+\Gamma_j(ik)]+y_2[\Delta_{ijk}+\Delta_i(jk)+\Delta_j(ik)]+y_3[\Theta_{ijk}+\Theta_i(jk)+\Theta_j(ik)] \\ &\qquad -\frac{1}{3\Omega}\Big[(ay_1+hy_2+gy_3)\{(1i,jk)+(1j,ik)\} \\ &\qquad\qquad +(hy_1+by_2+fy_3)\{(2i,jk)+(2j,ik)\} \\ &\qquad\qquad +(gy_1+fy_2+cy_3)\{(3i,jk)+(3j,ik)\}\Big]. \end{aligned}$$

We denote the coefficient of $-\frac{1}{3\Omega}$ in the last equation by $\Psi_\mu(ij)$. Then

$$\begin{aligned} \frac{\partial \eta_{ij}}{\partial x_k} &= y_{ijk}-\eta_k(ij)-y_1\{\Gamma_{ijk}+\Gamma_i(jk)+\Gamma_j(ik)+\Gamma_k(ij)\} \\ &\quad -y_2\{\Delta_{ijk}+\Delta_i(jk)+\Delta_j(ik)+\Delta_k(ij)\} \\ &\quad -y_3\{\Theta_{ijk}+\Theta_i(jk)+\Theta_j(ik)+\Theta_\mu(ij)\}-\frac{1}{3\Omega}\Psi_\kappa(ij). \end{aligned}$$

Now the general value of η_{ijk} is given by

$$6\eta_{ijk}=2\frac{\partial \eta_{ij}}{\partial x_k}+2\frac{\partial \eta_{jk}}{\partial x_i}+2\frac{\partial \eta_{ki}}{\partial x_j}-4\eta_i(jk)-4\eta_j(ki)-4\eta_k(ij).$$

When substitution is made for the derivatives of the quantities η_{ij}, η_{jk}, η_{ki}, according to the preceding result, there are two sets of terms, one set involving the four-index symbols through the quantities Ψ, the other set free from those symbols. In the combination

$$\Psi_i(jk)+\Psi_j(ki)+\Psi_k(ij),$$

the total coefficient of $ay_1+by_2+gy_3$

$$=\{(1i,jk)+(1j,ik)\}+\{(1j,ki)+(1k,ji)\}+\{(1k,ij)+(1i,kj)\}$$

which vanishes identically, because of the properties of the four-index symbol. Similarly, the respective coefficients of $hy_1+by_2+fy_3$, $gy_1+fy_2+cy_3$, vanish. Consequently, the aggregate of this set of terms is zero.

The remaining set of terms, after reduction, leads to the equation

$$\eta_{ijk} = y_{ijk} - \eta_i(jk) - \eta_j(ki) - \eta_k(ij) - y_1\bar{\Gamma}_{ijk} - y_2\bar{\Delta}_{ijk} - y_3\bar{\Theta}_{ijk},$$

where

$$\bar{\Phi}_{ijk} = \Phi_{ijk} + \Phi_i(jk) + \Phi_j(ki) + \Phi_k(ij),$$

when $\Phi = \Gamma, \Delta, \Theta$: the result holding for the values $i, j, k, = 1, 2, 3$, taken independently of one another. Also, it is convenient to express the derivatives of the magnitudes η_{ij} in terms of the magnitudes η_{ijk} rather than of the derivatives of the space-coordinates: the relation is

$$\frac{\partial \eta_{ij}}{\partial x_k} = \eta_{ijk} + \eta_i(jk) + \eta_j(ik) - \frac{1}{3\Omega}\Psi_k(ij),$$

where

$$\eta_a(\beta\gamma) = \eta_{1a}\Gamma_{\beta\gamma} + \eta_{2a}\Delta_{\beta\gamma} + \eta_{3a}\Theta_{\beta\gamma}$$

$$\begin{aligned}\Psi_a(\beta\gamma) = (ay_1 + hy_2 + gy_3)&\{(1\beta, \gamma a) + (1\gamma, \beta a)\} \\ + (hy_1 + by_2 + fy_3)&\{(2\beta, \gamma a) + (2\gamma, \beta a)\} \\ + (gy_1 + fy_2 + cy_3)&\{(3\beta, \gamma a) + (3\gamma, \beta a)\}.\end{aligned}$$

With these values for the quantities η_{ijk}, the expression for y''' is

$$y''' = \sum \eta_{111} p'^3.$$

The same result is obtained when, in the expression

$$\begin{aligned}y''' = y_1 p''' + y_2 q''' + y_3 r''' &+ \sum y_{111} p'^3 \\ &+ 3(y_{11}p' + y_{12}q' + y_{13}r')p'' \\ &+ 3(y_{12}p' + y_{22}q' + y_{23}r')q'' \\ &+ 3(y_{13}p' + y_{23}q' + y_{33}r')r'',\end{aligned}$$

which is quite general, we insert the geodesic values of $p'', q'', r'', p''', q''', r'''$, and use the relations expressing y_{ij} in terms of η_{ij} and y_{ijk} in terms of η_{ijk}.

Expressions for $\frac{d}{ds}\left(\frac{1}{\rho}\right), \frac{du_\lambda}{ds}, \frac{dv_\lambda}{ds}$.

174. We can at once obtain an expression for the arc-variation of the circular curvature along a geodesic. The relation

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \sum Y y'''$$

has been established (p. 494); accordingly, if we write

$$e_{ijk} = \sum Y \eta_{ijk},$$

where the quantities η_{ijk} are the coefficients in the expression for y''', and when we substitute the value of y''' in the relation, we have

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \sum e_{111} p'^3.$$

More explicit values, in terms of original magnitudes connected with the representation of the region and of magnitudes already used, can be obtained for these coefficients e_{ijk}. Let the value of η_{ijk} in terms of y_{ijk} be substituted in the expression for e_{ijk}. There are new magnitudes E_{ijk}, defined by

$$E_{ijk}=\sum Yy_{ijk}.$$

Also, we have

$$\sum Yy_l=0,$$

for $l=1, 2, 3$, so that all the terms in $\bar{\Gamma}_{ijk}$, $\bar{\Delta}_{ijk}$, $\bar{\Theta}_{ijk}$, disappear. Then we find

$$\left.\begin{aligned}
e_{111}&=E_{111}-3(\bar{A}\Gamma_{11}+\bar{H}\Delta_{11}+\bar{G}\Delta_{11})\\
e_{112}&=E_{112}-2(\bar{A}\Gamma_{12}+\bar{H}\Delta_{12}+\bar{G}\Delta_{12})-(\bar{H}\Gamma_{11}+\bar{B}\Delta_{11}+\bar{F}\Theta_{11})\\
e_{113}&=E_{113}-2(\bar{A}\Gamma_{13}+\bar{H}\Delta_{13}+\bar{G}\Delta_{13})-(\bar{G}\Gamma_{11}+\bar{F}\Delta_{11}+\bar{C}\Theta_{11})\\
e_{122}&=E_{122}-(\bar{A}\Gamma_{22}+\bar{H}\Delta_{22}+\bar{G}\Theta_{22})-2(\bar{H}\Gamma_{12}+\bar{B}\Delta_{12}+\bar{F}\Theta_{12})\\
e_{222}&=E_{222}-3(\bar{H}\Gamma_{22}+\bar{B}\Delta_{22}+\bar{F}\Theta_{22})\\
e_{223}&=E_{223}-(\bar{G}\Gamma_{22}+\bar{F}\Delta_{22}+\bar{C}\Theta_{22})-2(\bar{H}\Gamma_{23}+\bar{B}\Delta_{23}+\bar{F}\Theta_{23})\\
e_{133}&=E_{133}-(\bar{A}\Gamma_{33}+\bar{H}\Delta_{33}+\bar{G}\Theta_{33})-2(\bar{G}\Gamma_{13}+\bar{F}\Delta_{13}+\bar{C}\Theta_{13})\\
e_{233}&=E_{233}-(\bar{H}\Gamma_{33}+\bar{B}\Delta_{33}+\bar{F}\Theta_{33})-2(\bar{G}\Gamma_{23}+\bar{F}\Delta_{23}+\bar{C}\Theta_{23})\\
e_{333}&=E_{333}-3(\bar{G}\Gamma_{33}+\bar{F}\Delta_{33}+\bar{C}\Theta_{33})\\
e_{123}&=E_{123}-(\bar{A}\Gamma_{23}+\bar{H}\Delta_{23}+\bar{G}\Theta_{23})-(\bar{H}\Gamma_{31}+\bar{B}\Delta_{31}+\bar{F}\Theta_{31})-(\bar{G}\Gamma_{12}+\bar{F}\Delta_{12}+\Theta\bar{C}_{12})
\end{aligned}\right\}.$$

The expression for $\dfrac{d}{ds}\left(\dfrac{1}{\rho}\right)$ can be obtained otherwise. We have

$$\bar{A}=\sum Y\eta_{11}=\sum Yy_{11},$$

and therefore, differentiating along the regional geodesic,

$$\begin{aligned}
\frac{d\bar{A}}{ds}&=\sum Y(y_{111}p'+y_{112}q'+y_{113}r')+\sum Y'y_{11}\\
&=E_{111}p'+E_{112}q'+E_{113}r'-(\Gamma_{11}v_1+\Delta_{11}v_2+\Theta_{11}v_3),
\end{aligned}$$

by the results in § 172; and similarly

$$\frac{d\bar{H}}{ds}=E_{112}p'+E_{122}q'+E_{123}r'-(\Gamma_{12}v_1+\Delta_{12}v_2+\Theta_{12}v_3),$$

$$\frac{d\bar{B}}{ds}=E_{122}p'+E_{222}q'+E_{223}r'-(\Gamma_{22}v_1+\Delta_{22}v_2+\Theta_{22}v_3),$$

$$\frac{d\bar{G}}{ds}=E_{113}p'+E_{123}q'+E_{133}r'-(\Gamma_{13}v_1+\Delta_{13}v_2+\Theta_{13}v_3),$$

$$\frac{d\bar{F}}{ds}=E_{123}p'+E_{223}q'+E_{233}r'-(\Gamma_{23}v_1+\Delta_{23}v_2+\Theta_{23}v_3),$$

$$\frac{d\bar{C}}{ds}=E_{133}p'+E_{233}q'+E_{333}r'-(\Gamma_{33}v_1+\Delta_{33}v_2+\Theta_{33}v_3).$$

By the relations between e_{ijk} and E_{ijk}, we have

$$\begin{aligned}
E_{111}p' + E_{112}q' + E_{113}r' - (e_{111}p' + e_{112}q' + e_{113}r')
&= 3p'(\bar{A}\Gamma_{11} + \bar{H}\Delta_{11} + \bar{G}\Theta_{11}) \\
&\quad + q'(\bar{H}\Gamma_{11} + \bar{B}\Delta_{11} + \bar{F}\Theta_{11}) + 2q'(\bar{A}\Gamma_{12} + \bar{H}\Delta_{12} + \bar{G}\Theta_{12}) \\
&\quad + r'(\bar{G}\Gamma_{11} + \bar{F}\Delta_{11} + \bar{C}\Theta_{11}) + 2r'(\bar{A}\Gamma_{13} + \bar{H}\Delta_{13} + \bar{G}\Theta_{13}) \\
&= \Gamma_{11}v_1 + \Delta_{11}v_2 + \Theta_{11}v_3 + 2(\bar{A}\alpha + \bar{H}\xi + \bar{G}\phi),
\end{aligned}$$

with the symbols of § 172. Hence

$$\frac{d\bar{A}}{ds} = e_{111}p' + e_{112}q' + e_{113}r' + 2(\bar{A}\alpha + \bar{H}\xi + \bar{G}\phi).$$

Similarly, we find

$$\frac{d\bar{B}}{ds} = e_{122}p' + e_{222}q' + e_{223}r' + 2(\bar{H}\beta + \bar{B}\eta + \bar{F}\chi),$$

$$\frac{d\bar{C}}{ds} = e_{133}p' + e_{233}q' + e_{333}r' + 2(\bar{G}\gamma + \bar{F}\zeta + \bar{C}\psi),$$

$$\frac{d\bar{F}}{ds} = e_{123}p' + e_{223}q' + e_{233}r' + (\bar{H}\gamma + \bar{B}\zeta + \bar{F}\psi) + (\bar{G}\beta + \bar{F}\eta + \bar{C}\chi),$$

$$\frac{d\bar{G}}{ds} = e_{113}p' + e_{123}q' + e_{133}r' + (\bar{G}\alpha + \bar{F}\xi + \bar{C}\phi) + (\bar{A}\gamma + \bar{H}\zeta + \bar{G}\psi),$$

$$\frac{d\bar{H}}{ds} = e_{112}p' + e_{122}q' + e_{123}r' + (\bar{A}\beta + \bar{H}\eta + \bar{G}\chi) + (\bar{H}\alpha + \bar{B}\xi + \bar{F}\phi).$$

The circular curvature of a regional geodesic is given by

$$\frac{1}{\rho} = \sum \bar{A}p'^2\,;$$

and therefore, on differentiating along the geodesic,

$$\begin{aligned}
\frac{d}{ds}\left(\frac{1}{\rho}\right) = \frac{d\bar{A}}{ds}p'^2 + 2\frac{d\bar{H}}{ds}p'q' + \frac{d\bar{B}}{ds}q'^2 + 2\frac{d\bar{G}}{ds}p'r' + 2\frac{d\bar{F}}{ds}q'r' + \frac{d\bar{C}}{ds}r'^2 \\
+ 2v_1p'' + 2v_2q'' + 2v_3r''.
\end{aligned}$$

When substitution is made for the arc-derivatives of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{F}$, $\bar{G}$, $\bar{H}$, in terms of the quantities e_{ijk}, the aggregate of terms consists of two sets. The set involving the quantities e_{ijk}

$$= \sum e_{111}p'^3\,;$$

and the set independent of those quantities e_{ijk}

$$\begin{aligned}
&= 2(\bar{A}p + \bar{H}q + \bar{G}r')(\alpha p' + \beta q' + \gamma r') \\
&\quad + 2(\bar{H}p' + \bar{B}q' + \bar{F}r')(\xi p' + \eta q' + \zeta r') \\
&\quad + 2(\bar{G}p' + \bar{F}q' + \bar{C}r')(\phi p' + \chi q' + \psi r') \\
&= -2(v_1p'' + v_2q'' + v_3r'').
\end{aligned}$$

Consequently, we have

$$\frac{d}{ds}\left(\frac{1}{\rho}\right)=\sum e_{111}p'^3,$$

the former expression for the arc-rate of change of the circular curvature.

Occasionally, the ternary cubic in p', q', r', will be denoted by W; and there will be convenience in writing

$$w_1=\tfrac{1}{3}\frac{\partial W}{\partial p'}, \quad w_2=\tfrac{1}{3}\frac{\partial W}{\partial q'}, \quad w_3=\tfrac{1}{3}\frac{\partial W}{\partial r'},$$

so that w_1, w_2, w_3, are analogous in form to u_1, u_2, u_3, and v_1, v_2, v_3. Obviously

$$\left.\begin{aligned} w_1&=e_{111}p'^2+2e_{112}p'q'+e_{122}q'^2+2e_{113}p'r'+2e_{123}q'r'+e_{133}r'^2\\ w_2&=e_{112}p'^2+2e_{122}p'q'+e_{222}q'^2+2e_{123}p'r'+2e_{223}q'r'+e_{233}r'^2\\ w_3&=e_{113}p'^2+2e_{123}p'q'+e_{223}q'^2+2e_{133}p'r'+2e_{233}q'r'+e_{333}r'^2 \end{aligned}\right\}.$$

Later, it will be convenient to have magnitudes $\bar{w}_1$, $\bar{w}_2$, $\bar{w}_3$, defined by

$$\left.\begin{aligned} \Omega\bar{w}_1&=aw_1+hw_2+gw_3\\ \Omega\bar{w}_2&=hw_1+bw_2+fw_3\\ \Omega\bar{w}_3&=gw_1+fw_2+cw_3 \end{aligned}\right\}.$$

175. We have

$$u_1=Ap'+Hq'+Gr';$$

and therefore

$$\frac{du_1}{ds}=\frac{dA}{ds}p'+\frac{dH}{ds}q'+\frac{dG}{ds}r'+Ap''+Hq''+Br''.$$

When the values of the derivatives of A, H, G, as given in § 160 are substituted, as well as the values of p'', q'', r'', we find

$$\frac{du_1}{ds}=\alpha u_1+\xi u_2+\phi u_3.$$

The derivatives of u_2, u_3, are similarly obtained: the set of results is

$$\left.\begin{aligned} \frac{du_1}{ds}&=\alpha u_1+\xi u_2+\phi u_3\\ \frac{du_2}{ds}&=\beta u_1+\eta u_2+\chi u_3\\ \frac{du_3}{ds}&=\gamma u_1+\zeta u_2+\psi u_3 \end{aligned}\right\}.$$

Again, we have

$$v_1=\bar{A}p'+\bar{H}q'+\bar{G}r',$$

and therefore

$$\frac{dv_1}{ds}=\frac{d\bar{A}}{ds}p'+\frac{d\bar{H}}{ds}q'+\frac{d\bar{G}}{ds}r'+\bar{A}p''+\bar{H}q''+\bar{G}r''.$$

When the derivatives of $\bar{A}$, $\bar{H}$, $\bar{G}$, are substituted in terms of the quantities e_{ijk}, the aggregate involving these quantities

$$=w_1.$$

The aggregate of all the remaining terms

$$\begin{aligned}&=\bar{A}p''+\bar{H}q''+\bar{G}r''+2p'(\bar{A}a+\bar{H}\xi+\bar{G}\phi)\\&\qquad+q'\{(\bar{A}\beta+\bar{H}\eta+\bar{G}\chi)+(\bar{H}a+\bar{B}\xi+\bar{F}\phi)\}\\&\qquad+r'\{(\bar{A}\gamma+\bar{H}\zeta+\bar{G}\psi)+(\bar{G}a+\bar{F}\xi+\bar{C}\phi)\}\\&=av_1+\xi v_2+\phi v_3,\end{aligned}$$

on substituting the values of p'', q'', r''. Thus

$$\frac{dv_1}{ds}=w_1+av_1+\xi v_2+\phi v_3.$$

The derivatives of v_2, v_3, are similarly obtained: the set of results is

$$\left.\begin{aligned}\frac{dv_1}{ds}&=w_1+av_1+\xi v_2+\phi v_3\\\frac{dv_2}{ds}&=w_2+\beta v_1+\eta v_2+\chi v_3\\\frac{dv_3}{ds}&=w_3+\gamma v_1+\zeta v_2+\psi v_3\end{aligned}\right\}.$$

In the expression (§ 172) for the torsion, the combinations

$$u_2v_3-v_2u_3,\quad u_3v_1-v_3u_1,\quad u_1v_2-v_1u_2,$$

have occurred; and they will re-appear in the expression for the typical direction-cosine of the trinormal. We take them in the form

$$\Omega^{\frac{1}{2}}\bar{l}=u_2v_3-v_2u_3,\quad \Omega^{\frac{1}{2}}\bar{m}=u_3v_1-v_3u_1,\quad \Omega^{\frac{1}{2}}\bar{n}=u_1v_2-v_1u_2.$$

Then, by the use of the foregoing results,

$$\begin{aligned}\frac{d}{ds}(u_2v_3-v_2u_3)&=u_2w_3-w_2u_3+(a+\eta+\psi)(u_2v_3-u_3v_2)\\&\qquad-a(u_2v_3-v_2u_3)-\beta(u_3v_1-v_3u_1)-\gamma(u_1v_2-v_1u_2);\end{aligned}$$

also, by the formulæ on p. 494,

$$a+\eta+\psi=\frac{1}{2\Omega}\frac{d\Omega}{ds}:$$

and thus there is an expression for $\frac{d\bar{l}}{ds}$. Similar values can be obtained for $\frac{d\bar{m}}{ds}$, $\frac{d\bar{n}}{ds}$. The full set of results is

$$\left.\begin{aligned}\frac{d\bar{l}}{ds}&=\Omega^{-\frac{1}{2}}(u_2w_3-w_2u_3)-a\bar{l}-\beta\bar{m}-\gamma\bar{n}\\\frac{d\bar{m}}{ds}&=\Omega^{-\frac{1}{2}}(u_3w_1-w_3u_1)-\xi\bar{l}-\eta\bar{m}-\zeta\bar{n}\\\frac{d\bar{n}}{ds}&=\Omega^{-\frac{1}{2}}(u_1w_2-w_1u_2)-\phi\bar{l}-\chi\bar{m}-\psi\bar{n}\end{aligned}\right\}.$$

The quantities $\bar{l}$, $\bar{m}$, $\bar{n}$, are contragredient in character to the set u_1, u_2, u_3, and to the set v_1, v_2, v_3; and this contragredience appears in the derangement of the coefficients α, β, γ, ..., ψ, in the expressions of the derivatives.

Further, we had quantities $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, defined (p. 491) by equations of the form

$$\Omega\bar{v}_1 = av_1 + hv_2 + gv_3.$$

Hence

$$\begin{aligned}\frac{d\bar{v}_1}{ds} = &\ \frac{a}{\Omega}(w_1 + \alpha v_1 + \xi v_2 + \phi v_3) + v_1\frac{d}{ds}\left(\frac{a}{\Omega}\right)\\ &+\frac{h}{\Omega}(w_2 + \beta v_1 + \eta v_2 + \chi v_3) + v_2\frac{d}{ds}\left(\frac{h}{\Omega}\right)\\ &+\frac{g}{\Omega}(w_3 + \gamma v_1 + \eta v_2 + \psi v_3) + v_3\frac{d}{ds}\left(\frac{g}{\Omega}\right).\end{aligned}$$

From the formulæ in § 160, we have

$$\begin{aligned}\Omega\frac{d}{ds}\left(\frac{a}{\Omega}\right) &= -2(\alpha a + \beta h + \gamma g),\\ \Omega\frac{d}{ds}\left(\frac{h}{\Omega}\right) &= -(\alpha h + \beta b + \gamma f) - (\xi a + \eta h + \zeta g),\\ \Omega\frac{d}{ds}\left(\frac{g}{\Omega}\right) &= -(\alpha g + \beta f + \gamma c) - (\phi a + \chi h + \psi g);\end{aligned}$$

when the symbols $\bar{w}_1$, $\bar{w}_2$, $\bar{w}_3$, of p. 500 are used, and reduction is effected, the equation assumes the form

$$\frac{d\bar{v}_1}{ds} = \bar{w}_1 - \alpha\bar{v}_1 - \beta\bar{v}_2 - \gamma\bar{v}_3.$$

Similarly for the derivatives of $\bar{v}_2$, $\bar{v}_3$; the full set of results is

$$\left.\begin{aligned}\frac{d\bar{v}_1}{ds} &= \bar{w}_1 - \alpha\bar{v}_1 - \beta\bar{v}_2 - \gamma\bar{v}_3\\ \frac{d\bar{v}_2}{ds} &= \bar{w}_2 - \xi\bar{v}_1 - \eta\bar{v}_2 - \zeta\bar{v}_3\\ \frac{d\bar{v}_3}{ds} &= \bar{w}_3 - \phi\bar{v}_1 - \chi\bar{v}_2 - \psi\bar{v}_3\end{aligned}\right\}.$$

Again we note the contragredience in the arrangement of the coefficients α, ..., ψ, in the derivatives of $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, when compared with the derivatives of v_1, v_2, v_3.

Finally, in connection with the magnitudes w_1, w_2, w_3, as

$$Y' = -(y_1\bar{v}_1 + y_2\bar{v}_2 + y_3\bar{v}_3),$$

we have

$$\begin{aligned}\sum y_1 Y' &= -(A\bar{v}_1 + H\bar{v}_2 + G\bar{v}_3)\\ &= -v_1;\end{aligned}$$

and therefore

$$\sum y_1 Y'' + \sum y_1' Y' = -\frac{dv_1}{ds} = -(w_1 + av_1 + \xi v_2 + \phi v_3).$$

Also there has been the result (§ 172, *Ex.* 2)

$$-\sum y_1' Y' = av_1 + \xi v_2 + \phi v_3\,;$$

and therefore

$$\sum y_1 Y'' = -w_1.$$

In the same way, we have

$$\sum y_2 Y'' = -w_2, \quad \sum y_3 Y'' = -w_3.$$

Value of $\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)$ along a geodesic : magnitudes of rank four.

176. In the construction of an expression for $\frac{d}{ds}\left(\frac{1}{\rho}\right)$, it proved convenient to use the value of y''' with the formula

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \sum Y y'''.$$

But there is no like formula connecting the second arc-derivative of the circular curvature of the geodesic with y'''' ; and we therefore proceed directly from the expression already obtained for the first arc-derivative of the curvature.

At this stage, we adopt a triple-suffix notation for the derivatives, of the third order and of higher orders, of the space-variables of a point in the region. For all orders greater than two *, we now take

$$y_{lmn} = \frac{\partial^{l+m+n} y}{\partial p^l\, \partial q^m\, \partial r^n},$$

so that, for third-order derivatives in this notation †, $l+m+n=3$; and generally in this notation, the sum of the subscripts l, m, n, is the order of derivation.

Owing to the persistent recurrence of certain combinations of the symbols, we write

$$\mu_{ij} = \bar{A}_{i1} a_{j1} + \bar{A}_{i2} a_{j2} + \bar{A}_{i3} a_{j3},$$

* The single-suffix notation for first derivatives, and the double-suffix notation for second derivatives, have obvious advantages in their brevity. The triple-suffix notation for third-order derivatives, adopted earlier (§ 158), was an extension of the double-suffix notation ; when that extension is applied to derivatives of the fourth and higher orders, it is unwieldy, as compared with the notation now to be adopted. The slight modifications in third-order expressions, consequent on the change of notation, will be indicated specifically.

† When the symbol e_{lmn} denotes the same magnitude as the symbol e_{ijk} in § 174, l denotes the number of subscripts 1, m denotes the number of subscripts 2, and n the number of subscripts 3, in ijk.

the symbols $\bar{A}_{ij}$ denoting the secondary magnitudes of § 168, and the symbols a_{ij} similarly denoting the first minors in Ω (as in § 158). Also we shall write

$$(\mu_k\eta_{ij}) = \sum[\eta_{ij}\{\mu_{k1}(\eta_{11}p'+\eta_{12}q'+\eta_{13}r') \\ +\mu_{k2}(\eta_{21}p'+\eta_{22}q'+\eta_{23}r')+\mu_{k3}(\eta_{31}p'+\eta_{32}q'+\eta_{33}r')\}],$$

for all indices $i, j, k, =1, 2, 3$, in all combinations ; and likewise, for brevity,

$$(e_{lmn})_{ij} = \Gamma_{ij}e_{l+1,m,n} + \Delta_{ij}e_{l,m+1,n} + \Theta_{ij}e_{l,m,n+1} ;$$
$$(R_{lmn})_k = \bar{A}_{k1}R_{lmn} + \bar{A}_{k2}S_{lmn} + \bar{A}_{k3}T_{lmn},$$

where R_{lmn}, S_{lmn}, T_{lmn}, are the magnitudes of p. 463 ; and

$$(\bar{A}, ij)_1 = (\bar{A}, \bar{B}, \bar{C}, \bar{F}, \bar{G}, \bar{H}\between\Gamma_{ij}, \Delta_{ij}, \Theta_{ij}\between\alpha, \xi, \phi),$$
$$(\bar{A}, ij)_2 = (\bar{A}, \bar{B}, \bar{C}, \bar{F}, \bar{G}, \bar{H}\between\Gamma_{ij}, \Delta_{ij}, \Theta_{ij}\between\beta, \eta, \chi),$$
$$(\bar{A}, ij)_3 = (\bar{A}, \bar{B}, \bar{C}, \bar{F}, \bar{G}, \bar{H}\between\Gamma_{ij}, \Delta_{ij}, \Theta_{ij}\between\gamma, \zeta, \psi),$$

where α, β, γ ; ξ, η, ζ ; ϕ, χ, ψ ; are the symbols defined on p. 494.

We proceed from the expression

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = (e_{lmn}\between p', q', r')^3,$$

where the term in e_{lmn}, with $l+m+n=3$, is

$$\frac{3!}{l!\,m!\,n!}e_{lmn}p'^lq'^mr'^n,$$

and the values of the coefficients e_{lmn} (subject to the modifications that are due to the change of notation) are given in § 174. The symbols, occurring in the expression for the second derivative of the circular curvature of the regional geodesic, are defined by the relation

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) = (f_{lmn}\between p', q', r')^4,$$

where the term in f_{lmn}, with $l+m+n=4$, is

$$\frac{4!}{l!\,m!\,n!}f_{lmn}p'^lq'^mr'^n ;$$

and, as quantities of rank similar to f_{lmn}, we take

$$F_{lmn} = \sum Yy_{lmn}.$$

By the definition on p. 498, and with the now current notation, we have

$$e_{300} = \sum Yy_{300} - 3(\bar{A}\Gamma_{11} + \bar{H}\Delta_{11} + \bar{G}\Theta_{11}),$$

and therefore

$$e'_{300} = \sum\{Y(y_{400}p' + y_{310}q' + y_{301}r')\} + \sum Y'y_{300} - 3\frac{d}{ds}(\bar{A}\Gamma_{11} + \bar{H}\Delta_{11} + \bar{G}\Theta_{11}).$$

We have (§ 172)

$$-\Omega Y'=y_1\bar{v}_1+y_2\bar{v}_2+y_3\bar{v}_3,$$

and therefore

$$\begin{aligned}-\Omega\sum Y'y_{300}&=\bar{v}_1(\sum y_1y_{300})+\bar{v}_2(\sum y_2y_{300})+\bar{v}_3(\sum y_3y_{300})\\&=\bar{v}_1(AR_{300}+HS_{300}+GT_{300})\\&\qquad+\bar{v}_2(HR_{300}+BS_{300}+FT_{300})+\bar{v}_3(GR_{300}+FS_{300}+CT_{300})\\&=\Omega(v_1R_{300}+v_2S_{300}+v_3T_{300}),\end{aligned}$$

so that

$$-\sum Y'y_{300}=(R_{300})_1p'+(R_{300})_2q'+(R_{300})r';$$

consequently

$$\begin{aligned}e'_{300}=F_{400}p'+F_{310}q'+F_{301}r'-\{(R_{300})_1p'+(R_{300})_2q'+(R_{300})_3r'\}\\-3\frac{d}{ds}(\bar{A}\Gamma_{11}+\bar{H}\Delta_{11}+\bar{G}\Theta_{11}).\end{aligned}$$

Again, from the derivatives of $\bar{A}$, $\bar{H}$, $\bar{G}$, obtained in § 174,

$$\begin{aligned}&\frac{d}{ds}(\bar{A}\Gamma_{11}+\bar{H}\Delta_{11}+\bar{G}\Theta_{11})\\&=\Gamma_{11}\{e_{300}p'+e_{210}q'+e_{201}r'+2(a\bar{A}+\xi\bar{H}+\phi\bar{G})\}\\&\quad+\Delta_{11}\{e_{210}p'+e_{120}q'+e_{111}r'+(\beta\bar{A}+\eta\bar{H}+\chi\bar{G})+(a\bar{H}+\xi\bar{B}+\phi\bar{F})\}\\&\quad+\Theta_{11}\{e_{201}p'+e_{111}q'+e_{102}r'+(\gamma\bar{A}+\zeta\bar{H}+\psi\bar{G})+(a\bar{G}+\xi\bar{F}+\phi\bar{C})\}\\&\quad+\bar{A}\frac{d\Gamma_{11}}{ds}+\bar{H}\frac{d\Delta_{11}}{ds}+\bar{G}\frac{d\Theta_{11}}{ds}\\&=p'(\mathfrak{e}_{200})_{11}+q'(\mathfrak{e}_{110})_{11}+r'(\mathfrak{e}_{101})_{11}+(\bar{A},11)_1\\&\quad+\bar{A}\left(\frac{d\Gamma_{11}}{ds}+a\Gamma_{11}+\beta\Delta_{11}+\gamma\Theta_{11}\right)+\bar{H}\left(\frac{d\Delta_{11}}{ds}+\xi\Gamma_{11}+\eta\Delta_{11}+\zeta\Theta_{11}\right)\\&\qquad+\bar{G}\left(\frac{d\Theta_{11}}{ds}+\phi\Gamma_{11}+\chi\Delta_{11}+\psi\Theta_{11}\right).\end{aligned}$$

Also, from the derivatives of Γ, Δ, Θ, in § 162,

$$\begin{aligned}&\frac{d\Gamma_{11}}{ds}+a\Gamma_{11}+\beta\Delta_{11}+\gamma\Theta_{11}\\&=R_{300}p'+R_{210}q'+R_{201}r'+\frac{1}{\Omega}\Big[\ a\sum\{\eta_{11}(\eta_{11}p'+\eta_{12}q'+\eta_{13}r')\}\\&\qquad+h\sum\{\eta_{11}(\eta_{12}p'+\eta_{22}q'+\eta_{23}r')\}\\&\qquad+g\sum\{\eta_{11}(\eta_{13}p'+\eta_{23}q'+\eta_{33}r')\}\Big],\end{aligned}$$

with similar expressions for the coefficients of $\bar{A}$, $\bar{H}$, $\bar{G}$, in the last line. Hence, when the results are collected, we find

$$\begin{aligned}e'_{300}=F_{400}p'+F_{310}q'+F_{301}r'-3\{(\mathfrak{e}_{200})_{11}p'+(\mathfrak{e}_{110})_{11}q'+(\mathfrak{e}_{101})_{11}r'\}\\-\ \{(R_{300})_1p'+(R_{300})_2q'+(R_{300})_3r'\}\\-3\{(R_{300})_1p'+(R_{210})_1q'+(R_{201})_1r'\}-3(\bar{A},11)_1-\frac{3}{\Omega}(\mu_1\eta_{11}).\end{aligned}$$

From this expression for e'_{300}, expressions for e'_{030} and e'_{003} can be constructed by

the appropriate changes of parameters, subscripts, and correspondingly associated magnitudes. Thus

$$\begin{aligned} e'_{030} = F_{130}p' + F_{040}q' + F_{031}r' - 3\{(\mathfrak{e}_{110})_{22}p' + (\mathfrak{e}_{020})_{22}q' + (\mathfrak{e}_{011})_{22}r'\} \\ - \{(R_{030})_1 p' + (R_{030})_2 q' + (R_{030})_3 r'\} \\ - 3\{(R_{120})_2 p' + (R_{030})_2 q' + (R_{021})_2 r'\} - 3(\bar{A}, 22)_2 - \frac{3}{\Omega}(\mu_2\eta_{22}); \end{aligned}$$

and similarly for e'_{003}.

The corresponding expression for e'_{210} is obtained in the same manner as that for e'_{300}, and is

$$\begin{aligned} e'_{210} = F_{310}p' + F_{220}q' + F_{211}r' \\ - p'\{2(\mathfrak{e}_{200})_{12} + (\mathfrak{e}_{110})_{11}\} - q'\{2(\mathfrak{e}_{110})_{12} + (\mathfrak{e}_{020})_{11}\} - r'\{2(\mathfrak{e}_{101})_{12} + (\mathfrak{e}_{011})_{11}\} \\ - \{(R_{210})_1 p' + (R_{210})_2 q' + (R_{210})_3 r'\} \\ - p'\{2(R_{210})_1 + (R_{300})_2\} - q'\{2(R_{120})_1 + (R_{210})_2\} - r'\{2(R_{111})_1 + (R_{201})_2\} \\ - \{2(\bar{A}, 12)_1 + (\bar{A}, 11)_2\} - \frac{1}{\Omega}\{2(\mu_1\eta_{12}) + (\mu_2\eta_{11})\}. \end{aligned}$$

From this expression for e'_{210}, expressions for e'_{201}, e'_{120}, e'_{102}, e'_{021}, e'_{012}, can be constructed by the appropriate changes of parameters, subscripts, and correspondingly associated magnitudes. Thus

$$\begin{aligned} e'_{201} = F_{301}p' + F_{211}q' + F_{202}r' \\ - p'\{2(\mathfrak{e}_{200})_{13} + (\mathfrak{e}_{101})_{11}\} - q'\{2(\mathfrak{e}_{110})_{13} + (\mathfrak{e}_{002})_{11}\} - r'\{2(\mathfrak{e}_{101})_{13} + (\mathfrak{e}_{002})_{11}\} \\ - \{(R_{201})_1 p' + (R_{201})_2 q' + (R_{201})_3 r'\} \\ - p'\{2(R_{201})_1 + (R_{300})_3\} - q'\{2(R_{111})_1 + (R_{210})_3\} - r'\{2(R_{102})_1 + (R_{201})_3\} \\ - \{2(\bar{A}, 13)_1 + (\bar{A}, 11)_3\} - \frac{1}{\Omega}\{2(\mu_1\eta_{13}) + (\mu_3\eta_{11})\}; \end{aligned}$$

and similarly for the other four specified quantities e'.

The expression for e'_{111} is similarly obtained; and it remains unaltered by interchange of parameters, with the appropriate changes of subscripts and associated magnitudes. The value is

$$\begin{aligned} e'_{111} = F_{211}p' + F_{121}q' + F_{112}r' \\ - p'\{(\mathfrak{e}_{200})_{23} + (\mathfrak{e}_{110})_{31} + (\mathfrak{e}_{101})_{12}\} \\ - q'\{(\mathfrak{e}_{110})_{23} + (\mathfrak{e}_{020})_{31} + (\mathfrak{e}_{011})_{12}\} \\ - r'\{(\mathfrak{e}_{101})_{23} + (\mathfrak{e}_{011})_{31} + (\mathfrak{e}_{002})_{12}\} \\ - \{(R_{111})_1 p' + (R_{111})_2 q' + (R_{111})_3 r'\} \\ - p'\{(R_{111})_1 + (R_{201})_2 + (R_{210})_3\} \\ - q'\{(R_{021})_1 + (R_{111})_2 + (R_{120})_3\} \\ - r'\{(R_{012})_1 + (R_{102})_2 + (R_{111})_3\} \\ - \{(\bar{A}, 23)_1 + (\bar{A}, 31)_2 + (\bar{A}, 12)_3\} - \frac{1}{\Omega}\{(\mu_1\eta_{23}) + (\mu_2\eta_{31}) + (\mu_3\eta_{12})\}. \end{aligned}$$

These relations express the magnitudes e'_{lmn} in terms of the magnitudes already used and of the quantities F_{lmn}; and what is required for our purpose is the full connection between the quantities e' and the coefficients f_{lmn} in

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)=(f_{lmn}\between p', q', r')^4.$$

The left-hand side

$$=\frac{d}{ds}[(e_{lmn}\between p', q', r')^3]$$

$$=(e'_{lmn}\between p', q', r')^3+3(w_1p''+w_2q''+w_3r''),$$

with the significance of w_1, w_2, w_3, as defined in § 174. When, in this expression, the values of the quantities e', w_1, w_2, w_3, are inserted, and the coefficients are compared, we find the following (among other) relations, connecting the magnitudes f_{lmn} and F_{lmn}:

$$\begin{aligned} f_{400}=F_{400}&-6(\mathfrak{e}_{200})_{11}-4(R_{300})_1 \\ &-3(\bar{A}\between\Gamma_{11}, \Delta_{11}, \Theta_{11})^2-\frac{3}{\Omega}\{\mu_{11}(\textstyle\sum\eta_{11}^2)+\mu_{12}(\sum\eta_{11}\eta_{12})+\mu_{13}(\sum\eta_{11}\eta_{13})\},\end{aligned}$$

from which the values of f_{040}, f_{004}, are constructed by appropriate changes of parametric symbols:

$$\begin{aligned} f_{310}=F_{310}&-\{3(\mathfrak{e}_{200})_{12}+3(\mathfrak{e}_{110})_{11}\}-\{3(R_{210})_1+(R_{300})_2\} \\ &-3(\bar{A}\between\Gamma_{11}, \Delta_{11}, \Theta_{11}\between\Gamma_{12}, \Delta_{12}, \Theta_{12}) \\ &-\frac{3}{4\Omega}\{\mu_{11}(\textstyle\sum\eta_{11}\eta_{12})+\mu_{12}(\sum\eta_{11}\eta_{22})+\mu_{13}(\sum\eta_{11}\eta_{23})\} \\ &-\frac{6}{4\Omega}\{\mu_{11}(\textstyle\sum\eta_{12}\eta_{11})+\mu_{12}(\sum\eta_{12}\eta_{12})+\mu_{13}(\sum\eta_{12}\eta_{13})\} \\ &-\frac{3}{4\Omega}\{\mu_{12}(\textstyle\sum\eta_{11}\eta_{11})+\mu_{22}(\sum\eta_{11}\eta_{12})+\mu_{23}(\sum\eta_{11}\eta_{13})\},\end{aligned}$$

from which the values of f_{301}, f_{130}, f_{103}, f_{031}, f_{013}, are constructed by appropriate changes of parametric symbols:

$$\begin{aligned} f_{220}=F_{220}&-\{(\mathfrak{e}_{200})_{22}+4(\mathfrak{e}_{110})_{12}+(\mathfrak{e}_{020})_{11}\}-\{2(R_{120})_1+2(R_{210})_2\} \\ &-2(\bar{A}\between\Gamma_{12}, \Delta_{12}, \Theta_{12})^2-(\bar{A}\between\Gamma_{11}, \Delta_{11}, \Theta_{11}\between\Gamma_{22}, \Delta_{22}, \Theta_{22}) \\ &-\frac{1}{\Omega}\{\mu_{11}(\textstyle\sum\eta_{12}\eta_{12})+\mu_{12}(\sum\eta_{12}\eta_{22})+\mu_{13}(\sum\eta_{12}\eta_{23})\} \\ &-\frac{1}{2\Omega}\{\mu_{11}(\textstyle\sum\eta_{11}\eta_{22})+\mu_{12}(\sum\eta_{12}\eta_{22})+\mu_{13}(\sum\eta_{13}\eta_{22})\} \\ &-\frac{1}{\Omega}\{\mu_{12}(\textstyle\sum\eta_{12}\eta_{11})+\mu_{22}(\sum\eta_{12}\eta_{12})+\mu_{23}(\sum\eta_{12}\eta_{13})\} \\ &-\frac{1}{2\Omega}\{\mu_{12}(\textstyle\sum\eta_{12}\eta_{11})+\mu_{22}(\sum\eta_{11}\eta_{22})+\mu_{23}(\sum\eta_{11}\eta_{23})\},\end{aligned}$$

from which the values of f_{202}, f_{022}, are constructed by appropriate changes of parametric symbols: and

$$
\begin{aligned}
f_{211} = F_{211} & - \{(\mathfrak{e}_{200})_{23} + 2(\mathfrak{e}_{110})_{13} + 2(\mathfrak{e}_{101})_{12} + (\mathfrak{e}_{011})_{11}\} \\
& - \{2(R_{111})_1 + (R_{201})_2 + (\dot{R}_{210})_3\} \\
& - (\bar{A} \between \Gamma_{11}, \Delta_{11}, \Theta_{11} \between \Gamma_{23}, \Delta_{23}, \Theta_{23}) - 2(\bar{A} \between \Gamma_{12}, \Delta_{12}, \Theta_{12} \between \Gamma_{13}, \Delta_{13}, \Theta_{13}) \\
& - \frac{1}{2\Omega}\{\mu_{11}(\textstyle\sum \eta_{11}\eta_{23}) + \mu_{12}(\sum \eta_{12}\eta_{23}) + \mu_{13}(\sum \eta_{13}\eta_{23})\} \\
& - \frac{1}{2\Omega}\{\mu_{11}(\textstyle\sum \eta_{12}\eta_{13}) + \mu_{12}(\sum \eta_{13}\eta_{22}) + \mu_{13}(\sum \eta_{13}\eta_{23})\} \\
& - \frac{1}{2\Omega}\{\mu_{11}(\textstyle\sum \eta_{12}\eta_{13}) + \mu_{12}(\sum \eta_{12}\eta_{23}) + \mu_{13}(\sum \eta_{12}\eta_{33})\} \\
& - \frac{1}{2\Omega}\{\mu_{12}(\textstyle\sum \eta_{11}\eta_{13}) + \mu_{22}(\sum \eta_{12}\eta_{13}) + \mu_{23}(\sum \eta_{13}\eta_{13})\} \\
& - \frac{1}{4\Omega}\{\mu_{12}(\textstyle\sum \eta_{11}\eta_{13}) + \mu_{22}(\sum \eta_{11}\eta_{23}) + \mu_{23}(\sum \eta_{11}\eta_{23})\} \\
& - \frac{1}{2\Omega}\{\mu_{13}(\textstyle\sum \eta_{11}\eta_{12}) + \mu_{23}(\sum \eta_{12}\eta_{12}) + \mu_{33}(\sum \eta_{12}\eta_{13})\} \\
& - \frac{1}{4\Omega}\{\mu_{13}(\textstyle\sum \eta_{11}\eta_{12}) + \mu_{23}(\sum \eta_{11}\eta_{22}) + \mu_{33}(\sum \eta_{11}\eta_{23})\},
\end{aligned}
$$

from which the values of f_{121}, f_{112}, are constructed by appropriate changes of parametric symbols.

When, by means of these relations, we remove the magnitudes F_{lmn} from the expressions for the quantities e' so as to insert the magnitudes f_{lmn}, the full set of new expressions is:

$$
\left.
\begin{aligned}
e'_{300} &= f_{400}p' + f_{310}q' + f_{301}r' + 3(\alpha e_{300} + \xi e_{210} + \phi e_{201}) \\
e'_{030} &= f_{130}p' + f_{040}q' + f_{031}r' + 3(\beta e_{120} + \eta e_{030} + \chi e_{021}) \\
e'_{003} &= f_{103}p' + f_{013}q' + f_{004}r' + 3(\gamma e_{102} + \zeta e_{012} + \psi e_{003})
\end{aligned}
\right\},
$$

$$
\left.
\begin{aligned}
e'_{210} &= f_{310}p' + f_{220}q' + f_{211}r' \\
&\qquad + 2(\alpha e_{210} + \xi e_{120} + \phi e_{111}) + (\beta e_{300} + \eta e_{210} + \chi e_{201}) \\
e'_{201} &= f_{301}p' + f_{211}q' + f_{202}r' \\
&\qquad + 2(\alpha e_{201} + \xi e_{111} + \phi e_{102}) + (\gamma e_{300} + \zeta e_{210} + \psi e_{201})
\end{aligned}
\right\},
$$

$$
\left.
\begin{aligned}
e'_{120} &= f_{220}p' + f_{130}q' + f_{121}r' \\
&\qquad + 2(\beta e_{210} + \eta e_{120} + \chi e_{111}) + (\alpha e_{120} + \xi e_{030} + \phi e_{021}) \\
e'_{102} &= f_{202}p' + f_{112}q' + f_{103}r' \\
&\qquad + 2(\gamma e_{201} + \zeta e_{111} + \psi e_{102}) + (\alpha e_{102} + \xi e_{012} + \phi e_{003})
\end{aligned}
\right\};
$$

$$\left.\begin{aligned} e'_{021} &= f_{121}p' + f_{031}q' + f_{022}r' \\ &\quad + 2(\beta e_{111} + \eta e_{021} + \chi e_{102}) + (\gamma e_{120} + \zeta e_{030} + \psi e_{021}) \\ e'_{012} &= f_{112}p' + f_{022}q' + f_{013}r' \\ &\quad + 2(\gamma e_{111} + \zeta e_{021} + \psi e_{013}) + (\beta e_{102} + \eta e_{012} + \chi e_{003}) \end{aligned}\right\} ;$$

$$\begin{aligned} e'_{111} &= f_{211}p' + f_{121}q' + f_{112}r' \\ &\quad + (\alpha e_{111} + \xi e_{021} + \phi e_{012}) + (\beta e_{201} + \eta e_{111} + \chi e_{102}) + (\gamma e_{210} + \zeta e_{120} + \psi e_{111}). \end{aligned}$$

177. As in § 41 (but now with only three direction-variables), we write

$$T = \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) = (f_{lmn} \between p', q', r')^4 ;$$

and the partial derivatives of T with regard to the explicit occurrence of p', q', r', are represented in the scheme

$$t_1 = \tfrac{1}{4}\frac{\partial T}{\partial p'}, \quad t_2 = \tfrac{1}{4}\frac{\partial T}{\partial q'}, \quad t_3 = \tfrac{1}{4}\frac{\partial T}{\partial r'}$$

$$t_{11} = \tfrac{1}{12}\frac{\partial^2 T}{\partial p'^2}, \quad t_{22} = \tfrac{1}{12}\frac{\partial^2 T}{\partial q'^2}, \quad t_{33} = \tfrac{1}{12}\frac{\partial^2 T}{\partial r'^2}$$

$$t_{23} = \tfrac{1}{12}\frac{\partial^2 T}{\partial q'\,\partial r'}, \quad t_{31} = \tfrac{1}{12}\frac{\partial^2 T}{\partial r'\,\partial p'}, \quad t_{12} = \tfrac{1}{12}\frac{\partial^2 T}{\partial p'\,\partial q'}.$$

Also, we had (p. 500)

$$W = \frac{d}{ds}\left(\frac{1}{\rho}\right) = (e_{lmn} \between p', q', r')^3,$$

$$w_1 = \tfrac{1}{3}\frac{\partial W}{\partial p'}, \quad w_2 = \tfrac{1}{3}\frac{\partial W}{\partial q'}, \quad w_3 = \tfrac{1}{3}\frac{\partial W}{\partial r'};$$

and, further, it is convenient to use the magnitudes

$$w_{11} = \tfrac{1}{6}\frac{\partial^2 W}{\partial p'^2}, \quad w_{22} = \tfrac{1}{6}\frac{\partial^2 W}{\partial q'^2}, \quad w_{33} = \tfrac{1}{6}\frac{\partial^2 W}{\partial r'^2}$$

$$w_{23} = \tfrac{1}{6}\frac{\partial^2 W}{\partial q'\,\partial r'}, \quad w_{31} = \tfrac{1}{6}\frac{\partial^2 W}{\partial r'\,\partial p'}, \quad w_{12} = \tfrac{1}{6}\frac{\partial^2 W}{\partial p'\,\partial q'}.$$

Then, as

$$w_{11} = e_{300}p' + e_{210}q' + e_{201}r',$$

we have

$$\frac{dw_{11}}{ds} = e'_{300}p' + e'_{210}q' + e'_{201}r' + e_{300}p'' + e_{210}q'' + e_{201}r''.$$

With the symbols of p. 494, we have

$$-p'' = \alpha p' + \xi q' + \phi r', \quad -q'' = \beta p' + \eta q' + \chi r', \quad -r'' = \gamma p' + \zeta q' + \psi r' ;$$

and therefore, when these values are substituted as well as the values of e'_{300}, e'_{210}, e'_{201}, and reduction is effected, we find

$$\frac{dw_{11}}{ds} = t_{11} + 2(\alpha w_{11} + \xi w_{12} + \phi w_{13}).$$

Similarly

$$\frac{dw_{22}}{ds}=t_{22}+2(\beta w_{12}+\eta w_{22}+\chi w_{23}),$$

$$\frac{dw_{33}}{ds}=t_{33}+2(\gamma w_{13}+\zeta w_{23}+\psi w_{33}).$$

Proceeding in the same way from

$$w_{23}=e_{111}p'+e_{021}q'+e_{012}r',$$

and from the like expressions for w_{31}, w_{12}, we find

$$\frac{dw_{23}}{ds}=t_{23}+(\beta w_{13}+\eta w_{23}+\chi w_{33})+(\gamma w_{12}+\zeta w_{22}+\psi w_{23}),$$

$$\frac{dw_{31}}{ds}=t_{31}+(\gamma w_{11}+\zeta w_{12}+\psi w_{13})+(a w_{13}+\xi w_{23}+\phi w_{33}),$$

$$\frac{dw_{12}}{ds}=t_{12}+(a w_{12}+\xi w_{22}+\phi w_{23})+(\beta w_{11}+\eta w_{12}+\chi w_{13}).$$

Also, proceeding in the same manner from the three equations of the form

$$w_1=e_{300}p'^2+2e_{210}p'q'+2e_{201}p'r'+e_{120}q'^2+2e_{111}q'r'+e_{102}r'^2,$$

we find

$$\left.\begin{aligned}\frac{dw_1}{ds}&=t_1+a w_1+\xi w_2+\phi w_3\\ \frac{dw_2}{ds}&=t_2+\beta w_1+\eta w_2+\chi w_3\\ \frac{dw_3}{ds}&=t_3+\gamma w_1+\zeta w_3+\psi w_3\end{aligned}\right\}.$$

Ex. As a mere verification, we have

$$W=w_1p'+w_2q'+w_3r';$$

and therefore

$$\begin{aligned}T=\frac{dW}{ds}&=w_1'p'+w_2'q'+w_3'r'+w_1p''+w_2q''+w_3r''\\ &=\{w_1'-(a w_1+\xi w_2+\phi w_3)\}p'\\ &\qquad+\{w_2'-(\beta w_1+\eta w_2+\chi w_3)\}q'\\ &\qquad\quad+\{w_3'-(\gamma w_1+\zeta w_2+\psi w_3)\}r'\\ &=t_1p'+t_2q'+t_3r',\end{aligned}$$

which is an identity.

Trinormal of a regional geodesic: the tilt.

178. The trinormal of a regional geodesic lies in the tangent flat of the region. As the flat, itself three-dimensional and therefore incapable of containing within its own range more than three lines of any orthogonal system, already contains

the tangent and the binormal, the trinormal will be the remaining direction in the flat at right angles to the tangent and the binormal.

Thus the typical direction-cosine l_4 of the trinormal must be of the form

$$l_4 = y_1\bar{\alpha} + y_2\bar{\beta} + y_3\bar{\gamma},$$

where $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, are parameters to be determined. The trinormal is at right angles to the tangent, with the typical direction-cosine y', so that

$$\sum y'(y_1\bar{\alpha} + y_2\bar{\beta} + y_3\bar{\gamma}) = 0\,;$$

and it is at right angles to the binormal, with the typical direction-cosine l_3, so that

$$\sum l_3(y_1\bar{\alpha} + y_2\bar{\beta} + y_3\bar{\gamma}) = 0.$$

Because $\sum y' y_i = u_i$, for $i = 1, 2, 3$, the first condition is

$$\bar{\alpha}u_1 + \bar{\beta}u_2 + \bar{\gamma}u_3 = 0.$$

Because (§ 172)

$$\frac{1}{\sigma}\sum l_3 y_i = \frac{1}{\rho}u_i - v_i,$$

for $i = 1, 2, 3$, the second condition is

$$\bar{\alpha}\left(\frac{u_1}{\rho} - v_1\right) + \bar{\beta}\left(\frac{u_2}{\rho} - v_2\right) + \bar{\gamma}\left(\frac{u_3}{\rho} - v_3\right) = 0,$$

which, in virtue of the first condition, can be taken as

$$\bar{\alpha}v_1 + \bar{\beta}v_2 + \bar{\gamma}v_3 = 0.$$

Consequently

$$\frac{\bar{\alpha}}{u_2v_3 - u_3v_2} = \frac{\bar{\beta}}{u_3v_1 - u_1v_3} = \frac{\bar{\gamma}}{u_1v_2 - u_2v_1} = w,$$

where w is to be determined.

There is the further relation $\sum l_4^2 = 1$, so that

$$\sum A\bar{\alpha}^2 = 1\,;$$

hence

$$w^2 = \sum A(u_2v_3 - u_3v_2)^2 = \frac{\Omega}{\sigma^2},$$

from the value of the torsion on p. 492. Hence the typical direction-cosine of the trinormal is given by the equation

$$\frac{\Omega^{\frac{1}{2}}}{\sigma}l_4 = \begin{vmatrix} y_1, & y_2, & y_3 \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix}.$$

The quantities $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, except to a factor, are the quantities $\bar{l}$, $\bar{m}$, $\bar{n}$, of § 175: in fact,

$$\bar{\alpha} = \bar{l}\sigma, \quad \bar{\beta} = \bar{m}\sigma, \quad \bar{\gamma} = \bar{n}\sigma,$$

and we can write

$$\frac{l_4}{\sigma} = y_1\bar{l} + y_2\bar{m} + y_3\bar{n}.$$

We also have had the relations

$$l_3 = y_1 l + y_2 m + y_3 n,$$
$$y' = y_1 p' + y_2 q' + y_3 r',$$

and the three directions typified by y', l_3, l_4, are at right angles to one another; the equations can therefore be resolved, so as to express y_1, y_2, y_3, linearly in terms of the typical direction-cosines y', l_3, l_4. It is easy to verify the result of the resolution in the form

$$y_1 = y'u_1 + l_3\sigma\left(\frac{u_1}{\rho} - v_1\right) + l_4\Omega^{-\frac{1}{2}}\sigma \begin{vmatrix} A, & H, & G \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix},$$

$$y_2 = y'u_2 + l_3\sigma\left(\frac{u_2}{\rho} - v_2\right) + l_4\Omega^{-\frac{1}{2}}\sigma \begin{vmatrix} H, & B, & F \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix},$$

$$y_3 = y'u_3 + l_3\sigma\left(\frac{u_3}{\rho} - v_3\right) + l_4\Omega^{-\frac{1}{2}}\sigma \begin{vmatrix} G, & F, & C \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix}.$$

By these relations, one set of leading lines for the tangent flat, constituted from the directions at O of the three parametric curves of the region, can be changed into another set of leading lines for the flat, constituted from the tangent, the binormal, and the trinormal, of a regional geodesic in any contained direction. The former set proves convenient for the investigation of properties of the region, the latter for the development of the relations between the region and any of its geodesics.

Owing to the relations

$$\sum y_1\eta_{ij} = 0, \quad \sum y_2\eta_{ij} = 0, \quad \sum y_3\eta_{ij} = 0,$$

holding for all values of i, j, $=1, 2, 3$, and to the consequent relations

$$\sum y'\eta_{ij} = 0, \quad \sum l_3\eta_{ij} = 0, \quad \sum l_4\eta_{ij} = 0,$$

it follows that each of the six directions, with typical direction-cosines

$$\left(\sum \eta_{ij}^2\right)^{-\frac{1}{2}}\eta_{ij},$$

is orthogonal to the tangent flat. Also, because

$$\frac{Y}{\rho} = \sum \eta_{11}p'^2,$$

there is an immediate verification that the expressions obtained satisfy the conditions which represent the orthogonality of the prime normal of any regional geodesic to the tangent flat of the region.

Finally, all the principal lines of the geodesic, which are later in rank than the tangent, the prime normal, the binormal, and the trinormal, are orthogonal to the osculating block of the geodesic having these for its leading lines. Let the typical direction-cosines of these lines in successive rank be denoted by $l_5, l_6, \ldots, l_N$; then we have

$$\sum Yl_\mu=0, \quad \sum y_1 l_\mu=0, \quad \sum y_2 l_\mu=0, \quad \sum y_3 l_\mu=0,$$

for each of the values $\mu=5, 6, \ldots, N$.

179. The typical Frenet formula, connected with the trinormal of a curve, when the curve is a regional geodesic, is

$$\frac{dl_3}{ds}=\frac{l_4}{\tau}-\frac{Y}{\sigma}.$$

The typical direction-cosine of the binormal satisfies the equation

$$l_3=\frac{\sigma}{\rho}y'+\sigma Y';$$

and therefore

$$\frac{l_4}{\tau}=y'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)+\left(\frac{\sigma}{\rho^2}+\frac{1}{\sigma}\right)Y+\sigma' Y'+\sigma Y''.$$

An equivalent form of this relation is

$$\frac{l_4}{\sigma\tau}=y'\frac{d}{ds}\left(\frac{1}{\rho}\right)+Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)-l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right)+Y'',$$

being (§ 8) the corresponding equation for a geodesic in any amplitude, derived from the equation for any curve in space with the geodesic specialising relation $l_2=Y$.

The equation

$$\frac{\Omega^{\frac{1}{2}}}{\sigma}l_4=\begin{vmatrix} y_1, & y_2, & y_3 \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix}=\nabla$$

has been obtained for the typical direction-cosine l_4. When the two sides of this equation are multiplied by the corresponding sides of the second of the above forms of the Frenet equation, and the results are added for all the plenary space-dimensions, we have

$$\frac{\Omega^{\frac{1}{2}}}{\sigma^2\tau}=\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)\right\}\left(\sum y'\nabla\right)+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)\left(\sum Y\nabla\right)-\left\{\frac{d}{ds}\left(\frac{1}{\sigma}\right)\right\}\left(\sum l_3\nabla\right)+\left(\sum Y''\nabla\right).$$

Now, as ∇ is a mere multiple of l_4, we have

$$\sum y'\nabla=0, \quad \sum Y\nabla=0, \quad \sum l_3\nabla=0,$$

because the trinormal is at right angles to the tangent, to the prime normal, and to the binormal: the analytical verification of the relations is merely an inverse of the analysis in the preceding section. Also, the relations

$$\sum y_1 Y''=-w_1, \quad \sum y_2 Y''=-w_2, \quad \sum y_3 Y''=-w_3,$$

have been established (§ 175); and therefore

$$\sum Y''\nabla=-\begin{vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \\ w_1, & w_2, & w_3 \end{vmatrix}.$$

Consequently, the tilt of the regional geodesic is determined by the equation

$$-\frac{\Omega^{\frac{1}{2}}}{\sigma^2\tau}=\begin{vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \\ w_1, & w_2, & w_3 \end{vmatrix}.$$

Various inferences can be deduced by means of these formulæ, using symmetrical summations for all the dimensions of the plenary space.

(i) Reverting to the second form of the applied Frenet equation for the direction-cosine of the trinormal, multiplying throughout by l_4, and adding for all the dimensions, we have

$$\sum l_4 Y''=\frac{1}{\sigma\tau}.$$

(ii) Squaring the same second form of the Frenet equation for the trinormal, taken in the form

$$Y''=\frac{l_4}{\sigma\tau}+l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right)-Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)-y'\frac{d}{ds}\left(\frac{1}{\rho}\right),$$

and again adding for all the dimensions, we have

$$\sum Y''^2=\frac{1}{\sigma^2\tau^2}+\frac{\sigma'^2}{\sigma^4}+\frac{\rho'^2}{\rho^4}+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)^2,$$

a relation already (§ 8) noted as pertaining to a geodesic in any curved amplitude (tacitly supposed to be of at least three dimensions).

(iii) Let the same second form of the Frenet equation be multiplied throughout by y_1; and let the results be added for all the dimensions. As

$$\frac{l_4}{\sigma}=y_1\bar{l}+y_2\bar{m}+y_3\bar{n},$$

we have

$$\frac{1}{\sigma}\sum y_1 l_4 = A\bar{l} + H\bar{m} + G\bar{n}.$$

Also,

$$\sum y_1 y' = u_1, \quad \sum y_1 Y = 0, \quad \sum y_1 Y'' = -w_1,$$

$$\sum y_1 l_3 = Al + Hm + Gn = \sigma\left(\frac{1}{\rho}u_1 - v_1\right);$$

and therefore

$$\frac{1}{\tau}(A\bar{l} + H\bar{m} + G\bar{n}) = u_1\frac{d}{ds}\left(\frac{1}{\rho}\right) + \left(\frac{1}{\rho}u_1 - v_1\right)\frac{\sigma'}{\sigma} - w_1$$

$$= \left\{\frac{1}{\sigma}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}u_1 - \frac{\sigma'}{\sigma}v_1 - w_1.$$

Proceeding similarly from the same equation after multiplication throughout by y_2, and also similarly after multiplication by y_3, we find

$$\frac{1}{\tau}(H\bar{l} + B\bar{m} + F\bar{n}) = \left\{\frac{1}{\sigma}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}u_2 - \frac{\sigma'}{\sigma}v_2 - w_2,$$

$$\frac{1}{\tau}(G\bar{l} + F\bar{m} + C\bar{n}) = \left\{\frac{1}{\sigma}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}u_3 - \frac{\sigma'}{\sigma}v_3 - w_3.$$

There are the relations

$$\begin{aligned} au_1 + hu_2 + gu_3 &= \Omega p', \\ av_1 + hv_2 + gv_3 &= \Omega\bar{v}_1, \\ aw_1 + hw_2 + gw_3 &= \Omega\bar{w}_1, \end{aligned}$$

each typical of three relations in its own set; hence, when the foregoing equations are resolved for $\bar{l}$, $\bar{m}$, $\bar{n}$, we have

$$\left.\begin{aligned} \frac{\bar{l}}{\tau} &= \left\{\frac{1}{\sigma}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}p' - \frac{\sigma'}{\sigma}\bar{v}_1 - \bar{w}_1 \\ \frac{\bar{m}}{\tau} &= \left\{\frac{1}{\sigma}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}q' - \frac{\sigma'}{\sigma}\bar{v}_2 - \bar{w}_2 \\ \frac{\bar{n}}{\tau} &= \left\{\frac{1}{\sigma}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}r' - \frac{\sigma'}{\sigma}\bar{v}_3 - \bar{w}_3 \end{aligned}\right\}.$$

(iv) When these values of $\bar{l}$, $\bar{m}$, $\bar{n}$, are substituted in the value of l_4, cited in (iii) above, we have

$$\frac{l_4}{\sigma\tau} = \left\{\frac{1}{\sigma}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}y' - \frac{\sigma'}{\sigma}(y_1\bar{v}_1 + y_2\bar{v}_2 + y_3\bar{v}_3) - (y_1\bar{w}_1 + y_2\bar{w}_2 + y_3\bar{w}_3)$$

$$= \frac{1}{\sigma}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)y' + \frac{\sigma'}{\sigma}Y' - (y_1\bar{w}_1 + y_2\bar{w}_2 + y_3\bar{w}_3),$$

from the value of Y' obtained in § 172. When this expression for l_4 is compared with the first of the two forms of the Frenet equation for the trinormal, we obtain the relation

$$Y'' + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)Y = -(y_1\bar{w}_1 + y_2\bar{w}_2 + y_3\bar{w}_3).$$

In passing, it should be noted that Y'' is linear in Y, y_1, y_2, y_3, and therefore is the component of a vector in the osculating block of the geodesic—a result equally to be noted from the second form of the Frenet equation of the binormal.

Later (§ 183), this formula will be derived by direct arc-differentiation from

$$Y' = -(y_1\bar{v}_1 + y_2\bar{v}_2 + y_3\bar{v}_3).$$

(v) Let this last equation, in Y'', be squared; and let the results be added for all the dimensions of the plenary space. As

$$\sum YY'' = -\sum Y'^2 = -\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right),$$

the summed left-hand side

$$= \sum Y''^2 - \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^2$$

$$= \frac{1}{\sigma^2\tau^2} + \frac{\sigma'^2}{\sigma^4} + \frac{\rho'^2}{\rho^4}.$$

The summed right-hand side

$$= \sum A\bar{w}_1^2;$$

but

$$A\bar{w}_1 + H\bar{w}_2 + G\bar{w}_3 = w_1, \quad H\bar{w}_1 + B\bar{w}_2 + F\bar{w}_3 = w_2, \quad G\bar{w}_1 + F\bar{w}_2 + C\bar{w}_3 = w_3,$$

so that the right-hand side

$$= w_1\bar{w}_1 + w_2\bar{w}_2 + w_3\bar{w}_3$$

$$= \frac{1}{\Omega} \sum aw_1^2.$$

Consequently

$$\frac{1}{\sigma^2\tau^2} + \frac{\sigma'^2}{\sigma^4} + \frac{\rho'^2}{\rho^4} = \frac{1}{\Omega} \sum aw_1^2,$$

a result which can be regarded in two ways; it gives an expression for the tilt in terms of regional magnitudes: it provides a geometrical interpretation for the concomitant $\sum aw_1^2$.

(vi) Let the respective sides of the equations

$$Y' = -(y_1\bar{v}_1 + y_2\bar{v}_2 + y_3\bar{v}_3), \quad Y'' + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) Y = -(y_1\bar{w}_1 + y_2\bar{w}_2 + \bar{w}_3),$$

be multiplied together, and the products be added for all the dimensions of space. Then, because $\sum YY' = \frac{1}{2}\frac{d}{ds}(\sum Y^2) = 0$, the added left-hand side

$$= \sum Y'Y''$$

$$= \frac{1}{2}\frac{d}{ds}(\sum Y'^2) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) = -\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right).$$

The added right-hand side

$$= \sum A\bar{v}_1\bar{w}_1$$
$$= w_1\bar{v}_1 + w_2\bar{v}_2 + w_3\bar{v}_3$$
$$= \frac{1}{\Omega}\sum av_1w_1;$$

thus

$$\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3} = -\frac{1}{\Omega}\sum av_1w_1,$$

another result which gives an expression for a geometrical magnitude and also provides a geometrical interpretation of the concomitant $\sum av_1w_1$.

(vii) We have

$$\sum au_1w_1 = \Omega\sum w_1p' = \Omega\frac{d}{ds}\left(\frac{1}{\rho}\right).$$

This result can be derived also from the second form of the Frenet equation. Let the latter be multiplied throughout by y', and the results be added for all the space-dimensions; then

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = -\sum y'Y''$$
$$= \sum y'(y_1\bar{w}_1 + y_2\bar{w}_2 + y_3\bar{w}_3)$$
$$= u_1\bar{w}_1 + u_2\bar{w}_2 + u_3\bar{w}_3 = \frac{1}{\Omega}\sum au_1w_1.$$

(viii) We thus have expressions for the concomitants $\sum au_1w_1$, $\sum av_1w_1$, $\sum aw_1^2$, as already (§§ 171, 172) we have had expressions for $\sum au_1^2$, $\sum au_1v_1$, $\sum av_1^2$. We infer the results

$$\frac{1}{\Omega}\sum A(u_2w_3 - u_3w_2)^2 = \frac{1}{\sigma^2\tau^2} + \frac{\sigma'^2}{\sigma^4},$$

$$\frac{1}{\Omega}\sum A(v_2w_3 - v_3w_2)^2 = \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)\frac{1}{\sigma^2\tau^2} + \frac{1}{\rho^2\sigma^2}\left(\frac{\rho'}{\rho} - \frac{\sigma'}{\sigma}\right)^2,$$

which are easily verified.

Leading lines of the tangent flat.

180. As the typical direction-cosines of the binormal and the trinormal have the respective forms

$$l_3 = y_1l + y_2m + y_3n, \quad l_4 = y_1\bar{\alpha} + y_2\bar{\beta} + y_3\bar{\gamma},$$

both lines lying within the tangent flat of the region, the quantities l, m, n, can be taken as a set of direction-variables which will be denoted by p_3', q_3', r_3'; and the quantities $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$, can similarly be taken as a set of direction-variables, to be denoted by p_4', q_4', r_4'. As the direction-variables of the tangent to the geodesic

are p', q', r', these three sets of direction-variables belong to three lines which are at right angles to one another in pairs.

From the value of l_3, obtained in § 173, it follows that

$$\frac{l}{\sigma}=\frac{p'}{\rho}-\bar{v}_1,\quad \frac{m}{\sigma}=\frac{q'}{\rho}-\bar{v}_2,\quad \frac{n}{\sigma}=\frac{r'}{\rho}-\bar{v}_3.$$

Now

$$\sum A\bar{v}_1^{\,2}=\sum \dot{v}_1\bar{v}_1=\frac{1}{\Omega}\sum av_1^{\,2}=\frac{1}{\rho^2}+\frac{1}{\sigma^2}=\mu^2,$$

so that μ is a linear curvature, sometimes called the curvature of screw. If therefore we write

$$\bar{v}_1=\mu p_0',\quad \bar{v}_2=\mu q_0',\quad \bar{v}_3=\mu r_0',$$

we have $\sum Ap_0'^2=1$; and the quantities p_0', q_0', r_0', are direction-variables in the region. Thus we have

$$\frac{p_3'}{\sigma}=\frac{p'}{\rho}-\mu p_0',\quad \frac{q_3'}{\sigma}=\frac{q'}{\rho}-\mu q_0',\quad \frac{r_3'}{\sigma}=\frac{r'}{\rho}-\mu r_0';$$

and the direction p_0', q_0', r_0' lies in the plane determined by the tangent and the binormal of the regional geodesic as its leading lines. We have

$$\sum Ap'p_3'=0,$$

because the tangent and the binormal are at right angles; and therefore

$$\sum Ap'p_0'=\frac{1}{\mu\rho}=\sin\chi,\quad \sum Ap_3'p_0'=-\frac{1}{\sigma\mu}=-\cos\chi,$$

where

$$\tan\chi=\frac{\sigma}{\rho}.$$

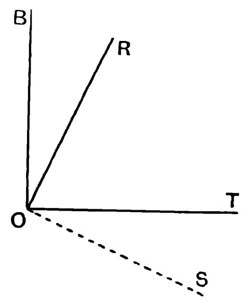

FIG. 19.

In the diagram, OT is the direction of the tangent, OB that of the binormal. For reference, OS is taken to be a direction such that $SOT=\frac{1}{2}\pi-\chi$, and OR to be a direction within the angle TOB at right angles to OS so that $TOR=\chi$.

From the value of l_4, obtained on p. 515, it follows that

$$\begin{aligned}\frac{\bar{a}}{\sigma\tau}&=\left\{\frac{1}{\sigma}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}p'-\frac{\sigma'}{\sigma}\bar{v}_1-\bar{w}_1\\&=p'\frac{d}{ds}\left(\frac{1}{\rho}\right)+\frac{\sigma'}{\sigma}\left(\frac{p'}{\rho}-\bar{v}_1\right)-\bar{w}_1\\&=p'\frac{d}{ds}\left(\frac{1}{\rho}\right)-l\frac{d}{ds}\left(\frac{1}{\sigma}\right)-\bar{w}_1,\end{aligned}$$

with two similar equations; and so there are three relations of the type

$$\frac{p_4'}{\sigma\tau}=p'\frac{d}{ds}\left(\frac{1}{\rho}\right)-p_3'\frac{d}{ds}\left(\frac{1}{\sigma}\right)-\bar{w}_1.$$

Now

$$\sum A\bar{w}_1^2 = \sum w_1\bar{w}_1 = \frac{1}{\Omega}\sum a w_1^2$$

$$= \frac{1}{\sigma^2\tau^2} + \frac{\sigma'^2}{\sigma^4} + \frac{\rho'^2}{\rho^4}$$

$$= \frac{1}{\sigma^2\tau^2} + \left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)\right\}^2 + \left\{\frac{d}{ds}\left(\frac{1}{\sigma}\right)\right\}^2 = \lambda^4,$$

where λ is used merely as an abbreviative symbol. Hence, if we write

$$\bar{w}_1 = \lambda^2 p_1', \quad \bar{w}_2 = \lambda^2 q_1', \quad \bar{w}_3 = \lambda^2 r_1',$$

we have $\sum A p_1'^2 = 1$; and p_1', q_1', r_1', are direction-variables. Thus the three foregoing equations become

$$\left.\begin{aligned} \frac{p_4'}{\sigma\tau} &= p'\frac{d}{ds}\left(\frac{1}{\rho}\right) - p_3'\frac{d}{ds}\left(\frac{1}{\sigma}\right) - p_1'\lambda^2 \\ \frac{q_4'}{\sigma\tau} &= q'\frac{d}{ds}\left(\frac{1}{\rho}\right) - q_3'\frac{d}{ds}\left(\frac{1}{\sigma}\right) - q_1'\lambda^2 \\ \frac{r_4'}{\sigma\tau} &= r'\frac{d}{ds}\left(\frac{1}{\rho}\right) - r_3'\frac{d}{ds}\left(\frac{1}{\sigma}\right) - r_1'\lambda^2 \end{aligned}\right\},$$

where λ^4 is the sum of the squares of the coefficients of p', p_3', p_4'. The three sets of direction-variables p', q', r'; p_3', q_3', r_3'; p_4', q_4', r_4'; belong to the tangent, the binormal, and the trinormal, a group of orthogonal leading lines in the tangent flat. Hence the direction p_1', q_1', r_1', also lies in the tangent flat; its inclination ϕ_e to the trinormal is given by

$$\cos\phi_e = -\frac{1}{\lambda^2\sigma\tau},$$

its inclination ϕ_b to the binormal by

$$\cos\phi_b = \frac{1}{\lambda^2}\frac{d}{ds}\left(\frac{1}{\sigma}\right),$$

and its inclination ϕ_t to the tangent by

$$\cos\phi_t = -\frac{1}{\lambda^2}\frac{d}{ds}\left(\frac{1}{\rho}\right).$$

Also, as there are three equations

$$\mu p_0' = \frac{p'}{\rho} - \frac{p_3'}{\sigma},$$

the inclination ϕ of the directions p_0', q_0', r_0', and p_1', q_1', r_1', is given by

$$\lambda^2\mu\cos\phi = \frac{1}{\rho}\frac{d}{ds}\left(\frac{1}{\rho}\right) + \frac{1}{\sigma}\frac{d}{ds}\left(\frac{1}{\sigma}\right) = -\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right),$$

which (p. 517) is the geometrical magnitude expressed analytically by the concomitant

$$\frac{1}{\Omega}\sum a v_1 w_1.$$

A diagram, similar to the preceding diagram for the direction of the line OS with the direction-variables p_0', q_0', r_0', in the same plane as the tangent and the binormal, can be constructed for the line with the direction-variables p_1', q_1', r_1', in the same flat as the tangent, the binormal, and the trinormal.

It thus appears that the magnitudes of the types $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$; $\bar{w}_1$, $\bar{w}_2$, $\bar{w}_3$; are proportional to significant direction-variables in the flat: the similar magnitudes constructed from u_1, u_2, u_3, viz.

$$\frac{1}{\Omega}(au_1+hu_2+gu_3),\quad \frac{1}{\Omega}(hu_1+bu_2+fu_3),\quad \frac{1}{\Omega}(gu_1+fu_2+cu_3),$$

are the actual direction-variables of the tangent to the regional geodesic.

Ex. Utilising the property that the tangent flat of a region at O contains the tangent, the binormal, and the trinormal, of every regional geodesic through O, we can obtain an approximation to the length of the perpendicular from a point on the geodesic near O, closer than that which was obtained in § 167.

The equations of the flat can be taken in the form

$$\| \; \bar{y}-y, \quad y', \quad l_3, \quad l_4 \; \|$$

and in the equivalent form

$$\bar{y}=y+\theta y'+\phi l_3+\chi l_4,$$

where θ, ϕ, χ, are the linear parameters for the flat. As in § 167, we denote by η the typical coordinate of the geodesic point Q near O from which the perpendicular Π is drawn to the flat, and we take the foregoing quantity $\bar{y}$ as the typical coordinate of the foot of that perpendicular, while $\bar{Y}$ is its typical direction-cosine: thus

$$\bar{Y}\Pi=\eta-\bar{y},$$

while the magnitude

$$\sum(\eta-\bar{y})^2$$

is to be a minimum among all possible values arising through the parameters θ, ϕ, ψ. The critical equations are

$$\sum(\eta-\bar{y})y'=0,\quad \sum(\eta-\bar{y})l_3=0,\quad \sum(\eta-\bar{y})l_4=0,$$

or, substituting for $\bar{y}$ its value $y+\theta y'+\phi l_3+\chi l_4$, these critical equations are

$$\theta=\sum y'(\eta-y),\quad \phi=\sum l_3(\eta-y),\quad \chi=\sum l_4(\eta-y).$$

Let t denote the small geodesic arc QO, so that

$$\eta-y=ty'+\tfrac{1}{2}t^2y''+\tfrac{1}{6}t^3y'''+\tfrac{1}{24}t^4y''''+\tfrac{1}{120}t^5y'''''+\dots,$$

terms of order higher than t^5 not occurring in the approximation for Π which is required*.

* Here, as before in § 167, the desired approximation is to be independent of all directional quantities of higher order, such as direction-cosines of principal lines of the geodesic.

Also, for any curve, by the continued use of the Frenet equations, we have relations of the form

$$\begin{aligned}
y'' &= \frac{1}{\rho} Y, \\
y''' &= c_3 l_3 + c_2 Y + c_1 y', \\
y'''' &= d_4 l_4 + d_3 l_3 + d_2 Y + d_1 y', \\
y''''' &= e_5 l_5 + e_4 l_4 + e_3 l_3 + e_2 Y + e_1 y',
\end{aligned}$$

where

$$\begin{aligned}
& c_3 = \frac{1}{\rho\sigma}, \quad c_2 = \frac{d}{ds}\left(\frac{1}{\rho}\right), \quad c_1 = -\frac{1}{\rho^2}; \\
& d_4 = \frac{1}{\rho\sigma\tau}, \quad d_3 = c_3' + \frac{c_2}{\sigma} = \frac{2}{\sigma}\frac{d}{ds}\left(\frac{1}{\rho}\right) + \frac{1}{\rho}\frac{d}{ds}\left(\frac{1}{\sigma}\right), \\
& d_2 = -\frac{c_3}{\sigma} + c_2' + \frac{c_1}{\rho} = \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) - \frac{1}{\rho}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right), \\
& d_1 = -\frac{c_2}{\rho} + c_1' = 3\frac{\rho'}{\rho^3}; \\
& e_5 = \frac{1}{\rho\sigma\tau\kappa}, \\
& e_2 = -\frac{d_3}{\sigma} + d_2' + \frac{d_1}{\rho} = \frac{d^3}{ds^3}\left(\frac{1}{\rho}\right) - \frac{3}{\sigma^2}\frac{d}{ds}\left(\frac{1}{\rho}\right) - \frac{3}{\rho\sigma}\frac{d}{ds}\left(\frac{1}{\sigma}\right) + 6\frac{\rho'}{\rho^4}.
\end{aligned}$$

Hence the values of θ, ϕ, χ, are given by

$$\begin{aligned}
\theta &= \sum y'(\eta - y) = t + \tfrac{1}{6}t^3 c_1 + \tfrac{1}{24}t^4 d_1 + \tfrac{1}{120}t^5 e_1 + \dots, \\
\phi &= \sum l_3(\eta - y) = \tfrac{1}{6}t^3 c_3 + \tfrac{1}{24}t^4 d_3 + \tfrac{1}{120}t^5 e_3 + \dots, \\
\chi &= \sum l_4(\eta - y) = \tfrac{1}{24}t^4 d_4 + \tfrac{1}{120}t^5 e_4 + \dots .
\end{aligned}$$

Accordingly, by the typical equation

$$\begin{aligned}
\bar{Y}\Pi &= \eta - \bar{y} \\
&= \eta - y - \theta y' - \phi l_3 - \chi l_4,
\end{aligned}$$

when these values of θ, ϕ, ψ, are substituted, we have

$$\begin{aligned}
\bar{Y}\Pi = & \; t y' + \tfrac{1}{2}t^2 \frac{Y}{\rho} + \tfrac{1}{6}t^3 y''' + \tfrac{1}{24}t^4 y'''' + \tfrac{1}{120}t^5 y''''' + \dots \\
& - y'(t + \tfrac{1}{6}t^3 c_1 + \tfrac{1}{24}t^4 d_1 + \tfrac{1}{120}t^5 e_1 + \dots) \\
& - l_3(\tfrac{1}{6}t^3 c_3 + \tfrac{1}{24}t^4 d_3 + \tfrac{1}{120}t^5 e_3 + \dots) \\
& - l_4(\tfrac{1}{24}t^4 d_4 + \tfrac{1}{120}t^5 e_3) \\
= & \; \tfrac{1}{2}t^2 \frac{Y}{\rho} + \tfrac{1}{6}t^3 c_2 Y + \tfrac{1}{24}t^4 d_2 Y + \tfrac{1}{120}t^5 (e_2 Y + e_5 l_5) + \dots,
\end{aligned}$$

the term in t^5 involving the typical direction-cosine l_5, and all higher powers of t involving l_5 and typical direction-cosines of higher rank. Let

$$T = \frac{1}{2\rho}t^2 + \tfrac{1}{6}t^3 c_2 + \tfrac{1}{24}t^4 d_2 + \tfrac{1}{120}t^5 e_2,$$

with the values for c_2, d_2, e_2, as given above; then

$$\bar{Y}\Pi = YT + \frac{l_5}{120\rho\sigma\tau\kappa}t^5,$$

accurately up to the fifth power of t inclusive.

Thus the direction of the perpendicular on the tangent flat from the neighbouring point Q of the geodesic is not exactly the same as that of the prime normal of the geodesic at O. As the directions, typified by Y and l_5, are at right angles. we can take

$$\Pi\cos\theta = T, \quad \Pi\sin\theta = \frac{t^5}{120\rho\sigma\tau\kappa},$$

where θ is the inclination of the perpendicular, from Q on the tangent flat, to the prime normal, so that

$$\cos\theta = \sum Y\bar{Y}, \quad \sin\theta = \sum l_5\bar{Y};$$

hence, approximately,

$$\theta = \frac{t^3}{60\sigma\tau\kappa}.$$

Also, up to the fifth power of the small arc-distance t inclusive, the length of the perpendicular Π is

$$\Pi = \frac{1}{2\rho}t^2 + \frac{c_2}{6}t^3 + \frac{d_2}{24}t^4 + \frac{e_2}{120}t^5,$$

with the given values of c_2, d_2, e_2.

Rectifying line of a geodesic.

181. Consider the developable region (denoted by D) which envelops the region along a regional geodesic originating in the direction p', q', r', through O. When D is developed into a flat, there is no stretching, nor tearing; in particular, no arc-length is altered in magnitude, and what was a curve of shortest length remains a curve of shortest length, that is, the geodesic in D, when the developable region is developed into a flat becomes a shortest length in that flat, that is, becomes a straight line. Thus the flat, which is the fundamental element in the construction of D, is called the rectifying flat of the geodesic: manifestly, it is the tangent flat of the region.

Accordingly, in this connection, we have to construct the envelope of the tangent flat of the region for the aggregate of flats taken along the regional geodesic. The equations of the flat have appeared in the form

$$\| \bar{y} - y, y_1, y_2, y_3 \| = 0.$$

But it has been proved that, as a set of three leading lines in the flat, it is possible to take the tangent, the binormal, and the trinormal of a geodesic; and therefore an aggregate of lines, orthogonal to these three, and constituting a complete orthogonal frame for the plenary space when combined with these three, is composed of the other principal lines in the frame of the geodesic, which have typical

direction-cosines denoted by $Y, l_5, l_6, \ldots, l_N$, respectively. Hence the tangent flat of the region can also be represented by the equations

$$\sum(\bar{y}-y)Y=0, \quad \sum(\bar{y}-y)l_5=0, \ldots, \quad \sum(\bar{y}-y)l_N=0,$$

which are $N-3$ in number, the same as the number of independent linear equations in the earlier form.

To obtain the envelope of this flat along the geodesic in the initial direction p', q', r', we associate, with these equations, the set of equations obtained by taking the first arc-derivatives along the geodesic. The first equation

$$\sum(\bar{y}-y)Y=0$$

thus provides an associated equation

$$\sum(\bar{y}-y)Y' - \sum y'Y=0,$$

that is, a new equation

$$\sum\left\{(\bar{y}-y)\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)\right\}=0.$$

The second equation $\sum(\bar{y}-y)l_5=0$ provides an associated equation

$$\sum\left\{(\bar{y}-y)\left(\frac{l_6}{\rho_5}-\frac{l_4}{\kappa}\right)\right\} - \sum y'l_5=0,$$

where $1/\kappa$ is the coil of the geodesic: that is, it provides a new equation

$$\sum(\bar{y}-y)l_4=0.$$

A subsequent equation $\sum(\bar{y}-y)l_\mu=0$, where $5<\mu<N$, provides an associated equation

$$\sum\left\{(\bar{y}-y)\left(\frac{l_{\mu+1}}{\rho_\mu}-\frac{l_{\mu-1}}{\rho_{\mu-1}}\right)\right\} - \sum y'l_\mu=0,$$

which, because of the value of μ, is satisfied by relations already retained. The final equation $\sum(\bar{y}-y)l_N=0$ provides an associated equation

$$\sum\left\{(\bar{y}-y)\left(-\frac{l_{N-1}}{\rho_{N-1}}\right)\right\} - \sum y'l_N=0,$$

which is satisfied by relations already retained. Hence the aggregate of new equations, which must be associated with the initially established set

$$\sum(\bar{y}-y)Y=0, \quad \sum(\bar{y}-y)l_\mu=0, \quad (\mu=5, 6, \ldots, N),$$

in order to provide the required envelope, is constituted by the pair

$$\sum\left\{(\bar{y}-y)\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)\right\}=0, \quad \sum(\bar{y}-y)l_4=0.$$

Thus there are $N-1$ equations in all; and therefore the locus, which they represent, is a line which is a generator of the developable region D. The line is called

the rectifying line of the regional geodesic; it is the line about which the tangent flat must be bent, in order to come into coincidence with the tangent flat at the next consecutive point of the geodesic.

Manifestly, this line lies in the tangent flat, and its equations satisfy all the equations of the initially established set. Also, within that flat, it is at right angles to the trinormal, because of the associated equation

$$\sum (\bar{y}-y)\, l_4 = 0.$$

Consequently, the line lies in the plane which has the tangent and the binormal of the geodesic for its leading lines; and its equations may be taken in the typical form

$$\bar{y}-y = Py' + Ql_3.$$

These equations have to be in accord with the remaining associated equation

$$\sum \left\{ (\bar{y}-y) \left(\frac{l_3}{\sigma} - \frac{y'}{\rho} \right) \right\} = 0 ;$$

and therefore

$$\frac{Q}{\sigma} - \frac{P}{\rho} = 0.$$

Hence the equations of the line are typified by

$$\bar{y}-y = T\left(\frac{y'}{\sigma} + \frac{l_3}{\rho} \right),$$

and its typical direction-cosine is

$$\frac{1}{\mu}\left(\frac{y'}{\sigma} + \frac{l_3}{\rho} \right);$$

and the line is the rectifying line of the geodesic.

In the diagram on p. 518, it is represented by the line OR at right angles to the line OS in the plane BOT through the tangent and the binormal of the regional geodesic.

Gremial and non-gremial principal lines of a geodesic.

182. The directions of the tangent, the binormal, and the trinormal, of any regional geodesic lie within the tangent flat of the region; they therefore can be described as *gremial* to the region (§ 51). The direction of the prime normal of the geodesic is orthogonal to the tangent flat of the region. These four directions are the earliest, in successive rank, among all the principal lines of the geodesic; and all the other principal lines of subsequent rank, being orthogonal to the osculating block of the geodesic and consequently orthogonal to the tangent flat of the region, can be described as *non-gremial* to the region.

In connection with each non-gremial principal line of any geodesic there occur

organic magnitudes, primarily associated with the geodesic, and dominated by the general character of the region without regard to the particular geodesic in any direction. For brevity, they will be called magnitudes of higher grade.

It has been proved (§ 178) that, for all values of i such that $5 \leqslant i \leqslant N$, the relations

$$\sum l_i y_1 = 0, \quad \sum l_i y_2 = 0, \quad \sum l_i y_3 = 0,$$

are satisfied. For the respective values of i, we introduce magnitudes A_i, B_i, C_i, F_i, G_i, H_i, with the definitions

$$\left.\begin{array}{ll} \sum l_i y_{11} = \sum l_i \eta_{11} = A_i, & \sum l_i y_{23} = \sum l_i \eta_{23} = F_i \\ \sum l_i y_{22} = \sum l_i \eta_{22} = B_i, & \sum l_i y_{31} = \sum l_i \eta_{31} = G_i \\ \sum l_i y_{33} = \sum l_i \eta_{33} = C_i, & \sum l_i y_{12} = \sum l_i \eta_{12} = H_i \end{array}\right\}.$$

Differentiating the relation $\sum l_i y_1 = 0$, for any value of $i \geqslant 6$, along the geodesic, we have

$$\begin{aligned} \sum l_i (y_{11} p' + y_{12} p' + y_{13} q') &= -\sum y_1 \frac{dl_i}{ds} \\ &= -\sum y_1 \left(\frac{l_{i+1}}{\rho_i} - \frac{l_{i-1}}{\rho_{i-1}} \right). \end{aligned}$$

The left-hand side

$$= A_i p' + H_i q' + G_i r'.$$

Because $\sum y_1 l_{i+1} = 0$, $\sum y_1 l_{i-1} = 0$, for $i \geqslant 6$, the right-hand side vanishes. Hence

$$A_i p' + H_i q' + G_i r' = 0,$$

for $i \geqslant 6$.

Proceeding similarly from the relations $\sum l_i y_2 = 0$, $\sum l_i y_3 = 0$, for the same range of values of i, we obtain similar equations: the full set is

$$\left.\begin{array}{l} A_i p' + H_i q' + G_i r' = 0 \\ H_i p' + B_i q' + F_i r' = 0 \\ G_i p' + F_i q' + C_i r' = 0 \end{array}\right\},$$

for all values of i greater than five.

But the results do not hold when $i = 5$. We still have the relations of the type $\sum l_5 y_1 = 0$; and therefore, differentiating along the geodesic, and denoting its fourth curvature (the coil) by $1/\kappa$, we have

$$\begin{aligned} \sum l_5 (y_{11} p' + y_{12} q' + y_{13} r') &= -\sum y_1 \frac{dl_5}{ds} \\ &= -\sum y_1 \left(\frac{l_6}{\rho_5} - \frac{l_4}{\kappa} \right) \\ &= \frac{1}{\kappa} \sum y_1 l_4 \\ &= \frac{1}{\kappa} (A\bar{\alpha} + H\bar{\beta} + G\bar{\gamma}), \end{aligned}$$

according to the value of l_4 in § 178. Similarly for the others; the full set of results is

$$u_5 = A_5 p' + H_5 q' + G_5 r' = \frac{1}{\kappa}(A\bar{a} + H\bar{\beta} + G\bar{\gamma}),$$

$$v_5 = H_5 p' + B_5 q' + F_5 r' = \frac{1}{\kappa}(H\bar{a} + B\bar{\beta} + F\bar{\gamma}),$$

$$w_5 = G_5 p' + F_5 q' + C_5 r' = \frac{1}{\kappa}(G\bar{a} + F\bar{\beta} + C\bar{\gamma}).$$

Now

$$1 = \sum l_4^2 = \sum (x_1\bar{a} + x_2\bar{\beta} + x_3\bar{\gamma})^2 = \sum A\bar{a}^2;$$

and therefore

$$\frac{1}{\kappa} = u_5\bar{a} + v_5\bar{\beta} + w_5\bar{\gamma},$$

that is, on the substitution of the values of $\bar{a}$, $\bar{\beta}$, $\bar{\gamma}$,

$$\frac{\Omega^{\frac{1}{2}}}{\sigma\kappa} = \begin{vmatrix} u_5, & u_1, & v_1 \\ v_5, & u_2, & v_2 \\ w_5, & u_3, & v_3 \end{vmatrix}.$$

Again,

$$\sum a u_5^2 = \frac{1}{\kappa^2}\sum a(A\bar{a} + H\bar{\beta} + G\bar{\gamma})^2$$
$$= \frac{\Omega}{\kappa^2}\sum \bar{a}(A\bar{a} + H\bar{\beta} + G\bar{\gamma}) = \frac{\Omega}{\kappa^2}.$$

Further,

$$\sum l_5 y_{ij} = \sum l_5(\eta_{ij} + y_1\Gamma_{ij} + y_2\Delta_{ij} + y_3\Theta_{ij}) = \sum l_5\eta_{ij};$$

and therefore

$$\begin{aligned} u_5p' + v_5q' + w_5r' &= A_5p'^2 + 2H_5p'q' + B_5q'^2 + 2G_5p'r' + 2F_5q'r' + C_5r'^2 \\ &= \sum\{l_5(\sum \eta_{ij}p_i'p_j')\} \\ &= \frac{1}{\rho}\sum l_5 Y = 0, \end{aligned}$$

a permanent relation. Also, substituting from the relations in (ii), of § 182, we have

$$\left.\begin{aligned} u_5 &= \frac{\tau}{\kappa}\left\{u_1\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - \sigma' v_1 - \sigma w_1\right\} \\ v_5 &= \frac{\tau}{\kappa}\left\{u_2\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - \sigma' v_2 - \sigma w_2\right\} \\ w_5 &= \frac{\tau}{\kappa}\left\{u_3\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - \sigma' v_3 - \sigma w_3\right\} \end{aligned}\right\},$$

which verify the relation

$$u_5p' + v_5q' + w_5r' = 0,$$

and, when substituted in the foregoing determinantal expression for $1/\kappa$, reproduce the former relation (p. 514)

$$-\frac{\Omega^{\frac{1}{2}}}{\sigma^2\tau}=\begin{vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \\ w_1, & w_2, & w_3 \end{vmatrix}.$$

Several preliminary results can be established in terms of the magnitudes u_5, v_5, w_5, before we proceed to their more detailed values (§ 189).

(i) Because

$$u_1\bar{v}_1+u_2\bar{v}_2+u_3\bar{v}_3=\frac{1}{\Omega}\sum au_1v_1=\frac{1}{\rho},$$

$$v_1\bar{v}_1+v_2\bar{v}_2+v_3\bar{v}_3=\frac{1}{\Omega}\sum av_1^2=\frac{1}{\rho^2}+\frac{1}{\sigma^2},$$

$$w_1\bar{v}_1+w_2\bar{v}_2+w_3\bar{v}_3=\frac{1}{\Omega}\sum av_1w_1=-\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right),$$

by § 179, we have

$$u_5\bar{v}_1+v_5\bar{v}_2+w_5\bar{v}_3=0.$$

(ii) Because

$$u_1\bar{w}_1+u_2\bar{w}_2+u_3\bar{w}_3=\frac{1}{\Omega}\sum au_1w_1=\frac{d}{ds}\left(\frac{1}{\rho}\right),$$

$$v_1\bar{w}_1+v_2\bar{w}_2+v_3\bar{w}_3=\frac{1}{\Omega}\sum av_1w_1=-\left(\frac{\rho'}{\rho_3}+\frac{\sigma'}{\sigma^3}\right),$$

$$w_1\bar{w}_1+w_2\bar{w}_2+w_3\bar{w}_3=\frac{1}{\Omega}\sum aw_1^2=\frac{1}{\sigma^2\tau^2}+\frac{\rho'^2}{\rho^4}+\frac{\sigma'^2}{\sigma^4},$$

also by § 179, we have

$$u_5\bar{w}_1+v_5\bar{w}_2+w_5\bar{w}_3=-\frac{1}{\sigma\tau\kappa}.$$

(iii) We had equations (p. 515) of the type

$$\frac{1}{\tau}(A\bar{\alpha}+H\bar{\beta}+G\bar{\gamma})=u_1\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\sigma'v_1-\sigma w_1,$$

so that, by the value of u_5 on p. 526,

$$u_5\frac{\kappa}{\tau}=u_1\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\sigma'v_1+\sigma w_1;$$

and the two similar equations are

$$v_5\frac{\kappa}{\tau}=u_2\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\sigma'v_2-\sigma w_2,$$

$$w_5\frac{\kappa}{\tau}=u_3\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\sigma'v_3-\sigma w_3.$$

From these three equations, which are linear and homogeneous in the four quantities $\frac{\kappa}{\tau}$, $\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)$, σ', σ, we find

$$\frac{\frac{\kappa}{\tau}}{\begin{vmatrix} u_1, & v_1, & w_1 \\ u_2, & v_2, & w_2 \\ u_3, & v_3, & w_3 \end{vmatrix}} = \frac{-\sigma}{\begin{vmatrix} u_5, & u_1, & v_1 \\ v_5, & u_2, & v_2 \\ w_5, & u_3, & v_3 \end{vmatrix}} = \frac{\sigma'}{\begin{vmatrix} u_5, & u_1, & w_1 \\ v_5, & u_2, & w_2 \\ w_5, & u_3, & w_3 \end{vmatrix}} = \frac{\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)}{\begin{vmatrix} u_5, & v_1, & w_1 \\ v_5, & v_2, & w_2 \\ w_5, & v_3, & w_3 \end{vmatrix}}.$$

The determinant in the first denominator has the value $-\Omega^{\frac{1}{2}}/\sigma^2\tau$, so that the common value of the fractions

$$= -\frac{\kappa\sigma^2}{\Omega^{\frac{1}{2}}}.$$

The geometrical significance of each of the other three determinants is therefore known. In particular, we thus verify the result for $\Omega^{\frac{1}{2}}/\sigma\kappa$ on p. 526; and we also have

$$\frac{\Omega^{\frac{1}{2}}}{\kappa}\frac{d}{ds}\left(\frac{1}{\sigma}\right) = \begin{vmatrix} u_1, & w_1, & u_5 \\ u_2, & w_2, & v_5 \\ u_3, & w_3, & w_5 \end{vmatrix}, \quad -\frac{\Omega^{\frac{1}{2}}}{\kappa\sigma^2}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) = \begin{vmatrix} v_1, & w_1, & u_5 \\ v_2, & w_2, & v_5 \\ v_3, & w_3, & w_5 \end{vmatrix}.$$

When u_5, v_5, w_5, vanish, the plenary space is quadruple.

183. The values obtained for u_5, v_5, w_5, lead to the theorem that, if they vanish simultaneously for all directions p', q', r', the region lies in a block, that is, in a quadruple homaloidal space. As

$$u_5 = \frac{1}{\kappa}(A\bar{\alpha} + H\bar{\beta} + G\bar{\gamma}), \quad v_5 = \frac{1}{\kappa}(H\bar{\alpha} + B\bar{\beta} + F\bar{\gamma}), \quad w_5 = \frac{1}{\kappa}(G\bar{\alpha} + F\bar{\beta} + C\bar{\gamma}),$$

the simultaneous evanescence of u_5, v_5, w_5, requires either

$$\frac{1}{\kappa} = 0,$$

or

$$\bar{\alpha} = 0, \quad \bar{\beta} = 0, \quad \bar{\gamma} = 0.$$

It will appear later (§ 192) that $1/\kappa$ can vanish for particular directions in a region, without vanishing generally; those are the directions of the curves of globular curvature*. If $1/\kappa$ vanishes for all directions at a point, the coil is zero

* When a region lies in a quadruple homaloidal space, there are no proper curves of globular curvature: an axis of the curve through the centre of spherical curvature, being the line drawn parallel to the trinormal, meets the corresponding axis for the consecutive point on the geodesic.

in all directions and so the osculating block of the geodesic is unique, thus establishing the theorem.

With the alternative inference, $\bar{\alpha}=0$, $\bar{\beta}=0$, $\bar{\gamma}=0$, simultaneously, it follows that the torsion is zero (p. 511). As this is to hold in all directions, it would follow that every geodesic is a plane curve. Also, if we take the formulæ of the type

$$u_5=\frac{\tau}{\kappa}\left\{u_1\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\sigma' v_1-\sigma w_1\right\},$$

when we have $u_5=0$, $v_5=0$, $w_5=0$, the alternative to a vanishing coil is

$$\tau\begin{vmatrix} u_1, & v_1, & w_1 \\ u_2, & v_2, & w_2 \\ u_3, & v_3, & w_3 \end{vmatrix}=0,$$

that is, the earlier result

$$\frac{1}{\sigma^2}=0.$$

The region is of a special character, because all its geodesics are plane curves: a globe is a particular instance.

Partial differential equations satisfied by space-coordinates.

184. By means of the preceding results, it is possible to formulate partial differential equations of the second order satisfied by the space-coordinates of a point in a region. The magnitudes η_{ij}, for all the combinations $i, j, =1, 2, 3$, are functions of position in the region and consequently are vectors when referred to the orthogonal frame of a regional geodesic; and they remain vectors for any internal modifications of the lines of reference in that frame. The three lines, constituted by the tangent, the binormal, and the trinormal, are leading lines for the tangent flat of the region: for them, there can be substituted any other set of three leading lines, in particular, the directions of the parametric curves of the region. Now we have

$$\sum y_1\eta_{ij}=0,\quad \sum y_2\eta_{ij}=0,\quad \sum y_3\eta_{ij}=0\,;$$

and therefore we can take, in the case of η_{11} for instance, a set of equations

$$\eta_{11}=YD_0+l_5D_5+l_6D_6+\ldots+l_ND_N,$$

where η_{11} is typical of the set $\eta_{11}{}^{(m)}$ for $m=1, \ldots, N$, while the quantities $D_0, D_5, D_6, \ldots, D_N$, are the same throughout the set of N equations thus typified. Owing to the relations of orthogonality among the lines with typical direction-cosines $Y, l_5, \ldots, l_N$, we have

$$D_0=\sum Y\eta_{11}=\bar{A},$$
$$D_\mu=\sum l_\mu\eta_{11}=A_\mu,\ \text{for}\ \mu=5, \ldots, N\,;$$

also, because

$$\sum Yy_k=0,\quad \sum l_\mu y_k=0,\quad (k=1, 2, 3),$$

these equations are in accordance with the cited relations

$$\sum y_k y_{11} = 0.$$

We proceed similarly with each of the other quantities η_{ij}; the full set of results is

$$\left.\begin{aligned}
\eta_{11} &= Y\bar{A} + l_5 A_5 + l_6 A_6 + \ldots\ldots + l_N A_N \\
\eta_{12} &= Y\bar{H} + l_5 H_5 + l_6 H_6 + \ldots\ldots + l_N H_N \\
\eta_{13} &= Y\bar{G} + l_5 G_5 + l_6 G_6 + \ldots\ldots + l_N G_N \\
\eta_{22} &= Y\bar{B} + l_5 B_5 + l_6 B_6 + \ldots\ldots + l_N B_N \\
\eta_{23} &= Y\bar{F} + l_5 F_5 + l_6 F_6 + \ldots\ldots + l_N F_N \\
\eta_{33} &= Y\bar{C} + l_5 C_5 + l_6 C_6 + \ldots\ldots + l_N C_N
\end{aligned}\right\}.$$

Let three magnitudes η_1, η_2, η_3, be taken according to the definitions

$$\left.\begin{aligned}
\eta_1 &= \eta_{11}p' + \eta_{12}q' + \eta_{13}r' \\
\eta_2 &= \eta_{12}p' + \eta_{22}q' + \eta_{23}r' \\
\eta_3 &= \eta_{13}p' + \eta_{23}q' + \eta_{33}r'
\end{aligned}\right\}.$$

Then, because

$$\begin{aligned}
A_\mu p' + H_\mu q' + G_\mu r' &= u_5, \text{ when } \mu = 5\,; \quad = 0, \text{ when } 5 < \mu \leqslant N, \\
H_\mu p' + B_\mu q' + F_\mu r' &= v_5, \quad \ldots\ldots\ldots\ldots\,; \quad = 0, \text{ when } 5 < \mu \leqslant N, \\
G_\mu p' + F_\mu r' + C_\mu r' &= w_5, \quad \ldots\ldots\ldots\ldots\,; \quad = 0, \text{ when } 5 < \mu \leqslant N,
\end{aligned}$$

while

$$\bar{A}p' + \bar{H}q' + \bar{G}r' = v_1, \quad \bar{H}p' + \bar{B}q' + \bar{F}r' = v_2, \quad \bar{G}p' + \bar{F}q' + \bar{C}r' = v_3,$$

we have

$$\eta_1 = Yv_1 + l_5 u_5, \quad \eta_2 = Yv_2 + l_5 v_5, \quad \eta_3 = Yv_3 + l_5 w_5.$$

Ex. 1. Multiplying these three equations by p', q', r', adding, and using the results

$$v_1 p' + v_2 q' + v_3 r' = \frac{1}{\rho}, \quad u_5 p' + v_5 q' + w_5 r' = 0,$$

we have the old equation

$$\frac{Y}{\rho} = \eta_1 p' + \eta_2 q' + \eta_3 r' = \sum \eta_{11} p'^2.$$

Ex. 2. There was (§ 172) the relation

$$Y' = -(y_1\bar{v}_1 + y_2\bar{v}_2 + y_3\bar{v}_3).$$

Hence

$$Y'' = -(\bar{v}_1 y_1' + \bar{v}_2 y_2' + \bar{v}_3 y_3') - \left(y_1\frac{d\bar{v}_1}{ds} + y_2\frac{d\bar{v}_2}{ds} + y_3\frac{d\bar{v}_3}{ds}\right).$$

Now

$$\begin{aligned}
y_1' &= y_{11}p' + y_{12}q' + y_{13}r' \\
&= (\eta_{11} + y_1\Gamma_{11} + y_2\Delta_{11} + y_3\Theta_{11})p' + (\eta_{12} + y_1\Gamma_{12} + y_2\Delta_{12} + y_3\Theta_{12})q' \\
&\qquad + (\eta_{13} + y_1\Gamma_{13} + y_2\Delta_{13} + y_3\Theta_{13})r' \\
&= \eta_1 + y_1 a + y_2\xi + y_3\phi,
\end{aligned}$$

with the significance for a, ξ, ϕ, given on p. 494; and similarly

$$y_2' = \eta_2 + y_1\beta + y_2\eta + y_3\chi,$$
$$y_3' = \eta_3 + y_1\gamma + y_2\zeta + y_3\psi.$$

Also, we had (p. 502) the results

$$\frac{d\bar{v}_1}{ds} = \bar{w}_1 - (a\bar{v}_1 + \beta\bar{v}_2 + \gamma\bar{v}_3),$$

$$\frac{d\bar{v}_2}{ds} = \bar{w}_2 - (\xi\bar{v}_1 + \eta\bar{v}_2 + \zeta\bar{v}_3),$$

$$\frac{d\bar{v}_3}{ds} = \bar{w}_3 - (\phi\bar{v}_1 + \chi\bar{v}_2 + \psi\bar{v}_3).$$

Consequently

$$Y'' = -(\eta_1\bar{v}_1 + \eta_2\bar{v}_2 + \eta_3\bar{v}_3) - (y_1\bar{w}_1 + y_2\bar{w}_2 + y_3\bar{w}_3).$$

Further, we have

$$\eta_1\bar{v}_1 + \eta_2\bar{v}_2 + \eta_3\bar{v}_3 = Y(v_1\bar{v}_1 + v_2\bar{v}_2 + v_3\bar{v}_3) + l_5(u_5\bar{v}_1 + v_5\bar{v}_2 + w_5\bar{v}_3);$$

but

$$v_1\bar{v}_1 + v_2\bar{v}_2 + v_3\bar{v}_3 = \frac{1}{\Omega}\sum av_1^2 = \frac{1}{\rho^2} + \frac{1}{\sigma^2},$$

by § 172, and

$$u_5\bar{v}_1 + v_5\bar{v}_2 + w_5\bar{v}_3 = 0,$$

by § 182, (i); and therefore

$$Y'' + Y\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) = -(y_1\bar{w}_1 + y_2\bar{w}_2 + y_3\bar{w}_3),$$

the result obtained in § 179, (iv).

Ex. 3. In the preceding example, it has been proved, incidentally, that

$$\eta_1\bar{v}_1 + \eta_2\bar{v}_2 + \eta_3\bar{v}_3 = Y\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right).$$

In the same way, the result

$$\eta_1\bar{w}_1 + \eta_2\bar{w}_2 + \eta_3\bar{w}_3 = -Y\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right) - \frac{l_5}{\sigma\tau\kappa},$$

can be established.

Ex. 4. The result, verified in *Ex.* 2, arises also when we use the form (§ 172)

$$l_3 = y_1 l + y_2 m + y_3 n,$$

for direct substitution in the Frenet equation for the binormal. With the established values of l, m, n, there given, we have

$$y_1' l + y_2' m + y_3' n = \frac{\sigma}{\rho}(y_1' p' + y_2' q' + y_3' r') - \sigma(y_1'\bar{v}_1 + y_2'\bar{v}_2 + y_3'\bar{v}_3).$$

The first term on the right-hand side

$$= \frac{\sigma}{\rho}\left(\frac{Y}{\rho} - y_1 p'' - y_2 q'' - y_3 r''\right),$$

on using the values for y_1', y_2', y_3', in Example 1. Also

$$\begin{aligned}&y_1'\bar{v}_1+y_2'\bar{v}_2+y_3'\bar{v}_3\\&\quad=\eta_1\bar{v}_1+\eta_2\bar{v}_2+\eta_3\bar{v}_3+y_1(\alpha\bar{v}_1+\beta\bar{v}_2+\gamma\bar{v}_3)+y_2(\xi\bar{v}_1+\eta\bar{v}_2+\zeta\bar{v}_3)+y_3(\phi\bar{v}_1+\chi\bar{v}_2+\psi\bar{v}_3):\end{aligned}$$

while

$$\begin{aligned}\eta_1\bar{v}_1+\eta_2\bar{v}_2+\eta_3\bar{v}_3&=Y(v_1\bar{v}_1+v_2\bar{v}_2+v_3\bar{v}_3)+l_5(u_5\bar{v}_1+v_5\bar{v}_2+w_5\bar{v}_3)\\&=Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right),\end{aligned}$$

by preceding results; and therefore

$$\begin{aligned}&y_1'\bar{v}_1+y_2'\bar{v}_2+y_3'\bar{v}_3\\&\quad=Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)+y_1(\alpha\bar{v}_1+\beta\bar{v}_2+\gamma\bar{v}_3)+y_2(\xi\bar{v}_1+\eta\bar{v}_2+\zeta\bar{v}_3)+y_3(\phi\bar{v}_1+\chi\bar{v}_2+\psi\bar{v}_3).\end{aligned}$$

Again,

$$l'=p'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)+\frac{\sigma}{\rho}p''-\sigma'\bar{v}_1-\sigma\{\bar{w}_1-(\alpha\bar{v}_1+\beta\bar{v}_2+\gamma\bar{v}_3)\},$$

and similarly for m', n'; hence

$$\begin{aligned}y_1l'+y_2m'+y_3n'=y'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)&+\frac{\sigma}{\rho}(y_1p''+y_2q''+y_3r'')\\&-\sigma'(y_1\bar{v}_1+y_2\bar{v}_2+y_3\bar{v}_3)-\sigma(y_1\bar{w}_1+y_2\bar{w}_2+y_3\bar{w}_3)\\&+\sigma\sum y_1(\alpha\bar{v}_1+\beta\bar{v}_2+\gamma\bar{v}_3).\end{aligned}$$

Consequently

$$\begin{aligned}l_3'&=(y_1'l+y_2'm+y_3'n)+(y_1l'+y_2m'+y_3n')\\&=y'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\frac{Y}{\sigma}-\sigma'(y_1\bar{v}_1+y_2\bar{v}_2+y_3\bar{v}_3)-\sigma(y_1\bar{w}_1+y_2\bar{w}_2+y_3\bar{w}_3)\\&=y'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\frac{Y}{\sigma}+\sigma'Y'-\sigma(y_1\bar{w}_1+y_2\bar{w}_2+y_3\bar{w}_3).\end{aligned}$$

But, by Frenet's equation,

$$l_3'=y'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)+\frac{\sigma}{\rho^2}Y+\sigma'Y'+\sigma Y'';$$

hence, as before

$$Y''+Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)=-(y_1\bar{w}_1+y_2\bar{w}_2+y_3\bar{w}_3).$$

Quartinormal of the geodesic: the coil.

185. Before proceeding to consider the analytical bearings of the magnitudes η_1, η_2, η_3, it is desirable to obtain an expression for the typical direction-cosine l_5 of the quartinormal as well as some value for the coil $1/\kappa$ of the regional geodesic.

We have had the relations (§ 178)

$$\frac{l_4}{\sigma}=y_1\bar{l}+y_2\bar{m}+y_3\bar{n},\quad l_4=y_1\bar{\alpha}+y_2\bar{\beta}+y_3\bar{\gamma},$$

where

$$\Omega^{\frac{1}{2}}\bar{l}=u_2v_3-u_3v_2,\quad \Omega^{\frac{1}{2}}\bar{m}=u_3v_1-u_1v_3,\quad \Omega^{\frac{1}{2}}\bar{n}=u_1v_2-u_2v_1.$$

Hence differentiating along the regional geodesic and using the values obtained (§ 175) for $\bar{l}'$, $\bar{m}'$, $\bar{n}'$, we have, by Frenet's equation,

$$\frac{1}{\sigma}\left(\frac{l_5}{\kappa}-\frac{l_3}{\tau}\right)+l_4\frac{d}{ds}\left(\frac{1}{\sigma}\right)=y_1'\bar{l}+y_2'\bar{m}+y_3'\bar{n}+y_1\frac{d\bar{l}}{ds}+y_2\frac{d\bar{m}}{ds}+y_3\frac{d\bar{n}}{ds}$$

$$=\eta_1\bar{l}+\eta_2\bar{m}+\eta_3\bar{n}+\Omega^{-\frac{1}{2}}\begin{vmatrix} y_1, & y_2, & y_3 \\ u_1, & u_2, & u_3 \\ w_1, & w_2, & w_3 \end{vmatrix}.$$

In the determinant

$$\begin{vmatrix} y_1, & y_2, & y_3 \\ u_1, & u_2, & u_3 \\ w_1, & w_2, & w_3 \end{vmatrix}$$

let the values of y_1, y_2, y_3, as expressed (§ 178) in terms of y', l_3, l_4, the variables of an alternative set of leading lines for the tangent flat, be substituted. The resulting coefficient of y' vanishes. The resulting coefficient of l_3

$$=-\sigma\begin{vmatrix} v_1, & v_2, & v_3 \\ u_1, & u_2, & u_3 \\ w_1, & w_2, & w_3 \end{vmatrix}=-\frac{\Omega^{\frac{1}{2}}}{\sigma\tau},$$

by the formula on p. 514. For the resulting coefficient of l_4, the part through y_1 is

$$=l_4\frac{\sigma}{\Omega^{\frac{1}{2}}}\begin{vmatrix} A, & H, & G \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix}=l_4(A\bar{\alpha}+H\bar{\beta}+G\bar{\gamma}),$$

and similarly for the parts through y_2 and y_3; thus in the determinant, the whole term in l_4

$$=l_4\begin{vmatrix} A\bar{\alpha}+H\bar{\beta}+G\bar{\gamma}, & u_1, & w_1 \\ H\bar{\alpha}+B\bar{\beta}+F\bar{\gamma}, & u_2, & w_2 \\ G\bar{\alpha}+F\bar{\beta}+C\bar{\gamma}, & u_3, & w_3 \end{vmatrix}=l_4\Omega^{\frac{1}{2}}\frac{d}{ds}\left(\frac{1}{\sigma}\right),$$

by the result in § 182, (iii). Consequently

$$\begin{vmatrix} y_1, & y_2, & y_3 \\ u_1, & u_2, & u_3 \\ w_1, & w_2, & w_3 \end{vmatrix}=\Omega^{\frac{1}{2}}\left\{-\frac{l_3}{\sigma\tau}+l_4\frac{d}{ds}\left(\frac{1}{\sigma}\right)\right\};$$

and therefore the equation for l_5 becomes

$$\frac{l_5}{\sigma\kappa}=\eta_1\bar{l}+\eta_2\bar{m}+\eta_3\bar{n},$$

or, what are two equivalent forms,

$$\frac{\Omega^{\frac{1}{2}}}{\sigma\kappa} l_5 = \begin{vmatrix} \eta_1, & u_1, & v_1 \\ \eta_2, & u_2, & v_2 \\ \eta_3, & u_3, & v_3 \end{vmatrix},$$

$$\frac{l_5}{\kappa} = \eta_1 \bar{\alpha} + \eta_2 \bar{\beta} + \eta_3 \bar{\gamma}.$$

The substitution of the values

$$\eta_1 = Y v_1 + l_5 u_5, \quad \eta_2 = Y v_2 + l_5 v_5, \quad \eta_3 = Y v_3 + l_5 w_5,$$

merely leads to the earlier relation

$$\frac{\Omega^{\frac{1}{2}}}{\sigma\kappa} = \begin{vmatrix} u_5, & u_1, & v_1 \\ v_5, & u_2, & v_2 \\ w_5, & u_3, & v_3 \end{vmatrix},$$

already (p. 526) established. The actual result, in any of the forms, gives the typical direction-cosine of the quartinormal.

We note that, squaring the foregoing equation for the determinant $(y_1 u_2 w_3)$, we have the former relation in § 179, (viii)

$$\Omega \left(\frac{1}{\sigma^2 \tau^2} + \frac{\sigma'^2}{\sigma^4} \right) = \sum A (u_2 w_3 - u_3 w_2)^2.$$

Gremial orientations associated with a geodesic.

186. A regional geodesic provides three gremial organic directions in a region, through its tangent, its binormal, and its trinormal; and thus there are three gremial orientations which belong to any geodesic in the region, composed from the three gremial directions in pairs.

For the gremial directions, the typical direction-cosines are

$$\begin{aligned} y' &= y_1 p' + y_2 q' + y_3 r', \\ l_3 &= y_1 l \ + y_2 m + y_3 n, \\ l_4 &= y_1 \bar{\alpha} \ + y_2 \bar{\beta} \ + y_3 \bar{\gamma}. \end{aligned}$$

As they are at right angles to one another, the three orientations are likewise at right angles to one another: each of the directions is at right angles to an orientation and is the intersection of the two other orientations.

For the orientation through the binormal and the trinormal, we take surface-variables L_{34}, M_{34}, N_{34}, such that

$$L_{34} = n\bar{\beta} - m\bar{\gamma}, \quad M_{34} = l\bar{\gamma} - n\bar{\alpha}, \quad N_{34} = m\bar{\alpha} - l\bar{\beta}.$$

With the values obtained (§§ 172, 178) for l, m, n, and $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, we have

$$L_{34}=\frac{\sigma^2}{\Omega^{\frac{1}{2}}}\left\{\left(\frac{r'}{\rho}-\bar{v}_3\right)(u_3v_1-u_1v_3)-\left(\frac{q'}{\rho}-\bar{v}_2\right)(u_1v_2-u_2v_1)\right\}$$

$$=\frac{\sigma^2}{\Omega^{\frac{1}{2}}}\left[u_1\left\{(v_2\bar{v}_2+v_3\bar{v}_3)-\frac{1}{\rho}(v_2q'+v_3r')\right\}-v_1\left\{(u_2\bar{v}_2+u_3\bar{v}_3)-\frac{1}{\rho}(u_2q'+u_3r')\right\}\right]$$

$$=\frac{\sigma^2}{\Omega^{\frac{1}{2}}}\left[u_1\left\{\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)-\frac{1}{\rho^2}\right\}-v_1\left(\frac{1}{\rho}-\frac{1}{\rho}\right)\right]=\frac{u_1}{\Omega^{\frac{1}{2}}};$$

and similarly

$$M_{34}=\frac{u_2}{\Omega^{\frac{1}{2}}},\quad N_{34}=\frac{u_3}{\Omega^{\frac{1}{2}}}.$$

The verification that these variables satisfy the permanent surface-condition (§§ 159, 193)

$$\sum aL_{34}^2=1$$

is immediate.

For the orientation through the tangent and the trinormal of the geodesic, we take surface-variables L_{14}, M_{14}, N_{14}, such that

$$L_{14}=q'\bar{\gamma}-r'\bar{\beta},\quad M_{14}=r'\bar{\alpha}-p'\bar{\gamma},\quad N_{14}=p'\bar{\beta}-q'\bar{\alpha}.$$

Their values are given by

$$L_{14}=\frac{\sigma}{\Omega^{\frac{1}{2}}}[q'(u_1v_2-u_2v_1)-r'(u_3v_1-u_1v_3)]$$

$$=\frac{\sigma}{\Omega^{\frac{1}{2}}}\{u_1(q'v_2+r'v_3)-v_1(q'u_2+r'u_3)\}$$

$$=\frac{\sigma}{\Omega^{\frac{1}{2}}}\left(\frac{u_1}{\rho}-v_1\right);$$

and similarly

$$M_{14}=\frac{\sigma}{\Omega^{\frac{1}{2}}}\left(\frac{u_2}{\rho}-v_2\right),\quad N_{14}=\frac{\sigma}{\Omega^{\frac{1}{2}}}\left(\frac{u_3}{\rho}-v_3\right).$$

It is easy to verify, by means of established results, that these surface-variables satisfy the permanent surface-condition

$$\sum a{L_{14}}^2=1.$$

For the orientation through the tangent and the binormal of the geodesic, we take surface-variables L_{13}, M_{13}, N_{13}, such that

$$L_{13}=r'm-q'n,\quad M_{13}=p'n-r'l,\quad N_{13}=q'l-p'm.$$

Using the earlier values of l, m, n, we have

$$L_{13}=\sigma(q'\bar{v}_3-r'\bar{v}_2)$$
$$=\frac{\sigma}{\Omega}\{v_1(gq'-hr')+v_2(fq'-br')+v_3(cq'-fr')\}$$
$$=\frac{\sigma}{\Omega}\{v_1(Hu_3-Gu_2)+v_2(Gu_1-Au_3)+v_3(Au_2-Hu_1)\}$$
$$=\frac{\sigma}{\Omega}\{A(u_2v_3-u_3v_2)+H(u_3v_1-u_1v_3)+G(u_1v_2-u_2v_1)\}$$
$$=\frac{1}{\Omega^{\frac{1}{2}}}(A\bar{\alpha}+H\bar{\beta}+G\bar{\gamma})=\frac{\kappa}{\Omega^{\frac{1}{2}}}u_5;$$

and similarly

$$M_{13}=\frac{\kappa}{\Omega^{\frac{1}{2}}}v_5,\quad N_{13}=\frac{\kappa}{\Omega^{\frac{1}{2}}}w_5.$$

Owing to the value of κ, these surface-variables satisfy the condition

$$\sum aL_{13}^2=1.$$

The quartinormal and the magnitudes $A_5, \ldots, H_5$.

187. From the formula (§ 185)

$$\frac{l_5}{\kappa}=\eta_1\bar{\alpha}+\eta_2\bar{\beta}+\eta_3\bar{\gamma},$$

by squaring, and adding for all the dimensions, we have

$$\frac{1}{\kappa^2}=\bar{\alpha}^2\sum\eta_1{}^2+2\bar{\alpha}\bar{\beta}\sum\eta_1\eta_2+\bar{\beta}^2\sum\eta_2{}^2+2\bar{\alpha}\bar{\gamma}\sum\eta_1\eta_3+2\bar{\beta}\bar{\gamma}\sum\eta_2\eta_3+\bar{\gamma}^2\sum\eta_3{}^2.$$

Now, with the notation of § 168 for the magnitudes $\sum\eta_{ij}\eta_{kl}$, we find

$$\sum\eta_1{}^2=\sum(\eta_{11}p'+\eta_{12}q'+\eta_{13}r')^2$$
$$=\kappa_{400}p'^2+2\kappa_{310}p'q'+\kappa_{220}q'^2+2\kappa_{301}p'r'+2\kappa_{211}q'r'+\kappa_{202}r'^2$$
$$-\tfrac{1}{3}(k_{33}q'^2-2k_{23}q'r'+k_{22}r'^2)$$
$$=\tfrac{1}{12}\frac{\partial^2Q}{\partial p'^2}-\tfrac{1}{3}(k_{33}q'^2-2k_{23}q'r'+k_{22}r'^2),$$

where Q denotes (§ 168) the homogeneous ternary quartic giving the value of $\frac{1}{\rho^2}$. Similarly

$$\sum\eta_2{}^2=\tfrac{1}{12}\frac{\partial^2Q}{\partial q'^2}-\tfrac{1}{3}(k_{11}r'^2-2k_{13}r'p'+k_{33}p'^2),$$
$$\sum\eta_3{}^2=\tfrac{1}{12}\frac{\partial^2Q}{\partial r'^2}-\tfrac{1}{3}(k_{22}p'^2-2k_{12}p'q'+k_{11}q'^2),$$

$$\sum \eta_2\eta_3 = \tfrac{1}{12}\frac{\partial^2 Q}{\partial q'\,\partial r'} + \tfrac{1}{3}(k_{11}q'r' - k_{12}p'r' - k_{13}p'q' + k_{23}p'^2),$$

$$\sum \eta_3\eta_1 = \tfrac{1}{12}\frac{\partial^2 Q}{\partial r'\,\partial p'} + \tfrac{1}{3}(k_{22}r'p' - k_{23}q'p' - k_{21}q'r' + k_{31}q'^2),$$

$$\sum \eta_1\eta_2 = \tfrac{1}{12}\frac{\partial^2 Q}{\partial p'\,\partial q'} + \tfrac{1}{3}(k_{33}p'q' - k_{31}r'q' - k_{32}r'p' + k_{12}r'^2).$$

When these values are substituted in the foregoing expression for $1/\kappa^2$, the aggregate of the terms involving the second derivatives of Q is

$$= \tfrac{1}{12}\left(\bar{\alpha}^2\frac{\partial^2 Q}{\partial p'^2} + 2\bar{\alpha}\bar{\beta}\frac{\partial^2 Q}{\partial p'\,\partial q'} + \bar{\beta}^2\frac{\partial^2 Q}{\partial q'^2} + 2\bar{\alpha}\bar{\gamma}\frac{\partial^2 Q}{\partial p'\,\partial r'} + 2\bar{\beta}\bar{\gamma}\frac{\partial^2 Q}{\partial q'\,\partial r'} + \bar{\gamma}^2\frac{\partial^2 Q}{\partial r'^2}\right);$$

and the aggregate of the terms involving the magnitudes k_{ij} can be arranged in the form

$$-\tfrac{1}{3}(k_{11}L_{14}{}^2 + 2k_{12}L_{14}M_{14} + k_{22}M_{14}{}^2 + 2k_{13}L_{14}N_{14} + 2k_{23}M_{14}N_{14} + k_{33}N_{14}{}^2)$$

Now we have $\sum aL_{14}{}^2 = 1$; and therefore (§ 217) the quantity within brackets is the Riemann sphericity of the region in the gremial orientation defined by the tangent and the trinormal of the geodesic. Denoting this sphericity of the region by S_{14}, we have

$$\frac{1}{\kappa^2} = \tfrac{1}{12}\left(\bar{\alpha}^2\frac{\partial^2 Q}{\partial p'^2} + 2\bar{\alpha}\bar{\beta}\frac{\partial^2 Q}{\partial p'\,\partial q'} + \bar{\beta}^2\frac{\partial^2 Q}{\partial q'^2} + 2\bar{\alpha}\bar{\gamma}\frac{\partial^2 Q}{\partial p'\,\partial r'} + 2\bar{\beta}\bar{\gamma}\frac{\partial^2 Q}{\partial q'\,\partial r'} + \bar{\gamma}^2\frac{\partial^2 Q}{\partial r'^2}\right) - \tfrac{1}{3}S_{14}.$$

Similarly, using the relations

$$u_5 = \sum l_5\eta_1, \quad v_5 = \sum l_5\eta_2, \quad w_5 = \sum l_5\eta_3,$$

the same formula for l_5/κ gives the equation

$$\begin{aligned}\frac{u_5}{\kappa} &= \bar{\alpha}\sum \eta_1{}^2 + \bar{\beta}\sum \eta_1\eta_2 + \bar{\gamma}\sum \eta_1\eta_3 \\ &= \tfrac{1}{12}\left(\bar{\alpha}\frac{\partial^2 Q}{\partial p'^2} + \bar{\beta}\frac{\partial^2 Q}{\partial p'\,\partial q'} + \bar{\gamma}\frac{\partial^2 Q}{\partial p'\,\partial r'}\right) \\ &\qquad + \tfrac{1}{3}q'(L_{14}k_{13} + M_{14}k_{23} + N_{14}k_{33}) - \tfrac{1}{3}r'(L_{14}k_{12} + M_{14}k_{22} + N_{14}k_{23});\end{aligned}$$

and the corresponding equations for v_5 and w_5, in the respective forms

$$\begin{aligned}\frac{v_5}{\kappa} &= \tfrac{1}{12}\left(\bar{\alpha}\frac{\partial^2 Q}{\partial p'\,\partial q'} + \bar{\beta}\frac{\partial^2 Q}{\partial q'^2} + \bar{\gamma}\frac{\partial^2 Q}{\partial q'\,\partial r'}\right) \\ &\qquad + \tfrac{1}{3}r'(L_{14}k_{11} + M_{14}k_{12} + N_{14}k_{13}) - \tfrac{1}{3}p'(L_{14}k_{13} + M_{14}k_{23} + N_{14}k_{33}), \\ \frac{w_5}{\kappa} &= \tfrac{1}{12}\left(\bar{\alpha}\frac{\partial^2 Q}{\partial p'\,\partial r'} + \bar{\beta}\frac{\partial^2 Q}{\partial q'\,\partial r'} + \bar{\gamma}\frac{\partial^2 Q}{\partial r'^2}\right) \\ &\qquad + \tfrac{1}{3}p'(L_{14}k_{12} + M_{14}k_{22} + N_{14}k_{23}) - \tfrac{1}{3}q'(L_{14}k_{11} + M_{14}k_{12} + N_{14}k_{13}).\end{aligned}$$

It is easy to verify that these satisfy the relation (p. 526)

$$u_5 p' + v_5 q' + w_5 r' = 0\,;$$

for

$$\tfrac{1}{12}\left(p'\frac{\partial^2 Q}{\partial p'^2} + q'\frac{\partial^2 Q}{\partial p'\,\partial q'} + r'\frac{\partial^2 Q}{\partial p'\,\partial r'}\right) = \tfrac{1}{4}\frac{\partial Q}{\partial p'} = Q_1,$$

$$\tfrac{1}{12}\left(p'\frac{\partial^2 Q}{\partial p'\,\partial q'} + q'\frac{\partial^2 Q}{\partial q'^2} + r'\frac{\partial^2 Q}{\partial q'\,\partial r'}\right) = \tfrac{1}{4}\frac{\partial Q}{\partial q'} = Q_2,$$

$$\tfrac{1}{12}\left(p'\frac{\partial^2 Q}{\partial p'\,\partial r'} + q'\frac{\partial^2 Q}{\partial q'\,\partial r'} + r'\frac{\partial^2 Q}{\partial r'^2}\right) = \tfrac{1}{4}\frac{\partial Q}{\partial r'} = Q_3,$$

and

$$\bar{a}Q_1 + \bar{\beta}Q_2 + \bar{\gamma}Q_3 = 0,$$

while the terms involving the magnitudes k_{ij} cancel one another in the relation.

Further, the same relation

$$\frac{l_5}{\kappa} = \eta_1\bar{a} + \eta_2\bar{\beta} + \eta_3\bar{\gamma}$$

can be used to obtain explicit expressions for the separate magnitudes A_5, B_5, C_5, F_5, G_5, H_5, which combine to occur in u_5, v_5, w_5. We have

$$\begin{aligned}
\frac{A_5}{\kappa} &= \frac{1}{\kappa}\sum \eta_{11} l_5 \\
&= \bar{a}\sum \eta_{11}\eta_1 + \bar{\beta}\sum \eta_{11}\eta_2 + \bar{\gamma}\sum \eta_{11}\eta_3 \\
&= \tfrac{1}{24}\left(\bar{a}\frac{\partial^3 Q}{\partial p'^3} + \bar{\beta}\frac{\partial^3 Q}{\partial p'^2\,\partial q'} + \bar{\gamma}\frac{\partial^3 Q}{\partial p'^2\,\partial r'}\right) + \tfrac{2}{3}\bar{\beta}(k_{33}q' - k_{23}r') + \tfrac{2}{3}\bar{\gamma}(k_{22}r' - k_{23}q'),
\end{aligned}$$

$$\frac{B_5}{\kappa} = \tfrac{1}{24}\left(\bar{a}\frac{\partial^3 Q}{\partial p'\,\partial q'^2} + \bar{\beta}\frac{\partial^3 Q}{\partial q'^3} + \bar{\gamma}\frac{\partial^3 Q}{\partial q'^2\,\partial r'}\right) + \tfrac{2}{3}\bar{\gamma}(k_{11}r' - k_{13}p') + \tfrac{2}{3}\bar{a}(k_{33}p' - k_{13}r'),$$

$$\frac{C_5}{\kappa} = \tfrac{1}{24}\left(\bar{a}\frac{\partial^3 Q}{\partial p'\,\partial r'^2} + \bar{\beta}\frac{\partial^3 Q}{\partial q'\,\partial r'^2} + \bar{\gamma}\frac{\partial^3 Q}{\partial r'^3}\right) + \tfrac{2}{3}\bar{a}(k_{22}p' - k_{12}q') + \tfrac{2}{3}\bar{\beta}(k_{11}q' - k_{12}p'),$$

$$\begin{aligned}
\frac{F_5}{\kappa} = \tfrac{1}{24}&\left(\bar{a}\frac{\partial^3 Q}{\partial p'\,\partial q'\,\partial r'} + \bar{\beta}\frac{\partial^3 Q}{\partial q'^2\partial r'} + \bar{\gamma}\frac{\partial^3 Q}{\partial q'\,\partial r'^2}\right) \\
&+ \tfrac{1}{3}\{\bar{a}(-2k_{23}p' + k_{13}q' + k_{12}r') + \bar{\beta}(k_{13}p' - k_{11}r') + \bar{\gamma}(k_{12}p' - k_{11}q')\},
\end{aligned}$$

$$\begin{aligned}
\frac{G_5}{\kappa} = \tfrac{1}{24}&\left(\bar{a}\frac{\partial^3 Q}{\partial p'^2\,\partial r'} + \bar{\beta}\frac{\partial^3 Q}{\partial p'\,\partial q'\,\partial r'} + \bar{\gamma}\frac{\partial^3 Q}{\partial p'\,\partial r'^2}\right) \\
&+ \tfrac{1}{3}\{\bar{a}(k_{23}q' - k_{22}r') + \bar{\beta}(k_{23}p' - 2k_{13}q' + k_{12}r') + \bar{\gamma}(-k_{22}p' + k_{12}q')\},
\end{aligned}$$

$$\begin{aligned}
\frac{H_5}{\kappa} = \tfrac{1}{24}&\left(\bar{a}\frac{\partial^3 Q}{\partial p'^2\,\partial q'} + \bar{\beta}\frac{\partial^3 Q}{\partial p'\partial q'^2} + \bar{\gamma}\frac{\partial^3 Q}{\partial p'\,\partial q'\,\partial r'}\right) \\
&+ \tfrac{1}{3}\{\bar{a}(-k_{33}q' + k_{23}r') + \bar{\beta}(-k_{33}p' + k_{13}r') + \bar{\gamma}(k_{23}p' + k_{13}q' - 2k_{12}r')\}.
\end{aligned}$$

Similarly, expressions can be obtained for

$$\frac{\Omega^{\frac{1}{2}}}{\sigma\kappa}\Theta_5,$$

for $\Theta = A, B, C, F, G, H$, in turn, by proceeding from the equivalent equation for the typical direction-cosine l_5,

$$\frac{\Omega^{\frac{1}{2}}}{\sigma\kappa} l_5 = \begin{vmatrix} \eta_1, & u_1, & v_1 \\ \eta_2, & u_2, & v_2 \\ \eta_3, & u_3, & v_3 \end{vmatrix}.$$

Orthogonal plane of a region associated with a geodesic.

188. We have had the typical relations

$$\eta_1 = Yv_1 + l_5 u_5, \quad \eta_2 = Yv_2 + l_5 v_5, \quad \eta_3 = Yv_3 + l_5 w_5 ;$$

consequently

$$\begin{vmatrix} \eta_1, & \eta_2, & \eta_3 \\ v_1, & v_2, & v_3 \\ u_5, & v_5, & w_5 \end{vmatrix} = 0,$$

a typical relation subsisting between each set of three quantities η_1, η_2, η_3, associated with the spatial axes of reference, the quantities constituting the coefficients of η_1, η_2, η_3, being the same throughout the N-relations. The typical relation may also be taken in the form

$$P\eta_1 + R\eta_3 = Q\eta_2.$$

We write

$$\left.\begin{aligned} \mathrm{a} &= \textstyle\sum \eta_1^2 = v_1^2 + u_5^2, & \mathrm{f} &= \textstyle\sum \eta_2\eta_3 = v_2 v_3 + v_5 w_5 \\ \mathrm{b} &= \textstyle\sum \eta_2^2 = v_2^2 + v_5^2, & \mathrm{g} &= \textstyle\sum \eta_3\eta_1 = v_3 v_1 + w_5 u_5 \\ \mathrm{c} &= \textstyle\sum \eta_3^2 = v_3^2 + w_5^2, & \mathrm{h} &= \textstyle\sum \eta_1\eta_2 = v_1 v_2 + u_5 v_5 \end{aligned}\right\}.$$

Now the determinant

$$\begin{vmatrix} v_1^2 + u_5^2, & v_1 v_2 + u_5 v_5, & v_1 v_3 + u_5 w_5 \\ v_1 v_2 + u_5 v_5, & v_2^2 + v_5^2, & v_2 v_3 + v_5 w_5 \\ v_1 v_3 + u_5 w_5, & v_2 v_3 + v_5 w_5, & v_3^2 + w_5^2 \end{vmatrix}$$

vanishes identically; and therefore

$$\begin{vmatrix} \mathrm{a}, & \mathrm{h}, & \mathrm{g} \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f} \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c} \end{vmatrix} = 0,$$

a relation * subsisting among the quantities a, b, c, f, g, h, which involve derivatives

* Formally it is the same as the corresponding relation (*G.F.D.*, vol. i, § 214) among the like quantities for a surface in quadruple space: there, the constituents in the determinant are independent of direction-variables on the surface; here, they involve direction-variables in the region.

of the second order. To develop inferences from this relation, we require minors of the determinant; and we take

$$\left.\begin{aligned} \bar{\mathbf{a}} &= \mathbf{bc} - \mathbf{f}^2, & \bar{\mathbf{f}} &= \mathbf{gh} - \mathbf{af} \\ \bar{\mathbf{b}} &= \mathbf{ca} - \mathbf{g}^2, & \bar{\mathbf{g}} &= \mathbf{hf} - \mathbf{bg} \\ \bar{\mathbf{c}} &= \mathbf{ab} - \mathbf{h}^2, & \bar{\mathbf{h}} &= \mathbf{fg} - \mathbf{ch} \end{aligned}\right\}.$$

The earlier values of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, shew that $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$, are necessarily positive quantities in general; when the radicals $\bar{\mathbf{a}}^{\frac{1}{2}}$, $\bar{\mathbf{b}}^{\frac{1}{2}}$, $\bar{\mathbf{c}}^{\frac{1}{2}}$, occur, a positive sign will be prescribed for them as algebraical magnitudes. Let quantities θ, ϕ, ψ, be selected, such that

$$v_1 = \mathbf{a}^{\frac{1}{2}} \cos\theta, \quad v_2 = \mathbf{b}^{\frac{1}{2}} \cos\phi, \quad v_3 = \mathbf{c}^{\frac{1}{2}} \cos\psi,$$
$$u_5 = \mathbf{a}^{\frac{1}{2}} \sin\theta, \quad v_5 = \mathbf{b}^{\frac{1}{2}} \sin\phi, \quad w_5 = \mathbf{c}^{\frac{1}{2}} \sin\psi.$$

It will appear immediately, from the values of $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$, that no two of the quantities θ, ϕ, ψ, are equal; so, as a standard relation, we postulate

$$\theta > \phi > \psi.$$

Then

$$\mathbf{f} = (\mathbf{bc})^{\frac{1}{2}} \cos(\phi - \psi), \quad \mathbf{g} = (\mathbf{ca})^{\frac{1}{2}} \cos(\theta - \psi), \quad \mathbf{h} = (\mathbf{ab})^{\frac{1}{2}} \cos(\theta - \phi);$$

and therefore, after the preceding postulate,

$$v_5 v_3 - w_5 v_2 = (\mathbf{bc})^{\frac{1}{2}} \sin(\phi - \psi) = (\mathbf{bc} - \mathbf{f}^2)^{\frac{1}{2}} = \bar{\mathbf{a}}^{\frac{1}{2}},$$
$$u_5 v_3 - w_5 v_1 = (\mathbf{ca})^{\frac{1}{2}} \sin(\theta - \psi) = (\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}} = \bar{\mathbf{b}}^{\frac{1}{2}},$$
$$u_5 v_2 - v_5 v_1 = (\mathbf{ab})^{\frac{1}{2}} \sin(\theta - \phi) = (\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}} = \bar{\mathbf{c}}^{\frac{1}{2}}.$$

Now

$$\begin{aligned} \mathbf{af} - \mathbf{gh} &= (v_1^2 + u_5^2)(v_2 v_3 + v_5 w_5) - (v_3 v_1 + w_5 u_5)(v_1 v_2 + u_5 v_5) \\ &= (u_5 v_2 - v_5 v_1)(u_5 v_3 - w_5 v_1) = (\bar{\mathbf{b}}\bar{\mathbf{c}})^{\frac{1}{2}}, \end{aligned}$$

that is,

$$\bar{\mathbf{f}} = -(\bar{\mathbf{b}}\bar{\mathbf{c}})^{\frac{1}{2}}.$$

Similarly for $\bar{\mathbf{g}}$, $\bar{\mathbf{h}}$; the full set of values is

$$\left.\begin{aligned} \bar{\mathbf{f}} &= -(\bar{\mathbf{b}}\bar{\mathbf{c}})^{\frac{1}{2}} \\ \bar{\mathbf{g}} &= (\bar{\mathbf{c}}\bar{\mathbf{a}})^{\frac{1}{2}} \\ \bar{\mathbf{h}} &= -(\bar{\mathbf{a}}\bar{\mathbf{b}})^{\frac{1}{2}} \end{aligned}\right\},$$

for the standard relation postulated.

Returning to the typical linear relation

$$P\eta_1 + R\eta_3 = Q\eta_2$$

between the quantities η_1, η_2, η_3, we can express the coefficients P, Q, R, in terms of the preceding magnitudes. Let it be multiplied by η_1, and the results for all

the dimensions be added; let it also be multiplied by η_3, and the results similarly be added; then, in turn,

$$P\mathfrak{a}+R\mathfrak{g}=Q\mathfrak{h},$$
$$P\mathfrak{g}+R\mathfrak{c}=Q\mathfrak{f}.$$

Thus

$$P(\mathfrak{ac}-\mathfrak{g}^2)=Q(\mathfrak{ch}-\mathfrak{fg})=-Q\bar{\mathfrak{h}}=Q(\bar{\mathfrak{a}}\bar{\mathfrak{b}})^{\frac{1}{2}},$$

so that

$$P\bar{\mathfrak{b}}^{\frac{1}{2}}=Q\bar{\mathfrak{a}}^{\frac{1}{2}};$$

and, similarly,

$$R\bar{\mathfrak{b}}^{\frac{1}{2}}=Q\bar{\mathfrak{c}}^{\frac{1}{2}}.$$

Hence the typical linear relation becomes

$$\eta_1\bar{\mathfrak{a}}^{\frac{1}{2}}-\eta_2\bar{\mathfrak{b}}^{\frac{1}{2}}+\eta_3\bar{\mathfrak{c}}^{\frac{1}{2}}=0;$$

the rational equivalent is

$$\begin{vmatrix} \mathfrak{a}, & \mathfrak{h}, & \mathfrak{g}, & \eta_1 \\ \mathfrak{h}, & \mathfrak{b}, & \mathfrak{f}, & \eta_2 \\ \mathfrak{g}, & \mathfrak{f}, & \mathfrak{c}, & \eta_3 \\ \eta_1, & \eta_2, & \eta_3, & 0 \end{vmatrix}=0,$$

which can be established otherwise.

When the values of η_1, η_2, η_3, in terms of Y and l_5, are substituted in the linear relation, the two equations

$$\bar{\mathfrak{a}}^{\frac{1}{2}}v_1-\bar{\mathfrak{b}}^{\frac{1}{2}}v_2+\bar{\mathfrak{c}}^{\frac{1}{2}}v_3=0,$$
$$\bar{\mathfrak{a}}^{\frac{1}{2}}u_5-\bar{\mathfrak{b}}^{\frac{1}{2}}v_5+\bar{\mathfrak{c}}^{\frac{1}{2}}w_5=0,$$

can be inferred immediately. Also, we have (p. 526)

$$\frac{\Omega^{\frac{1}{2}}}{\sigma\kappa}\begin{vmatrix} u_5, & v_5, & w_5 \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix}=-\bar{\mathfrak{a}}^{\frac{1}{2}}u_1+\bar{\mathfrak{b}}^{\frac{1}{2}}u_2-\bar{\mathfrak{c}}^{\frac{1}{2}}u_3.$$

The values of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$, are expressible (§ 168) in terms of magnitudes appertaining to the region and of the direction-variables p', q', r'.

Because the equations

$$\sum \eta_1 y_i=0, \quad (i=1, 2, 3),$$

are satisfied, the quantities typified by

$$\eta_1\mathfrak{a}^{-\frac{1}{2}}$$

are direction-cosines of a line at right angles to the tangent flat of the region; and likewise for the two sets of quantities, typified respectively by

$$\eta_2\mathfrak{b}^{-\frac{1}{2}}, \quad \eta_3\mathfrak{c}^{-\frac{1}{2}}.$$

All three directions lie in the plane

$$\| \; \bar{y}-y, \quad Y, \quad l_5 \; \|=0,$$

which is orthogonal to the tangent flat: it is, of course, a plane changing from geodesic to geodesic.

For the inclination of the direction typified by $\eta_1\mathbf{a}^{-\frac{1}{2}}$ to the prime normal, we have

$$\sum Y(\eta_1\mathbf{a}^{-\frac{1}{2}})=v_1\mathbf{a}^{-\frac{1}{2}}=\cos\theta\,;$$

that is, the preceding angle θ is the inclination of that direction to the prime normal of the geodesic, while $\frac{1}{2}\pi-\theta$ is its inclination to the quartinormal of the geodesic. Similarly ϕ and ψ are the respective inclinations of the directions typified by $\eta_2\mathbf{b}^{-\frac{1}{2}}$, $\eta_3\mathbf{c}^{-\frac{1}{2}}$, to the prime normal; and the postulation

$$\theta>\phi>\psi$$

assumes that the direction typified by $\eta_2\mathbf{b}^{-\frac{1}{2}}$ lies between the other two directions, all of them lying in the orthogonal plane.

Values of u_5, v_5, w_5.

189. In the next place, let the values of u_5, v_5, w_5, in terms of θ, ϕ, ψ, be substituted in the relation

$$u_5p'+v_5q'+w_5r'=0\,;$$

then

$$\mathbf{a}^{\frac{1}{2}}p'\sin\theta+\mathbf{b}^{\frac{1}{2}}q'\sin\phi+\mathbf{c}^{\frac{1}{2}}r'\sin\psi=0,$$

and therefore

$$\mathbf{a}^{\frac{1}{2}}p'\sin\theta+\mathbf{b}^{\frac{1}{2}}q'\{\sin\theta\cos(\theta-\phi)-\cos\theta\sin(\theta-\phi)\}$$
$$+\mathbf{c}^{\frac{1}{2}}r'\{\sin\theta\cos(\theta-\psi)-\cos\theta\sin(\theta-\psi)\}=0.$$

The aggregated coefficient of $\sin\theta$ on the left-hand side

$$=\mathbf{a}^{\frac{1}{2}}p'+\mathbf{b}^{\frac{1}{2}}q'\frac{\mathbf{h}}{(\mathbf{ab})^{\frac{1}{2}}}+\mathbf{c}^{\frac{1}{2}}r'\frac{\mathbf{g}}{(\mathbf{ac})^{\frac{1}{2}}}$$
$$=\frac{1}{\mathbf{a}^{\frac{1}{2}}}(\mathbf{a}p'+\mathbf{h}q'+\mathbf{g}r')\,;$$

but

$$\mathbf{a}p'+\mathbf{h}q'+\mathbf{g}r'=\sum\eta_1(\eta_1p'+\eta_2q'+\eta_3r')=\frac{1}{\rho}\sum Y\eta_1=\frac{v_1}{\rho}=\frac{\mathbf{a}^{\frac{1}{2}}}{\rho}\cos\theta,$$

so that the coefficient of $\sin\theta$

$$=\frac{\cos\theta}{\rho}.$$

The aggregated coefficient of $-\cos\theta$ on the left-hand side

$$=\mathfrak{b}^{\frac{1}{2}}q'\frac{\bar{\mathfrak{c}}^{\frac{1}{2}}}{(\mathfrak{ab})^{\frac{1}{2}}}+\mathfrak{c}^{\frac{1}{2}}r'\frac{\bar{\mathfrak{b}}^{\frac{1}{2}}}{(\mathfrak{ac})^{\frac{1}{2}}}$$
$$=\frac{1}{\mathfrak{a}^{\frac{1}{2}}}(\bar{\mathfrak{c}}^{\frac{1}{2}}q'+\bar{\mathfrak{b}}^{\frac{1}{2}}r').$$

The equation becomes

$$\sin\theta\frac{\cos\theta}{\rho}-\frac{1}{\mathfrak{a}^{\frac{1}{2}}}(\bar{\mathfrak{c}}^{\frac{1}{2}}q'+\bar{\mathfrak{b}}^{\frac{1}{2}}r')\cos\theta=0,$$

and therefore

$$\frac{1}{\rho}\mathfrak{a}^{\frac{1}{2}}\sin\theta=\bar{\mathfrak{c}}^{\frac{1}{2}}q'+\bar{\mathfrak{b}}^{\frac{1}{2}}r',$$

thus giving a value for u_5. Similarly for v_5 and w_5; in all, we have

$$\left.\begin{aligned}\frac{1}{\rho}u_5 &= \phantom{-\bar{\mathfrak{c}}^{\frac{1}{2}}p'-\bar{\mathfrak{a}}^{\frac{1}{2}}q'}\ \bar{\mathfrak{c}}^{\frac{1}{2}}q'+\bar{\mathfrak{b}}^{\frac{1}{2}}r'\\ \frac{1}{\rho}v_5 &= -\bar{\mathfrak{c}}^{\frac{1}{2}}p' \qquad\qquad +\bar{\mathfrak{a}}^{\frac{1}{2}}r'\\ \frac{1}{\rho}w_5 &= -\bar{\mathfrak{b}}^{\frac{1}{2}}p'-\bar{\mathfrak{a}}^{\frac{1}{2}}q'\end{aligned}\right\},$$

values manifestly satisfying the earlier condition

$$\bar{\mathfrak{a}}^{\frac{1}{2}}u_5-\bar{\mathfrak{b}}^{\frac{1}{2}}v_5+\bar{\mathfrak{c}}^{\frac{1}{2}}w_5=0.$$

Ex. Verify that the condition

$$\bar{\mathfrak{a}}^{\frac{1}{2}}v_1-\bar{\mathfrak{b}}^{\frac{1}{2}}v_2+\bar{\mathfrak{c}}^{\frac{1}{2}}v_3=0$$

is satisfied identically by the foregoing values of v_1, v_2, v_3, in terms of θ, ϕ, ψ, without any residuary relation.

Again, we had (§ 182) three equations of the type

$$u_5=\frac{\tau}{\kappa}\left\{u_1\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\sigma'v_1-\sigma w_1\right\};$$

let these values of u_5, v_5, w_5, be substituted in the foregoing condition, and let the cited similar condition in v_1, v_2, v_3, be used. Then

$$\bar{\mathfrak{a}}^{\frac{1}{2}}w_1-\bar{\mathfrak{b}}^{\frac{1}{2}}w_2+\bar{\mathfrak{c}}^{\frac{1}{2}}w_3=(\bar{\mathfrak{a}}^{\frac{1}{2}}u_1-\bar{\mathfrak{b}}^{\frac{1}{2}}u_2+\bar{\mathfrak{c}}^{\frac{1}{2}}u_3)\left\{\frac{1}{\sigma}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}$$
$$=-\frac{\Omega}{\sigma^2\kappa}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right),$$

by the relation in § 182.

We thus have expressions for the magnitudes

$$\bar{\mathfrak{a}}^{\frac{1}{2}}z_1-\bar{\mathfrak{b}}^{\frac{1}{2}}z_2+\bar{\mathfrak{c}}^{\frac{1}{2}}z_3,$$

which are ternariants connected with the region, when

$$\text{(i) } z_1, z_2, z_3, = u_1, u_2, u_3, \qquad \text{(ii) } z_1, z_2, z_3, = w_1, w_2, w_3,$$

both these quantities involving the coil, and when

$$\text{(iii) } z_1, z_2, z_3, = v_1, v_2, v_3, \qquad \text{(iv) } z_1, z_2, z_3, = u_5, v_5, w_5,$$

both these quantities being zero.

Further, we have (p. 526) obtained the result

$$\sum a u_5^2 = \frac{\Omega}{\kappa^2},$$

as well as the results (p. 527)

$$\sum a u_1 u_5 = \Omega(u_5 p' + v_5 q' + w_5 q') = 0,$$
$$u_5 \bar{v}_1 + v_5 \bar{v}_2 + w_5 \bar{v}_3 = 0;$$

hence multiplying the three cited values of u_5, v_5, w_5, from § 182 by

$$a u_5 + h v_5 + g w_5, \quad h u_5 + b v_5 + f w_5, \quad g u_5 + f v_5 + c w_5,$$

respectively, and adding, we have

$$\frac{\Omega}{\kappa^2} = -\frac{\sigma\tau}{\kappa} \sum a u_5 w_1,$$

that is,

$$\sum a u_5 w_1 = -\frac{\Omega}{\sigma\tau\kappa},$$

which can also be written in the form

$$u_5 \bar{w}_1 + v_5 \bar{w}_2 + w_5 \bar{w}_3 = -\frac{1}{\sigma\tau\kappa}.$$

It follows from these results, and from earlier results in §§ 171, 172, 179, that all the concomitants

$$\begin{array}{llll}
\sum a u_1^2, & & & \\
\sum a u_1 v_1, & \sum a v_1^2, & & \\
\sum a u_1 w_1, & \sum a v_1 w_1, & \sum a w_1^2, & \\
\sum a u_1 u_5, & \sum a v_1 u_5, & \sum a w_1 u_5, & \sum a u_5^2,
\end{array}$$

are known as to their geometrical significance; and, by means of the relations

$$\eta_1 = Y v_1 + l_5 u_5, \quad \eta_2 = Y v_2 + l_5 v_5, \quad \eta_3 = Y v_3 + l_5 w_5,$$

the values of the concomitants

$$\sum a u_1 \eta_1, \quad \sum a v_1 \eta_1, \quad \sum a w_1 \eta_1, \quad \sum a u_5 \eta_1,$$

can be deduced immediately.

Ex. To illustrate the use of these formulæ in the derivation of other concomitants, we take

$$\frac{Y}{\rho}=\eta_1 p'+\eta_2 q'+\eta_3 r',$$

so that

$$\frac{1}{\rho^2}=\sum\left(\frac{Y}{\rho}\right)^2=\mathfrak{a}p'^2+2\mathfrak{h}p'q'+\mathfrak{b}q'^2+2\mathfrak{g}p'r'+2\mathfrak{f}q'r'+\mathfrak{c}r'^2,$$

apparently a ternary quadratic form in p', q', r', though the coefficients $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$, themselves involve those variables. When thus regarded as a quadratic ternary form, with coefficients in which the implicit occurrence of p', q', r', is ignored, its discriminant

$$\begin{vmatrix} \mathfrak{a}, & \mathfrak{h}, & \mathfrak{g} \\ \mathfrak{h}, & \mathfrak{b}, & \mathfrak{f} \\ \mathfrak{g}, & \mathfrak{f}, & \mathfrak{c} \end{vmatrix}$$

is known to vanish.

There is also the permanent arc-relation

$$1=Ap'^2+2Hp'q'+Bq'^2+2Gp'r'+2Fq'r'+Cr'^2,$$

where the coefficients A, B, C, F, G, H, in the quadratic form are definitely independent of p', q', r', being functions of position alone. Its discriminant is Ω, denoting the determinant

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix};$$

and the geometrical significance of Ω is associated (§ 159) with the volume of a rudimentary parallelepiped with conterminous small edges in the region.

Now two ternary quadratic forms T_1 and T_2 possess four invariants, being the coefficients of the different combinations of λ_1 and λ_2 in the discriminant of $\lambda_1 T_1+\lambda_2 T_2$. It remains, in order to have all the invariants of the two forms under consideration, to determine the geometrical significance of these invariants. We have had the relations

$$\sum a v_1{}^2=\Omega\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right),\quad \sum a v_1 v_5=0,\quad \sum a u_5{}^2=\frac{\Omega}{\kappa^2};$$

and therefore

$$\begin{aligned}\frac{\Omega^2}{\kappa^2}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)&=\left(\sum a v_1{}^2\right)\left(\sum a u_5{}^2\right)-\left(\sum a v_1 u_5\right)^2\\ &=\Omega\sum A\,(v_2 w_5-v_3 v_5)^2.\end{aligned}$$

The values of the arguments in the quadratic form on the right-hand side are known (p. 543) in terms of the minors $\bar{\mathfrak{a}}$, $\bar{\mathfrak{b}}$, $\bar{\mathfrak{c}}$, $\bar{\mathfrak{f}}$, $\bar{\mathfrak{g}}$, $\bar{\mathfrak{h}}$; when their values are substituted, and the relations among the minors are used, we have

$$A\bar{\mathfrak{a}}+2H\bar{\mathfrak{h}}+B\bar{\mathfrak{b}}+2G\bar{\mathfrak{g}}+2F\bar{\mathfrak{f}}+C\bar{\mathfrak{c}}=\Omega\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)\frac{1}{\kappa^2},$$

which gives the geometrical significance of one of the two intermediate invariants.

Again, with the relations

$$\eta_1 = Yv_1 + l_5 u_5, \quad \eta_2 = Yv_2 + l_5 v_5, \quad \eta_3 = Yv_3 + l_5 w_5,$$

we have

$$\begin{aligned} a\eta_1^2 + 2h\eta_1\eta_2 + b\eta_2^2 + 2g\eta_1\eta_3 + 2f\eta_2\eta_3 + c\eta_3^2 \\ &= Y^2(\textstyle\sum av_1^2) + 2Yl_5(\sum av_1u_5) + l_5^2(\sum au_5^2) \\ &= \Omega\left\{Y^2\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) + l_5^2\frac{1}{\kappa^2}\right\}. \end{aligned}$$

Consequently, adding for all the dimensions of the plenary space, we have

$$a\mathbf{a} + 2h\mathbf{h} + b\mathbf{b} + 2g\mathbf{g} + 2f\mathbf{f} + c\mathbf{c} = \Omega\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}+\frac{1}{\kappa^2}\right),$$

which gives the geometrical significance of the other of the two intermediate invariants, when the expression for $1/\rho^2$ is regarded as a quadratic form.

The quantity $\sum a\eta_1^2$ can also be regarded as a concomitant of the form $\sum Ap'^2$, when η_1, η_2, η_3, are taken as variables contragradient to p', q', r' : it is a contravariant. The corresponding contravariant of the quadratic form $\sum \mathbf{a}p'^2$

$$\begin{aligned} &= \bar{\mathbf{a}}\eta_1^2 + 2\bar{\mathbf{h}}\eta_1\eta_2 + \bar{\mathbf{b}}\eta_2^2 + 2\mathbf{g}\eta_1\eta_3 + 2\mathbf{f}\eta_2\eta_3 + \mathbf{c}\eta_3^2 \\ &= Y^2(\textstyle\sum\bar{\mathbf{a}}v_1^2) + 2Yl_5(\sum\bar{\mathbf{a}}v_1u_5) + l_5^2(\sum\bar{\mathbf{a}}u_5^2), \end{aligned}$$

which vanishes because

$$\sum\bar{\mathbf{a}}v_1^2 = (\bar{\mathbf{a}}^{\frac{1}{2}}v_1 - \bar{\mathbf{b}}^{\frac{1}{2}}v_2 + \bar{\mathbf{c}}^{\frac{1}{2}}v_3)^2 = 0,$$

$$\sum\bar{\mathbf{a}}v_1u_5 = (\bar{\mathbf{a}}^{\frac{1}{2}}v_1 - \bar{\mathbf{b}}^{\frac{1}{2}}v_2 + \bar{\mathbf{c}}^{\frac{1}{2}}v_3)(\bar{\mathbf{a}}^{\frac{1}{2}}u_5 - \bar{\mathbf{b}}^{\frac{1}{2}}v_5 + \bar{\mathbf{c}}^{\frac{1}{2}}w_5) = 0,$$

$$\sum\bar{\mathbf{a}}u_5^2 = (\bar{\mathbf{a}}^{\frac{1}{2}}u_5 - \bar{\mathbf{b}}^{\frac{1}{2}}v_5 + \bar{\mathbf{c}}^{\frac{1}{2}}w_5)^2 = 0.$$

Curves of circular curvature.

190. Curves of circular curvature, which have occurred on surfaces in plenary space of more than three dimensions and are of fundamental organic importance on surfaces in triple homaloidal space, arise similarly in regions. When the homaloidal space containing the region is quadruple, the curves of curvature have the same kind of organic importance in the region as they have for surfaces in triple space ; their significance does not appear so prominently when the plenary space has more than four dimensions. Their derivation is the same for all configurations, emerging from one or other of the two properties. For a region, as for a surface, a curve of curvature can be defined, either as a curve such that the prime normals of consecutive geodesic tangents intersect, or as a curve whose geodesic tangent at each point is such as to give a maximum or minimum among the circular curvatures of the regional geodesics through the point. We consider these in turn, to prove that they lead to the same analytical and geometrical result.

In the first definition, connected with the fact of intersection of prime normals of geodesics, we refer the osculating block of the region to the organic lines of

the geodesics. The osculating plane of a curve at any point contains two consecutive tangents and the prime normal. Let A, B, C, D, be four consecutive points connected with geodesic tangents. The lines AB, BC, and the prime normal of the geodesic at B, lie in one plane which can be taken as the plane through BC and that prime normal at B; the lines BC, CD, and the prime normal of the geodesic tangent at C, lie in one plane which can be taken as the plane through BC and that prime normal at C. If the normals at B and at C meet, these two planes are one and the same, so that the lines AB, BC, CD, and the prime normals at B and at C, all lie in one plane. Thus the osculating plane of the geodesic at B coincides with the osculating plane of the same geodesic at C: in other words, the osculating plane of the geodesic at the point is stationary, so that the torsion of the geodesic must vanish, and therefore

$$\frac{1}{\sigma}=0.$$

For the analytical derivation of the result from this property, we proceed from the typical equation of the prime normal in the form

$$\bar{y}-y=YD,$$

where D is a parametric distance along the normal. If the line is intersected by the prime normal at a consecutive point $y+dy$, the new direction-cosine being $Y+dY$, we now take $\bar{y}$ to be the typical coordinate of the intersection, so that, if $D+dD$ be the intercept on the second normal between the point $y+dy$ and the point of intersection, we have

$$\bar{y}-(y+dy)=(Y+dY)(D+dD).$$

Thus there are equations

$$-dy=D\,.\,dY+Y\,.\,dD+dY\,.\,dD,$$

one such equation holding for each space-coordinate. Now, always

$$\sum Y^2=1, \quad \sum Y\,dY=0;$$

and because the prime normal is at right angles to every regional direction, we have

$$\sum Y\,dy=0.$$

Hence, multiplying by Y, and adding, we have

$$dD=0,$$

as is to be expected from the geometry of the small triangle: that is,

$$-dy=D\,dY,$$

or, proceeding to the limit as BA decreases,

$$-y'=DY'=D\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right).$$

Multiply by y', and add for all the equations: also by l_3, and similarly add: then we have

$$D=\rho, \quad \frac{\rho}{\sigma}=0.$$

As ρ does not vanish, we obtain

$$\frac{1}{\sigma}=0,$$

the property in question.

Under the second definition, associated with maximum or minimum circular curvature of the geodesics in various directions through O, we have to find the maximum or the minimum values of $1/\rho$ for varying values of p', q', r', subject to the permanent relation $\sum Ap'^2=1$. Now

$$\frac{\partial}{\partial p'}\left(\frac{1}{\rho}\right)=2v_1, \quad \frac{\partial}{\partial q'}\left(\frac{1}{\rho}\right)=2v_2, \quad \frac{\partial}{\partial r'}\left(\frac{1}{\rho}\right)=2v_3;$$

and therefore the critical relations are

$$v_1=\mu u_1, \quad v_2=\mu u_2, \quad v_3=\mu u_3,$$

μ being a multiplier left undetermined in forming these relations. To find μ, multiply these relations by p', q', r', respectively, and add: then

$$\frac{1}{\rho}=\mu,$$

so that the equations of the directions of the curves of curvature are

$$v_1-\frac{u_1}{\rho}=0, \quad v_2-\frac{u_2}{\rho}=0, \quad v_3-\frac{u_3}{\rho}=0.$$

The equations

$$\frac{1}{\sigma}\sum y_1l_3=\frac{u_1}{\rho}-v_1, \quad \frac{1}{\sigma}\sum y_2l_3=\frac{u_2}{\rho}-v_2, \quad \frac{1}{\sigma}\sum y_3l_3=\frac{u_3}{\rho}-v_3,$$

were established: the binormal direction l_3 lies in the tangent flat, so that not all the quantities $\sum y_il_3$ (for $i=1, 2, 3$) vanish. Hence

$$\frac{1}{\sigma}=0,$$

the same result as before.

The equations can be taken in the form

$$\frac{v_1}{u_1}=\frac{v_2}{u_2}=\frac{v_3}{u_3};$$

but the coefficients $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{F}$, $\bar{G}$, $\bar{H}$, in v_1, v_2, v_3, involve p', q', r', implicitly, so that the equations are less simple for a plenary space in general than for a quadruple plenary space.

Another mode of regarding the equations

$$v_1 - \frac{u_1}{\rho} = 0, \quad v_2 - \frac{u_2}{\rho} = 0, \quad v_3 - \frac{u_3}{\rho} = 0,$$

analytically is to note that, in fact, they are the three equations which initially are required to form the (vanishing) discriminant of the ternary quartic homogeneous form

$$Q - \frac{1}{\rho^2}(\sum A p'^2)^2,$$

where Q is the quartic of § 168. Thus the equations might serve two purposes. In their first form, they could provide the directions at O of the curves of circular curvature of the region. By the vanishing of the discriminant of the indicated quartic ternary form, they would lead to the algebraic equation for the principal radii of circular curvature of the region at O.

When the equations are used in the form

$$\frac{v_1}{v_3} = \frac{u_1}{u_3}, \quad \frac{v_2}{v_3} = \frac{u_2}{u_3},$$

they suggest an inference that there are sixteen directions of curves of curvature at any point O in the region. Apparently the actual formation of the discriminant of a ternary quartic of the most general type has not yet been achieved; and therefore a confirmation of the inference, through the degree of the discriminant in the magnitude $1/\rho^2$, cannot thus be obtained analytically.

Curves of spherical curvature.

191. A region possesses curves of spherical curvature, in their source analogous to curves of circular curvature. They can be defined as curves, along which consecutive geodesic tangents have intersecting radii of spherical curvature. Thus consider the regional geodesic in the direction p', q', r'; along its prime normal and its binormal, measure respective lengths α and β, thus obtaining a point

$$\bar{y}_0 = y + \alpha Y + \beta l_3.$$

At a consecutive point $p + dp$, $q + dq$, $r + dr$, which lies on the geodesic, let a similar line be drawn, with such (small) modifications in α and β so as to make them $\alpha + \alpha' ds$ and $\beta + \beta' ds$, chosen so that (if possible) the two lines thus drawn shall

intersect. When the point of intersection is denoted by $\bar{y}_0$ as its typical coordinate, we have

$$\begin{aligned}0&=y'+\alpha' Y+\alpha Y'+\beta' l_3+\beta l_3'\\&=y'+\alpha' Y+\alpha\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)+\beta' l_3+\beta\left(\frac{l_4}{\tau}-\frac{Y}{\sigma}\right),\end{aligned}$$

a relation holding in association with each space-coordinate. Hence

$$1-\frac{\alpha}{\rho}=0,\quad \alpha'-\frac{\beta}{\sigma}=0,\quad \frac{\alpha}{\sigma}+\beta'=0,\quad \frac{\beta}{\tau}=0.$$

From the first two of these equations, we have

$$\alpha=\rho,\quad \beta=\sigma\alpha'=\sigma\rho',$$

shewing that the point of intersection (if any) is the centre of spherical curvature of the geodesic.

The third equation now gives

$$\frac{\rho}{\sigma}+\sigma\rho''+\sigma'\rho'=0.$$

But, always,
$$R^2=\rho^2+\sigma^2\rho'^2,$$
and therefore
$$RR'=\sigma\rho'\left(\frac{\rho}{\sigma}+\sigma\rho''+\sigma'\rho'\right),$$

so that R' vanishes in the present circumstances: consequently, the spherical curvature is stationary along the direction (which is not the same property as a maximum or a minimum among its values for all directions).

Finally, the fourth equation gives

$$\frac{1}{\tau}=0,$$

because β is not zero: or the tilt of a geodesic touching a curve of spherical curvature vanishes at the point of contact. As the tilt is the arc-rate of change of the inclination of osculating flats of the geodesic, and as the intersection of consecutive radii of spherical curvature entails a stationary position of the osculating flat, it follows that a vanishing tilt is a geometrical accompaniment of the intersecting of consecutive radii of spherical curvature.

To state analytical equations for the direction of a curve of spherical curvature, we use the general results of § 179 concerning the tilt. When $1/\tau$ vanishes, as here is necessary, we have

$$\left.\begin{aligned}u_1\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-(\sigma' v_1+\sigma w_1)=0\\u_2\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-(\sigma' v_2+\sigma w_2)=0\\u_3\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-(\sigma' v_3+\sigma w_3)=0\end{aligned}\right\}.$$

As equations, to be associated with $\sum Ap'^2=1$ for the determination of the direction of a curve of spherical curvature, we take

$$\frac{1}{u_1}\left(\frac{\sigma'}{\sigma}v_1+w_1\right)=\frac{1}{u_2}\left(\frac{\sigma'}{\sigma}v_2+w_2\right)=\frac{1}{u_3}\left(\frac{\sigma'}{\sigma}v_3+w_3\right),$$

utilising the value of σ'/σ obtained in § 182 : or we may take one of these two equations, together with

$$\begin{vmatrix} u_1, & v_1, & w_1 \\ u_2, & v_2, & w_2 \\ u_3, & v_3, & w_3 \end{vmatrix}=0.$$

Curves of globular curvature.

192. Similarly, a region possesses curves of globular curvature, these being defined as curves, the successive regional geodesic tangents to which have intersecting radii of globular curvature.

As for curves of spherical curvature, we consider a regional geodesic at O in the initial direction p', q', r' ; along the prime normal, we measure a distance α ; from the extremity of α, along the axial line of the geodesic which is parallel to the binormal, we measure a distance β ; and from the extremity of β, along the axial line of the geodesic which is parallel to the trinormal, we measure a distance γ. Then we obtain a point with a typical coordinate $\bar{\bar{y}}$, where

$$\bar{\bar{y}}-y=\alpha Y+\beta l_3+\gamma l_4.$$

We draw a corresponding line at a consecutive point of the locus, this point lying on the geodesic : and we suppose the quantities α, β, γ, to receive such small continuous changes as to allow the two lines to intersect. Then, if $\bar{\bar{y}}$ is taken to be the typical coordinate of the intersection, the quantities α, β, γ, and their arc-derivatives must satisfy the typical condition

$$-y'=\alpha' Y+\alpha\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)+\beta' l_3+\beta\left(\frac{l_4}{\tau}-\frac{Y}{\sigma}\right)+\gamma' l_4+\gamma\left(\frac{l_5}{\kappa}-\frac{l_3}{\tau}\right),$$

a condition holding for each space-coordinate. Consequently, we have the equations

$$\begin{aligned}
-1&=-\frac{\alpha}{\rho},\\
0&=\alpha'-\frac{\beta}{\sigma},\\
0&=\frac{\alpha}{\sigma}+\beta'-\frac{\gamma}{\tau},\\
0&=\frac{\beta}{\tau}+\gamma',\\
0&=\frac{\gamma}{\kappa}.
\end{aligned}$$

From the first three of these equations, we have

$$\alpha=\rho,$$
$$\beta=\sigma\rho',$$
$$\gamma=\tau\left(\frac{\rho}{\sigma}+\sigma\rho''+\sigma'\rho'\right)=\tau\frac{RR'}{\sigma\rho'},$$

shewing that the point, attained by the measured distances, is the centre of globular curvature.

The fourth equation now gives

$$\gamma'+\frac{\sigma\rho'}{\tau}=0,$$

and therefore

$$\gamma\gamma'+RR'=0.$$

The radius of globular curvature of any curve is given * by

$$G^2=R^2+\left(\tau\frac{RR'}{\sigma\rho'}\right)^2=R^2+\gamma^2\,;$$

and thus the preceding relation, valid along the curve-locus in question, becomes

$$GG'=0.$$

The argument has assumed that the plenary space has five dimensions at least: and, in a plenary space of four dimensions, no such investigation is needed, because consecutive radii of globular curvature do meet, for all geodesics. Hence, for our instance,

$$G'=0:$$

that is, the magnitude of the globular curvature is stationary for consecutive geodesic tangents to a curve of globular curvature.

Finally, there is the fifth equation; because γ does not vanish in general, we have

$$\frac{1}{\kappa}=0:$$

or the coil of the regional geodesic tangent to a curve of globular curvature vanishes at the point of contact.

To obtain parametric differential equations of the curves of globular curvature, we use results already established (§ 182) in the form

$$u_5=\frac{\sigma}{\kappa\Omega^{\frac{1}{2}}}\begin{vmatrix} A, & H, & G \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix},\quad v_5=\frac{\sigma}{\kappa\Omega^{\frac{1}{2}}}\begin{vmatrix} H, & B, & F \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix},\quad w_5=\frac{\sigma}{\kappa\Omega^{\frac{1}{2}}}\begin{vmatrix} G, & F, & C \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix}.$$

* *G.F.D.*, vol. i, § 200.

As the coil is to vanish for regional geodesics touching the curves of globular curvature, we have

$$u_5=0, \quad v_5=0, \quad w_5=0,$$

which are only two equations because of the relation

$$u_5 p' + v_5 q' + w_5 r' = 0.$$

Again, if we take, as in § 188,

$$\left.\begin{array}{lll} \mathfrak{a}=\sum \eta_1{}^2\ , & \mathfrak{b}=\sum \eta_2{}^2\ , & \mathfrak{c}=\sum \eta_3{}^2 \\ \mathfrak{f}=\sum \eta_2\eta_3, & \mathfrak{g}=\sum \eta_3\eta_1, & \mathfrak{h}=\sum \eta_1\eta_2 \end{array}\right\},$$

so that the symmetrical determinant of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$, vanishes, and if we write

$$\bar{\mathfrak{a}} = \mathfrak{bc} - \mathfrak{f}^2, \quad \bar{\mathfrak{b}} = \mathfrak{ca} - \mathfrak{g}^2, \quad \bar{\mathfrak{c}} = \mathfrak{ab} - \mathfrak{h}^2,$$

we have (§ 189)

$$\frac{1}{\rho} u_5 = \bar{\mathfrak{c}}^{\frac{1}{2}} q' + \bar{\mathfrak{b}}^{\frac{1}{2}} r', \quad \frac{1}{\rho} v_5 = -\bar{\mathfrak{c}}^{\frac{1}{2}} p' + \bar{\mathfrak{a}}^{\frac{1}{2}} r', \quad \frac{1}{\rho} w_5 = -\bar{\mathfrak{b}}^{\frac{1}{2}} p' - \bar{\mathfrak{a}}^{\frac{1}{2}} q' \,;$$

and therefore the equations of the curves of globular curvature in the region are

$$p'\bar{\mathfrak{a}}^{-\frac{1}{2}} = -q'\bar{\mathfrak{b}}^{-\frac{1}{2}} = r'\bar{\mathfrak{c}}^{-\frac{1}{2}}.$$

Ex. From the formula for l_5 in § 187, or otherwise, shew that the equations

$$\begin{vmatrix} p'\sum \eta_{11}\eta_{ij} + q'\sum \eta_{12}\eta_{ij} + r'\sum \eta_{13}\eta_{ij}, & u_1, & v_1 \\ p'\sum \eta_{21}\eta_{ij} + q'\sum \eta_{22}\eta_{ij} + r'\sum \eta_{23}\eta_{ij}, & u_2, & v_2 \\ p'\sum \eta_{31}\eta_{ij} + q'\sum \eta_{32}\eta_{ij} + r'\sum \eta_{33}\eta_{ij}, & u_3, & v_3 \end{vmatrix} = 0$$

are satisfied along the curves of globular curvature, for all the combinations $ij = 11, 12, 22, 13, 23, 33$.

PRINTED IN GREAT BRITAIN
BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW